316-466 Monetary Economics — Midsemester Practice

1. **Asset pricing**. Consider the Lucas (1978) asset pricing model where a representative consumer has preferences

$$E_0\left\{\sum_{t=0}^{\infty}\beta^t U(c_t)\right\}, \qquad 0<\beta<1$$

with strictly increasing, concave period utility $U(c)$ and with flow budget constraints

$$c_t + p_t s_{t+1} \leq (p_t + y_t)s_t, \qquad s_0 = 1 \text{ given}$$

Here s_t denotes the number of shares of a long-lived (durable asset), with fixed supply normalized to one. The relative price of this asset in terms of consumption is p_t. The dividends issued on this asset follow an AR(1) process. Specifically

$$y_{t+1} = (1-\rho)\bar{y} + \rho y_t + \epsilon_{t+1} \qquad \bar{y}>0, \qquad 0\leq\rho\leq 1$$

with $E_t\{\epsilon_{t+1}\} = 0$ and $E_t\{\epsilon_{t+1}^2\} = \sigma_\epsilon^2$. [If it bothers you that this process will go negative with some probability and that this means consumption would also have to be negative, don't be overly concerned: I can make that probability vanishingly small by making $\bar{y}$ very large relative to σ_ϵ, so just ignore your concerns when answering the questions below].

(a) Derive a first order condition characterizing the price of this asset. Write this condition so that it expresses the price as a stochastic difference equation.

(b) Show that the only bounded solution to this stochastic difference equation is

$$p_t = E_t\left\{\sum_{j=1}^{\infty}\beta^t \frac{U'(c_{t+j})}{U'(c_t)} y_{t+j}\right\}$$

(c) What is equilibrium consumption in this model? What is the long run average consumption? What is the long run variance of consumption?

(d) Solve for the equilibrium asset price assuming *risk neutrality*, $U(c) = ac$ for $a>0$. Discuss how equilibrium asset prices depend on the serial correlation of dividends ρ, on the discount factor β, on the mean endowment $\bar{y}$ and on the position of the current endowment relative to its mean, $y_t - \bar{y}$. Give *economic* intuition for all your answers.

(e) Solve for the equilibrium return on this asset

$$R_{t+1} = \frac{p_{t+1} + y_{t+1}}{p_t}$$

How does the equilibrium return depend on the persistence ρ? How does it depend on dividend *growth*? [You may find it easier to answer these questions in the case when $\bar{y} = 0$]. Also compute the expected return $E_t\{R_{t+1}\}$. Is this return time-varying? Why or why not? [Hint: is there a risk premium?]. Again, give economic intuition for all your answers.

(f) Repeat parts (d) and (e) assuming *log preferences*, $U(c) = \log(c)$. How, if at all, do your answers change?

2. **Stochastic growth model**. Consider the Brock-Mirman (1972) model where a social planner has preferences

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t U(c_t) \right\}, \qquad 0 < \beta < 1$$

with strictly increasing, concave period utility $U(c)$ and with resource constraint

$$c_t + k_{t+1} \leq z_t A k_t^{\alpha} + (1-\delta) k_t \qquad k_0 > 0 \text{ given}$$

where $A > 0$ and $0 < \alpha < 1$. Here k_t denotes the physical capital stock and z_t denotes an exogenous stochastic productivity shock. Specifically, suppose

$$\log(z_{t+1}) = \rho \log(z_t) + \epsilon_{t+1} \qquad 0 \leq \rho \leq 1$$

with $\epsilon_{t+1} \sim \mathcal{N}(0, \sigma_{\epsilon}^2)$.

(a) Derive a first order condition characterizing the optimal accumulation of capital.

(b) Suppose $U(c) = \log(c)$ and that $\delta = 1$. Now guess that optimal consumption is

$$c_t = (1 - \alpha\beta) z_t A k_t^{\alpha}$$

What is the capital accumulation equation implied by this consumption function? Interpret these "policy" functions in terms of the usual saddle-path diagram. Show that in this case ($U(c) = \log(c)$ and that $\delta = 1$) this consumption function satisfies the first order conditions obtained in part (a).

(c) What kind of stochastic process does the log capital stock follow? Characterize the long run mean and variance of the log capital stock. How quickly does the capital stock revert to its mean? How does this mean reversion depend on α? Interpret your results. You might like to compare this model to the Solow (1956) growth model — in what ways is it similar?

3. **Stochastic growth model (again)**. Consider the growth model as in Question 2.

(a) Assume now that

$$U(c) = \frac{1}{1-\sigma}(c^{1-\sigma} - 1), \qquad \sigma > 0$$

and that $0 < \delta < 1$. Solve for the non-stochastic steady state (with $\log(\bar{z}) = 0$ so that $\bar{z} = 1$).

(b) Now log-linearize the first order conditions around the non-stochastic steady state. Likewise, log-linearize the resource constraint. Combine these two results to show that the capital stock satisfies a 2nd order stochastic difference equation of the form

$$E_t\{F\Delta\hat{k}_{t+2} + G\Delta\hat{k}_{t+1} + H\Delta\hat{k}_t + J\Delta\hat{z}_{t+1}\} = E_t\{K\hat{z}_{t+1} + L\hat{k}_{t+1}\}$$

where $\Delta\hat{k}_{t+1} \equiv \hat{k}_{t+1} - \hat{k}_t$ and $\hat{k}_t \equiv \log(k_t/\bar{k})$, etc, and where F, G, H etc denote matrices of coefficients that depend on parameters of the model (both directly and via the steady state). State F, G, H etc explicitly in terms of parameters and your results from part (a).

(c) Guess that

$$\hat{k}_{t+1} = P\hat{k}_t + Q\hat{z}_t$$

and show how to solve for P and Q as a function of the parameters of the model. Go as far as you can in solving for P and Q. What issue do you encounter when solving for P? How is this issue usually resolved in models of this kind?

4. **Money in the utility function**. Consider a deterministic money in the utility function model where a representative consumer has preferences

$$\sum_{t=0}^{\infty} \beta^t U(c_t, m_t)$$

with period budget constraint

$$P_t c_t + M_{t+1} + q_t B_{t+1} \leq P_t \bar{y} + M_t + B_t - T_t$$

where $m_t \equiv M_{t+1}/P_t$ denotes real balances, B_{t+1} denotes nominal one-period bond holdings with relative price q_t and T_t denotes nominal taxes. The output of goods is a constant $\bar{y} > 0$ each period. The initial conditions $M_0 > 0$ and $B_0 = 0$ are given. The budget constraint of the government is

$$\mathcal{M}_{t+1} + T_t = \mathcal{M}_t$$

(a) Derive first order conditions that characterize the consumer's choice of bond and money holdings.

(b) Suppose that the monetary policy is $\mathcal{M}_t = M_0$ all t. Suppose also that preferences are given by

$$U(c, m) = \left[\omega c^{1-\rho} + (1-\omega)m^{1-\rho}\right]^{1/(1-\rho)}$$

where $0 < \omega < 1$ and $\rho > 0$. Solve for the consumption, taxes, bond holdings, bond price/nominal interest rate, and price level in a stationary equilibrium (where the price level is constant). Give economic interpretations to all these solutions.

(c) Is it possible for there to also exist non-stationary equilibria where the price level is not constant? Explain how this might occur or why it cannot occur. What possible interpretations would non-stationary equilibria have in this model?

Chris Edmond, 8 September 2003