316-466 Monetary Economics — Homework 1 Solutions

1. Since consumers will never waste any valuable resources, I will write all constraints with equality. The constraints of the consumer can be combined to give

$$c_t + k_{t+1} = w_t n_t + (1 + r_t - \delta)k_t \tag{1}$$

with some initial capital k_0 given. The Lagrangian for this problem can be written

$$L = \sum_{t=0}^{T} \{\beta^t u(c_t, 1 - n_t) + \lambda_t[w_t n_t + (1 + r_t - \delta)k_t - c_t - k_{t+1}]\}$$

where $\lambda_t \geq 0$ denote the Lagrange multipliers. The FONC for this problem include

$$\begin{aligned} \beta^t u_c(c_t, \ell_t) &= \lambda_t \\ \beta^t u_\ell(c_t, \ell_t) &= \lambda_t w_t \\ -\lambda_t + (1 + r_{t+1} - \delta)\,\lambda_{t+1} &\leq 0 \qquad \text{with equality if } k_{t+1} > 0 \end{aligned}$$

Since $\lambda_T = \beta^T u_c(c_T, \ell_T) > 0$, consumption is valued in the last period. So capital accumulation in the last period incurs a cost, λ_T, with no offsetting benefit. The optimal thing to do is have $k_{T+1} = 0$. Combining the first order conditions gives

$$\frac{u_\ell(c_t, \ell_t)}{u_c(c_t, \ell_t)} = w_t$$

This says that at an optimum, the MRS between leisure and consumption is equal to the relative price of leisure in terms of consumption, the real wage. The marginal cost is the marginal utility of foregone leisure. The marginal benefit is the return per unit of labor (the real wage) valued according to what that real wage can buy (a marginal unit of consumption). Similarly

$$\beta \frac{u_c(c_{t+1}, \ell_{t+1})}{u_c(c_t, \ell_t)} = \frac{1}{1 + r_{t+1} - \delta}$$

equates the intertemporal marginal rate of substitution with the relative price of consumption. A consumer that sacrifices a unit of consumption incurs the cost $u_c(c_t, \ell_t)$ and receives the return $1 + r_{t+1} - \delta$ per unit of consumption sacrificed. However, next period, this return is valued by $\beta u_c(c_{t+1}, \ell_{t+1})$. The real interest rate, the real return on savings, is not the rental rate of capital r_{t+1}. The real interest rate is the rental rate less the depreciation rate, $r_{t+1} - \delta$. The consumer has to be compensated for the physical depreciation of capital. Notice that the relative prices of consumption are

$$\beta \frac{u_c(c_{t+1}, \ell_{t+1})}{u_c(c_t, \ell_t)} = \frac{p_{t+1}}{p_t} = \frac{1}{1 + r_{t+1} - \delta}$$

So the rate at which the Arrow-Debreu prices shrink over time is determined by $r_{t+1} - \delta$ (see below).

2. Write equation (1) as

$$k_1 = w_0 n_0 + (1 + r_0 - \delta)k_0 - c_0 \tag{2}$$

but

$$\begin{aligned} k_2 &= w_1 n_1 + (1 + r_1 - \delta)k_1 - c_1 \\ &= w_1 n_1 + (1 + r_1 - \delta)[w_0 n_0 + (1 + r_0 - \delta)k_0 - c_0] - c_1 \end{aligned}$$

and

$$\begin{aligned} k_3 &= w_2 n_2 + (1 + r_2 - \delta)k_2 - c_2 \\ &= w_2 n_2 + (1 + r_2 - \delta)\{w_1 n_1 + (1 + r_1 - \delta)[w_0 n_0 + (1 + r_0 - \delta)k_0 - c_0] - c_1\} - c_2 \end{aligned}$$

and so on. Define the Arrow-Debreu prices

$$p_t \equiv \prod_{s=0}^{t} \frac{1}{1 + r_s - \delta}$$

with $p_0 = 1$ adopted as a normalization. Notice that this implies

$$\frac{p_{t+1}}{p_t} = \frac{\prod_{s=0}^{t+1} \frac{1}{1+r_t-\delta}}{\prod_{s=0}^{t} \frac{1}{1+r_t-\delta}} = \frac{1}{1 + r_{t+1} - \delta}$$

Multiplying the tth iteration of (2) by p_t, this allows us to write

$$p_t k_{t+1} = \sum_{s=0}^{t} p_s(w_s n_s - c_s) + k_0$$

or for $t = T$,

$$p_T k_{T+1} = \sum_{t=0}^{T} p_t(w_t n_t - c_t) + k_0$$

The Lagrangian for this problem is

$$L = \sum_{t=0}^{T} \beta^t u(c_t, 1 - n_t) + \mu \left[\sum_{t=0}^{T} p_t(w_t n_t - c_t) + k_0 - p_T k_{T+1} \right]$$

Notice that now there is a single Lagrange multiplier $\mu \geq 0$ for the single intertemporal budget constraint. The FONC for this problem include

$$\begin{aligned} \beta^t u_c(c_t, \ell_t) &= \mu p_t \\ \beta^t u_\ell(c_t, \ell_t) &= \mu p_t w_t \\ -\mu p_T &\leq 0 \qquad \text{with equality if } k_{t+1} > 0 \end{aligned}$$

Notice that there are only first order conditions for consumption, labor supply, and the terminal capital stock. Once again, $k_{T+1} = 0$ because otherwise a cost would be paid with no offsetting benefit. The product μp_t plays the role λ_t played in Question 1. Obviously, we get

$$\frac{u_\ell(c_t, \ell_t)}{u_c(c_t, \ell_t)} = \frac{\mu p_t}{\mu p_t} w_t = w_t$$

as before. Similarly, by taking the optimality conditions for consumption at any two adjacent dates t and $t+1$ we get

$$\frac{\beta^t u_c(c_t, \ell_t)}{p_t} = \frac{\beta^{t+1} u_c(c_{t+1}, \ell_{t+1})}{p_{t+1}} = \mu$$

Hence

$$\beta \frac{u_c(c_{t+1}, \ell_{t+1})}{u_c(c_t, \ell_t)} = \frac{p_{t+1}}{p_t} = \frac{1}{1 + r_{t+1} - \delta}$$

as before. Thus we have an identical set of first order conditions as in Question 1.

3. The FONC for the firm are

$$\begin{aligned} p_t F_k(k_t, n_t) &= p_t r_t \\ p_t F_n(k_t, n_t) &= p_t w_t \end{aligned}$$

for each t. If $p_t > 0$, these are the same as

$$\begin{aligned} F_k(k_t, n_t) &= r_t \\ F_n(k_t, n_t) &= w_t \end{aligned}$$

The FONC require that the firm hire factors of production until the (value) marginal product of the factor is equal to the factor's rental rate — the marginal cost of hiring an additional unit. Since there is nothing linking the firm's objective function or constraint set over different time periods, the firm does not really face an intertemporal problem. It can just solve the one-shot problem of maximizing static profits

$$F(k, n) - wn - rk$$

each period. (The firm has nothing like the capital accumulation decision of the consumer that links the consumer's flow constraints at two adjacent dates t and $t+1$; see below, however). Suppose that a differentiable $f(\mathbf{x})$ exhibits constant returns to scale in $\mathbf{x} = (x_1, \cdots, x_n)$. Then $f(\lambda \mathbf{x}) \equiv \lambda f(\mathbf{x})$ for any constant $\lambda > 0$. But then differentiating with respect to λ on both sides gives

$$\begin{aligned} \frac{d}{d\lambda} f(\lambda \mathbf{x}) &= \sum_{i=1}^{n} \frac{\partial}{\partial(\lambda x_i)} f(\lambda \mathbf{x}) x_i \qquad \text{(by the chain rule)} \\ \frac{d}{d\lambda} \lambda f(\mathbf{x}) &= f(\mathbf{x}) \end{aligned}$$

This implies the representation

$$f(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial}{\partial(\lambda x_i)} f(\lambda \mathbf{x}) x_i$$

for any $\lambda > 0$. So setting $\lambda = 1$

$$f(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(\mathbf{x}) x_i$$

Now take $\mathbf{x} = (k, n)$ and $f(\mathbf{x}) = F(k, n)$. The result we've just established (which is an implication of Euler's Theorem for homogeneous functions) gives

$$F(k, n) = F_k(k, n)k + F_n(k, n)n$$

But then profit maximization gives

$$F(k, n) = rk + wn$$

So economic profits are indeed zero. [I did not require you to derive Euler's result, just to state it and use it].

4. Suppose now that the consumer does not own capital but the firm does. The firm's static profits each period are now output less labor costs less any investment

$$F(k_t, n_t) - w_t n_t - i_t$$

where

$$k_{t+1} = (1 - \delta)k_t + i_t$$

The Lagrangian for the firm's intertemporal optimization problem is

$$L = \sum_{t=0}^{T} p_t[F(k_t, n_t) - w_t n_t - k_{t+1} + (1 - \delta)k_t]$$

So the FONC include

$$\begin{aligned} p_{t+1}[1 + F_k(k_{t+1}, n_{t+1}) - \delta] &= p_t \\ p_t F_n(k_t, n_t) &= p_t w_t \end{aligned}$$

The static labor demand choice yields the familiar requirement $F_n(k_t, n_t) = w_t$. The choice of capital requires

$$\frac{p_{t+1}}{p_t} = \frac{1}{1 + F_k(k_{t+1}, n_{t+1}) - \delta}$$

If we interpret the marginal product of capital $F_k(k_{t+1}, n_{t+1})$ as the shadow rental rate for the firm, we have the same allocations as before.

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