# 316-466 Monetary Economics — Homework 2 Solutions

1. $M_t(s^{t-1})$ was chosen in the previous period before the realization of the state $s_t$ at the beginning of period $t$ so it depends on $s^{t-1}$ and not on the entire history $s^t$. Put differently, $M_t(s^{t-1})$ is *predetermined* at date $t$. Similarly, in the previous period, an entire portfolio of Arrow securities was chosen, $\{B_t(s^{t-1}, s')\}_{s' \in \mathcal{S}}$. At the beginning of this period $t$, exactly one $s_t \in \mathcal{S}$ was realized and the consumer's wealth depends on the quantity of Arrow securities corresponding to this particular state, i.e., $\{B_t(s^{t-1}, s')\}_{s'=s_t} = B_t(s^{t-1}, s_t)$.

2. The Lagrangian can be written

$$\begin{aligned} L &= \sum_{t=0}^{\infty}\sum_{s^t}\beta^t u\left[c_t(s^t), \frac{M_{t+1}(s^t)}{P_t(s^t)}\right] f(s^t \mid s_0) \\ &\quad + \sum_{t=0}^{\infty}\sum_{s^t}\lambda_t(s^t)\left[P_t(s^t)y_t(s^t) + M_t(s^{t-1}) + B_t(s^{t-1}, s_t) - T_t(s^t)\right. \\ &\quad \left. - P_t(s^t)c_t(s^t) - M_{t+1}(s^t) - \sum_{s'} q_t(s^t, s')B_{t+1}(s^t, s')\right] \end{aligned}$$

where $\lambda_t(s^t) \geq 0$ denotes the Lagrange multiplier for date $t$ given history $s^t$. The key FONC are

$$\begin{aligned} \lambda_t(s^t) &= \beta^t \frac{u_c[c_t(s^t), m_t(s^t)]}{P_t(s^t)} f(s^t \mid s_0) \\ \lambda_t(s^t) &= \beta^t \frac{u_m[c_t(s^t), m_t(s^t)]}{P_t(s^t)} f(s^t \mid s_0) + \sum_{s'} \lambda_{t+1}(s^t, s') \\ q_t(s^t, s')\lambda_t(s^t) &= \lambda_{t+1}(s^t, s') \end{aligned}$$

The first of these is the standard condition relating the Lagrange multiplier to the marginal utility of (a unit of account's worth of) consumption. The second condition says that holding money from this period to next means giving up some consumption today, paying the marginal cost $\lambda_t(s^t)$, but getting in return some utility dividends plus having some money to spend tomorrow. Notice that in nominal terms, the payoff of money is certain and a dollar saved today is available for spending no matter what state is realized tomorrow. Hence we have to sum over the possible states $s' \in \mathcal{S}$ that can follow the history $s^t$. Holding a state contingent bond that pays of in state $s' \in \mathcal{S}$ given $s^t$ incurs the marginal cost $q_t(s^t, s')\lambda_t(s^t)$ which must be matched by the marginal benefit $\lambda_{t+1}(s^t, s')$. Notice however that there is no sum here — the state contingent bonds are risky assets even in nominal terms because they pay off if and only if the relevant state does in fact realize.

3. Suppose that we want to replicate a safe nominal bond that pays a unit of account irrespective of which state realizes at $t+1$. This bond is replicated by holding exactly

one of each Arrow security $s' \in \mathcal{S}$ and so its price should (by no arbitrage) be the sum of the Arrow security prices. The price of this bond is, therefore,

$$\begin{aligned}
\sum_{s'} q_t(s^t, s') &= \sum_{s'} \frac{\lambda_{t+1}(s^t, s')}{\lambda_t(s^t)} \\
&= \sum_{s'} \frac{\beta^{t+1}}{\beta^t} \frac{u_c[c_{t+1}(s^t, s'), m_{t+1}(s^t, s')]}{u_c[c_t(s^t), m_t(s^t)]} \frac{P_t(s^t)}{P_{t+1}(s^t, s')} \frac{f(s^t, s' \mid s_0)}{f(s^t \mid s_0)} \\
&= \sum_{s'} \beta \frac{u_c[c_{t+1}(s^t, s'), m_{t+1}(s^t, s')]}{u_c[c_t(s^t), m_t(s^t)]} \frac{P_t(s^t)}{P_{t+1}(s^t, s')} f(s' \mid s_t) \\
&= E_t \left\{ \beta \frac{u_c(c_{t+1}, m_{t+1})}{u_c(c_t, m_t)} \frac{P_t}{P_{t+1}} \right\}
\end{aligned}$$

If we denote the nominal interest rate on this safe nominal bond by $i_t(s^t)$ we have

$$\frac{1}{1 + i_t(s^t)} = \sum_{s'} \beta \frac{u_c[c_{t+1}(s^t, s'), m_{t+1}(s^t, s')]}{u_c[c_t(s^t), m_t(s^t)]} \frac{P_t(s^t)}{P_{t+1}(s^t, s')} f(s' \mid s_t)$$

The *real* return on this bond is risky because the inflation rate between $t$ and $t+1$ is not known at $t$. Letting $r_{t+1}$ denote the random real interest rate on a bond maturing at date $t+1$, we have

$$\begin{aligned}
1 &= E_t \left\{ \beta \frac{u_c(c_{t+1}, m_{t+1})}{u_c(c_t, m_t)} (1 + r_{t+1}) \right\} \\
1 &= E_t \left\{ \beta \frac{u_c(c_{t+1}, m_{t+1})}{u_c(c_t, m_t)} (1 + i_t) \frac{P_t}{P_{t+1}} \right\}
\end{aligned}$$

Informally, these asset pricing conditions imply a "Fisher equation" relationship between expected inflation and real and nominal interest rates. Notice that with this nominal interest rate, we can simplify the first order condition for money to

$$\frac{u_m[c_t(s^t), m_t(s^t)]}{u_c[c_t(s^t), m_t(s^t)]} = \frac{i_t(s^t)}{1 + i_t(s^t)}$$

4. The tax rates must satisfy

$$T_t(s^t) = -\mu_t(s^t)\mathcal{M}_t(s^{t-1})$$

Since $\mathcal{M}_t(s^{t-1}) \geq 0$, the tax rates are negative if the money supply is growing, $\mu_t(s^t) > 0$, and positive if the money supply is shrinking. Hence consumers are being given money in the form of lump-sum *transfers* if the money growth rate is positive and they are having money taken away from them in the form of lump-sum taxes if the money growth rate is negative.

5. In equilibrium

$$\begin{aligned}
c_t(s^t) &= y_t(s^t) \\
B_{t+1}(s^t, s') &= 0 \qquad \text{each } s' \\
M_{t+1}(s^t) &= [1 + \mu_t(s^t)]\mathcal{M}_t(s^{t-1})
\end{aligned}$$

These quantities are not very interesting. This sort of model is usually used only to talk about prices and interest rates.

6. (a) **Non stochastic steady state**. With the assumed utility function and with $c_t = y_t$, the key first order conditions can be written

$$1 = E_t\left\{\beta\frac{y_t}{y_{t+1}}(1+r_{t+1})\right\} \tag{1}$$

$$1 = E_t\left\{\beta\frac{y_t}{y_{t+1}}(1+i_t)\frac{P_t}{P_{t+1}}\right\} \tag{2}$$

and

$$\frac{1-\gamma}{\gamma}\frac{y_t}{m_t} = \frac{i_t}{1+i_t} \tag{3}$$

In a non-stochastic steady state, with $\mu_t = \bar{\mu}$, $y_t = \bar{y}$ and $m_{t+1} = m_t = \bar{m}$, we have constant money growth

$$M_{t+1} = (1+\bar{\mu})M_t$$

which implies

$$\frac{M_{t+1}}{P_t} = (1+\bar{\mu})\frac{M_t}{P_{t-1}}\frac{P_{t-1}}{P_t}$$

but since real balances are also constant, this implies

$$\frac{P_t}{P_{t-1}} = 1+\bar{\pi} = 1+\bar{\mu}$$

Hence the inflation rate is equal to the constant money growth rate. From the the first order conditions for real and nominal bonds, (1) and (2), we have

$$1 = \beta(1+\bar{r})$$

$$1 = \beta\frac{(1+\bar{\imath})}{(1+\bar{\mu})}$$

The first of these conditions implies that the steady state equilibrium real interest rate equals the rate of time preference

$$\bar{r} = \frac{1-\beta}{\beta}$$

and the second condition then implies that the steady state equilibrium nominal interest rate equals the real rate plus expected inflation

$$1+\bar{\imath} = (1+\bar{r})(1+\bar{\mu}) = \frac{1+\bar{\mu}}{\beta}$$

In this example, monetary policy can affect the nominal interest rate but not the real rate (and cannot affect consumption). From the money demand condition, (3), this implies

$$\bar{m} = \frac{1-\gamma}{\gamma}\frac{1+\bar{\mu}}{1+\bar{\mu}-\beta}\bar{y}$$

Real money demand is decreasing in the money growth rate (because higher money growth raises equilibrium expected inflation and increases the opportunity cost of holding real balances). Notice that real money demand is only well behaved if $1+\bar{\mu}-\beta>0$. We implicitly need the (exogenous!) money growth rate to not be too negative. The monetary authority cannot deflate the economy too fast. Put differently, not all monetary policies are necessarily consistent with a well-behaved steady state.

The nominal money supply is

$$M_t=(1+\bar{\mu})^t M_0 \tag{4}$$

while the price level is

$$P_t=(1+\bar{\mu})^t P_0 \tag{5}$$

Both the nominal money supply and the price level grow (or shrink) at the same geometric rate $\bar{\mu}$. We're not done here, however. In these last two equations, the initial money supply $M_0$ is a primitive of the model and so (4) constitutes a "bona fide" solution. The initial price level $P_0$ is an endogenous variable, however, and still has to be solved for. It is given by

$$\bar{m}=\frac{M_1}{P_0}=\frac{(1+\bar{\mu})M_0}{P_0}$$

so that

$$P_0=(1+\bar{\mu})\left(\frac{M_0}{\bar{m}}\right)=\frac{\gamma}{1-\gamma}(1+\bar{\mu}-\beta)\left(\frac{M_0}{\bar{y}}\right)$$

For the price level to be positive, i.e., for money to be valued in equilibrium (for a steady state "monetary equilibrium" to exist) we need the restriction on money growth policies $1+\bar{\mu}-\beta>0$ as before. Notice that the initial price level is higher the faster is money growth and is directly proportional to both the level of the money supply and to the level of real output/consumption. The proportionality to the initial money supply implies that the model exhibits (long run) monetary neutrality in the sense that changing $M_0$ simply bids up the price level and leaves all the real variables unchanged. The proportionality to $\bar{y}$ is a consequence of the homothetic utility function inherited by the price level from the real money demand.

(b) **Local dynamics.** I write the log-linear model as

$$\begin{aligned}\hat{m}_t &= -\eta\hat{\imath}_t+\hat{c}_t\\ E_t\{\hat{c}_{t+1}-\hat{c}_t\} &= \hat{\imath}_t-E_t\{\hat{\pi}_{t+1}\}\\ \hat{m}_{t+1} &= \hat{m}_t+\hat{\mu}_{t+1}-\hat{\pi}_{t+1}\end{aligned}$$

plus the exogenous shocks. In this system, the constant $\eta$ comes from the log-linearization. It is given by $\eta=(1+\bar{\imath})/\bar{\imath}^2$ where $\bar{\imath}$ is the steady state nominal interest rate that we've already solved for.

In equilibrium, with $\hat{c}_t = \hat{y}_t$ and with $E_t\{\hat{y}_{t+1}\} = \phi\hat{y}_t$, these equations can be simplified to

$$\begin{aligned} \hat{m}_t &= -\eta\hat{\imath}_t + \hat{y}_t & (6)\\ (\phi-1)\hat{y}_t &= \hat{\imath}_t - E_t\{\hat{\pi}_{t+1}\} & (7)\\ \hat{m}_{t+1} &= \hat{m}_t + \hat{\mu}_{t+1} - \hat{\pi}_{t+1} & (8) \end{aligned}$$

These equations describe a *linear rational expectations* model in a single endogenous state variable $\hat{m}_t$. If we can find a solution for $\hat{m}_t$ in terms of the exogenous state variables $\hat{y}_t$ and $\hat{\mu}_t$ we'll be done. To do this, begin by rewriting (8) as

$$\hat{\pi}_{t+1} = -(\hat{m}_{t+1} - \hat{m}_t) + \hat{\mu}_{t+1}$$

Taking conditional expectations on both sides gives

$$E_t\{\hat{\pi}_{t+1}\} = -E_t\{\hat{m}_{t+1} - \hat{m}_t\} + \rho\hat{\mu}_t$$

This then implies from the bond pricing Euler equation, (7),

$$(\phi-1)\hat{y}_t = \hat{\imath}_t + E_t\{\hat{m}_{t+1} - \hat{m}_t\} - \rho\hat{\mu}_t$$

Use the money demand condition (6) to eliminate the nominal interest rate gives

$$(\phi-1)\hat{y}_t = \frac{1}{\eta}(\hat{y}_t - \hat{m}_t) + \hat{y}_t + E_t\{\hat{m}_{t+1} - \hat{m}_t\} - \rho\hat{\mu}_t$$

And simplifying gives

$$\begin{aligned} \hat{m}_t &= \frac{\eta}{1+\eta}E_t\{\hat{m}_{t+1}\} + \frac{\eta}{1+\eta}\hat{x}_t \\ \hat{x}_t &\equiv \frac{\eta(1-\phi)+1}{\eta}\hat{y}_t - \rho\hat{\mu}_t \end{aligned}$$

This is a linear stochastic difference equation in one unknown variable $\hat{m}_t$ with exogenous forcing process $\hat{x}_t$. Once we've solved it, we're largely done. By iterating forwards recursively and imposing the transversality condition, it's easy to show that the unique bounded solution to this stochastic difference equation is

$$\begin{aligned} \hat{m}_t &= E_t\left\{\frac{\eta}{1+\eta}\sum_{s=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^s \hat{x}_{t+s}\right\} \\ &= E_t\left\{\frac{\eta}{1+\eta}\sum_{s=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^s \left[\frac{\eta(1-\phi)+1}{\eta}\hat{y}_{t+s} - \rho\hat{\mu}_{t+s}\right]\right\} \end{aligned}$$

Although this constitutes a solution to the difference equation and allows us to calculate lots of things, it's a little more attractive to simplify this answer by using the linear nature of the shock processes to compute the conditional expectations. Noting that

$$\begin{aligned} E_t\{\hat{y}_{t+s}\} &= \phi^s\hat{y}_t \\ E_t\{\hat{\mu}_{t+s}\} &= \rho^s\hat{\mu}_t \end{aligned}$$

and computing the infinite sums gives

$$\begin{aligned}
\hat{m}_t &= E_t\left\{\frac{\eta}{1+\eta}\sum_{s=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^s\left[\frac{\eta(1-\phi)+1}{\eta}\hat{y}_{t+s}-\rho\hat{\mu}_{t+s}\right]\right\} \\
&= \frac{\eta}{1+\eta}\sum_{s=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^s\left[\frac{\eta(1-\phi)+1}{\eta}\phi^s\hat{y}_t-\rho^{s+1}\hat{\mu}_t\right] \\
&= \hat{y}_t-\frac{\rho\eta}{1+\eta-\rho\eta}\hat{\mu}_t
\end{aligned}$$

This gives the endogenous state variable $\hat{m}_t$ as a function of the exogenous state variables $\hat{y}_t$ and $\hat{\mu}_t$. Relatively high money growth reduces real balances through the expected inflation channel. Relatively high output increases money demand.

Notice that what we've done is find an *equilibrium stochastic process* for real balances. Plugging this result back into the equations above allows us to solve for other endogenous variables, in this case, equilibrium stochastic processes for nominal interest rates and inflation

$$\begin{aligned}
\hat{\pi}_{t+1} &= -(\hat{m}_{t+1}-\hat{m}_t)+\hat{\mu}_{t+1} \\
&= -(\hat{y}_{t+1}-\hat{y}_t)+\hat{\mu}_{t+1}-\frac{\rho\eta}{1+\eta-\rho\eta}(\hat{\mu}_t-\hat{\mu}_{t+1})
\end{aligned}$$

This implies

$$E_t\{\hat{\pi}_{t+1}\}=(1-\phi)\hat{y}_t+\frac{\rho}{1+\eta-\rho\eta}\hat{\mu}_t$$

The first of these terms is just the expected growth rate of output, while the second is the expected growth rate of nominal money adjusted for changes in real balances. Notice that this implies

$$\begin{aligned}
\hat{\imath}_t &= (\phi-1)\hat{y}_t+E_t\{\hat{\pi}_{t+1}\} \\
&= \frac{\rho}{1+\eta-\rho\eta}\hat{\mu}_t
\end{aligned}$$

so the nominal interest rate just depends on money growth via expected inflation.

Once we have all these solutions, we can compute lots of other interesting things, like moments of these equilibrium stochastic processes and impulse response functions. For example, if we wanted to know whether this model predicts nominal interest rates that are more volatile than money growth rates (or not), we would simply compute

$$\frac{\mathrm{Var}(\hat{\imath}_t)}{\mathrm{Var}(\hat{\mu}_t)}=\left(\frac{\rho}{1+\eta-\rho\eta}\right)^2<1$$

so that we immediately have the result that this model predicts that nominal rates are smoother than money growth rates.

Obviously this model exhibits both monetary neutrality (nothing real depends on the level of the money supply) and monetary superneutrality (only the nominal variables and real balances depend on the money growth rate).

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