316-466 Monetary Economics — Homework 3 Solutions

1. Plugging the final good firm's production function into its objective gives

$$\bar{P}_t(s^t)\left[\int_0^1 y_t(i,s^t)^\theta di\right]^{1/\theta} - \int_0^1 P_t(i,s^{t-1})y_t(i,s^t)di$$

The key FONC associated with this maximization problem is

$$0 = \bar{P}_t(s^t)\frac{1}{\theta}\left[\int_0^1 y_t(i,s^t)^\theta di\right]^{(1-\theta)/\theta}\theta y_t(i,s^t)^{\theta-1} - P_t(i,s^{t-1})$$

Simplifying and using the definition of $y_t(s^t)$ gives

$$y_t(s^t)^{\theta-1}y_t(i,s^t)^{\theta-1} = \frac{P_t(i,s^{t-1})}{\bar{P}_t(s^t)}$$

Hence

$$(1) \qquad y_t(i,s^t) = \left[\frac{\bar{P}_t(s^t)}{P_t(i,s^{t-1})}\right]^{1/(1-\theta)} y_t(s^t)$$

The demand for intermediate i is decreasing in the real relative price $P_t(i,s^{t-1})/\bar{P}_t(s^t)$ and has constant elasticity $-1/(1-\theta)$, i.e.,

$$\log[y_t(i,s^t)] = -\frac{1}{1-\theta}\log\left[\frac{P_t(i,s^{t-1})}{\bar{P}_t(s^t)}\right] + \log[y_t(s^t)]$$

The ratio $P_t(i,s^{t-1})/\bar{P}_t(s^t)$ is a real relative price since both numerator and denominator are measured in units of account. Notice that as $\theta \nearrow 1$, the price elasticity of demand goes to $-\infty$. As the intermediate goods become perfect substitutes, any increase in the relative price charged by an intermediate firm will cause the demand for its product to plummet to zero.

Now plugging the demand curve (1) into the zero profits condition gives

$$\begin{aligned} 0 &= \bar{P}_t(s^t)y_t(s^t) - \int_0^1 P_t(i,s^{t-1})\left(\frac{\bar{P}_t(s^t)}{P_t(i,s^{t-1})}\right)^{1/(1-\theta)} y_t(s^t)di \\ &= \left[\bar{P}_t(s^t) - \bar{P}_t(s^t)^{1/(1-\theta)}\int_0^1 P_t(i,s^{t-1})^{-\theta/(1-\theta)}di\right] y_t(s^t) \end{aligned}$$

Hence for any scale of operations $y_t(s^t) > 0$, the ideal price index must satisfy

$$0 = \bar{P}_t(s^t) - \bar{P}_t(s^t)^{1/(1-\theta)}\int_0^1 P_t(i,s^{t-1})^{-\theta/(1-\theta)}di$$

or

$$\bar{P}_t(s^t)^{\theta/(\theta-1)} = \int_0^1 P_t(i,s^{t-1})^{\theta/(\theta-1)}di$$

or

$$\bar{P}_t(s^t) = \left(\int_0^1 P_t(i,s^{t-1})^{\theta/(\theta-1)}di\right)^{(\theta-1)/\theta}$$

2. Since the cost minimization problem is static, let me write it as

$$v(i) = \min_{k(i),n(i)} \left\{ rk(i) + wn(i) \quad | \quad 1 = k(i)^\alpha n(i)^{1-\alpha} \right\}$$

The key FONC for this problem include

$$\begin{aligned} w &= \lambda(i)(1-\alpha)k(i)^\alpha n(i)^{-\alpha} \\ r &= \lambda(i)\alpha k(i)^{\alpha-1} n(i)^{1-\alpha} \end{aligned}$$

where $\lambda(i)$ denotes the Lagrange multiplier associated with the constraint $1 = k(i)^\alpha n(i)^{1-\alpha}$. Hence:

$$\frac{w}{r} = \frac{1-\alpha}{\alpha} \frac{k(i)^\alpha n(i)^{-\alpha}}{k(i)^{\alpha-1} n(i)^{1-\alpha}} = \frac{1-\alpha}{\alpha} \frac{k(i)}{n(i)}$$

Since the factor prices (w, r) are common to all firms and the production function has constant returns, the capital labor ratios are identical across intermediate firms

$$\frac{k(i)}{n(i)} = \frac{k(0)}{n(0)} = \frac{\alpha}{1-\alpha} \frac{w}{r} \qquad \text{all } i \in [0,1]$$

This in turn implies that the Lagrange multipliers are the same across intermediate firms, since

$$\lambda(i) = w \frac{1}{1-\alpha} \left(\frac{n(i)}{k(i)} \right)^\alpha = w \frac{1}{1-\alpha} \left(\frac{n(0)}{k(0)} \right)^\alpha = \lambda(0) \qquad \text{all } i \in [0,1]$$

Now multiplying the FONC by $n(i)$ and $k(i)$ (respectively) and summing up gives

$$\begin{aligned} v(i) = rk(i) + wn(i) &= \lambda(i)[\alpha k(i)^\alpha n(i)^{1-\alpha} + (1-\alpha)k(i)^\alpha n(i)^{1-\alpha}] \\ &= \lambda(i) k(i)^\alpha n(i)^{1-\alpha} \\ &= \lambda(i) \end{aligned}$$

But since $\lambda(i) = \lambda(0)$ all i, we now know $v(i) = v(0)$ all i too.

3. Plugging the demand function (1) into the intermediate's objective function and writing $x_t (= P_t(i, s^{t-1}))$ for the single object of choice gives

$$\max_{x_t} \sum_{\tau=t}^{t+N-1} \sum_{s^\tau} Q_{\tau,t-1}\left(s^\tau | s^{t-1}\right) \left[x_t - \bar{P}_\tau(s^\tau) v_\tau(s^\tau)\right] \left(\frac{\bar{P}_\tau(s^\tau)}{x_t} \right)^{1/(1-\theta)} y_\tau(s^\tau)$$

The FONC for this problem is

$$\begin{aligned} 0 = \sum_{\tau=t}^{t+N-1} \sum_{s^\tau} Q_{\tau,t-1}\left(s^\tau | s^{t-1}\right) &\left\{ \left(\frac{\bar{P}_\tau(s^\tau)}{x_t} \right)^{1/(1-\theta)} \right. \\ &\left. + \frac{1}{1-\theta} [x_t - \bar{P}_\tau(s^\tau) v_\tau(s^\tau)] \left(\frac{\bar{P}_\tau(s^\tau)}{x_t} \right)^{1/(1-\theta)} \frac{1}{x_t} \right\} y_\tau(s^\tau) \end{aligned}$$

The common term in $(1/x_t)^{1/(1-\theta)}$ can be divided out of both sides and we can simplify to get

$$\begin{aligned}
&\theta x_t \sum_{\tau=t}^{t+N-1} \sum_{s^\tau} Q_{\tau,t-1}\left(s^\tau | s^{t-1}\right) \bar{P}_\tau(s^\tau)^{1/(1-\theta)} y_\tau(s^\tau) \\
&= \sum_{\tau=t}^{t+N-1} \sum_{s^\tau} Q_{\tau,t-1}\left(s^\tau | s^{t-1}\right) \bar{P}_\tau(s^\tau)^{(2-\theta)/(1-\theta)} v_\tau(s^\tau) y_\tau(s^\tau)
\end{aligned}$$

or

$$P_t(i, s^{t-1}) = x_t = \frac{1}{\theta} \frac{\sum_{\tau=t}^{t+N-1} \sum_{s^\tau} Q_{\tau,t-1}\left(s^\tau | s^{t-1}\right) \bar{P}_\tau(s^\tau)^{(2-\theta)/(1-\theta)} v_\tau(s^\tau) y_\tau(s^\tau)}{\sum_{\tau=t}^{t+N-1} \sum_{s^\tau} Q_{\tau,t-1}\left(s^\tau | s^{t-1}\right) \bar{P}_\tau(s^\tau)^{1/(1-\theta)} y_\tau(s^\tau)}$$

This is the natural dynamic stochastic generalization of a constant $1/\theta$ markup of price over marginal cost. Consider the special case when $N = 1$ so that firms are setting prices every period. Then

$$\begin{aligned}
P_t(i, s^{t-1}) &= \frac{1}{\theta} \frac{\sum_{\tau=t}^{t} \sum_{s^\tau} Q_{\tau,t-1}\left(s^\tau | s^{t-1}\right) \bar{P}_\tau(s^\tau)^{(2-\theta)/(1-\theta)} v_\tau(s^\tau) y_\tau(s^\tau)}{\sum_{\tau=t}^{t} \sum_{s^\tau} Q_{\tau,t-1}\left(s^\tau | s^{t-1}\right) \bar{P}_\tau(s^\tau)^{1/(1-\theta)} y_\tau(s^\tau)} \\
&= \frac{1}{\theta} \frac{Q_{t,t-1}\left(s^t | s^{t-1}\right) \bar{P}_t(s^t)^{(2-\theta)/(1-\theta)} v_t(s^t) y_t(s^t)}{Q_{t,t-1}\left(s^t | s^{t-1}\right) \bar{P}_t(s^t)^{1/(1-\theta)} y_t(s^t)} \\
&= \frac{1}{\theta} \bar{P}_t(s^t) v_t(s^t)
\end{aligned}$$

so that in this case, we have the usual constant markup rule for a monopolist facing a constant elasticity demand curve. Price is a constant markup over nominal marginal cost, $\bar{P}_t(s^t)v_t(s^t)$. With N symmetric groups setting their prices in this fashion, the ideal price index can be written

$$\begin{aligned}
\bar{P}_t(s^t) &= \left(\int_0^1 P_t(i, s^{t-1})^{\theta/(\theta-1)} di\right)^{(\theta-1)/\theta} \\
&= \left(\frac{1}{N} \sum_{i=1}^{N} P_{t-i+1}(s^{t-i})^{\theta/(\theta-1)}\right)^{(\theta-1)/\theta}
\end{aligned}$$

(i.e., within each group of firms, all firms set the same price).

4. The consumer's income in any date and state consists of nominal wage income, rental income on capital (which depreciates at rate δ), money held over from the previous period, payments from any holdings of the Arrow security for state s_t as well as lump sum profits from intermediate firms (which the consumer implicitly owns) less lump-sum taxes paid to the government. She can spend this income on consumption of the final good, or can choose various assets to save with — specifically, physical capital, money, and a complete set of state contingent nominal bonds.

5. The Lagrangian for the consumer's problem can be written

$$\begin{aligned} L &= \sum_{t=0}^{\infty}\sum_{s^t}\Bigg\{\beta^t U\left(c_t(s^t), \frac{M_{t+1}(s^t)}{P_t(s^t)}, 1-n_t(s^t)\right) f(s^t \mid s_0) \\ &\quad +\lambda_t(s^t)\left\{\bar{P}_t(s^t)[w_t(s^t)n_t(s^t) + (r_t(s^t)+1-\delta)k_t(s^{t-1})] + M_t(s^{t-1}) + B_t(s^{t-1}, s_t)\right. \\ &\quad \left. + \Pi_t(s^t) - T_t(s^t) - \bar{P}_t(s^t)[c_t(s^t) + k_{t+1}(s^t)] - M_{t+1}(s^t) - \sum_{s'} Q_{t+1,t}(s^t, s')B_{t+1}(s^t, s')\right\} \end{aligned}$$

where $\lambda_t(s^t) \geq 0$ denotes the Lagrange multiplier for date t given history s^t. The key FONC are

$$(2) \qquad c_t(s^t) \;:\; \lambda_t(s^t) = \beta^t \frac{U_{c,t}(s^t)}{\bar{P}_t(s^t)} f(s^t \mid s_0)$$

$$(3) \qquad M_{t+1}(s^t) \;:\; \lambda_t(s^t) = \beta^t \frac{U_{m,t}(s^t)}{\bar{P}_t(s^t)} f(s^t \mid s_0) + \sum_{s'} \lambda_{t+1}(s^t, s')$$

$$(4) \qquad n_t(s^t) \;:\; w_t(s^t)\lambda_t(s^t) = \beta^t \frac{U_{\ell,t}(s^t)}{\bar{P}_t(s^t)} f(s^t \mid s_0)$$

$$(5) \qquad k_{t+1}(s^t) \;:\; \bar{P}_t(s^t)\lambda_t(s^t) = \sum_{s'} \bar{P}_{t+1}(s^t, s')\lambda_{t+1}(s^t, s')[r_{t+1}(s^{t+1}, s') + 1 - \delta]$$

$$(6) \qquad B_{t+1}(s^t, s') \;:\; Q_{t+1,t}(s^t, s')\lambda_t(s^t) = \lambda_{t+1}(s^t, s')$$

Combining (2) and (4) gives the MRS between leisure and consumption equal to the real wage rate. This determines labor supply

$$\frac{U_{\ell,t}(s^t)}{U_{c,t}(s^t)} = w_t(s^t)$$

Combining (3) with (2) at dates t and $t+1$ gives the MRS between real balances and consumption equal to the opportunity cost of holding money. This particular expression is often used to highlight the analogy between real balances and a *durable* consumption good

$$\beta^t \frac{[U_{c,t}(s^t) - U_{m,t}(s^t)]}{\bar{P}_t(s^t)} f(s^t \mid s_0) = \sum_{s'} \beta^{t+1} \frac{U_{c,t+1}(s^t, s')}{\bar{P}_{t+1}(s^t, s')} f(s^t, s' \mid s_0)$$

or

$$U_{c,t}(s^t) - U_{m,t}(s^t) = \beta \sum_{s'} U_{c,t+1}(s^t, s') \frac{\bar{P}_t(s^t)}{\bar{P}_{t+1}(s^t, s')} f(s' \mid s_t)$$

Combining (5) with (2) at dates t and $t+1$ gives the standard capital accumulation condition

$$U_{c,t}(s^t) = \beta \sum_{s'} U_{c,t+1}(s^t, s')[r_{t+1}(s^t, s') + 1 - \delta] f(s' \mid s_t)$$

Now writing $Q_{\tau,t}(s^\tau|s^t)$ for the price at date t state s^t of a unit of account delivered in date $\tau > t$ state s^τ, we have the following relationship with the one-period intertemporal prices

$$\begin{aligned}
Q_{\tau,t}(s^\tau|s^t) &= Q_{\tau,\tau-1}(s^{\tau-1}, s_\tau) \times \cdots \times Q_{t+2,t+1}(s^{t+1}, s_{t+2}) \times Q_{t+1,t}(s^t, s_{t+1}) \\
&= \frac{\lambda_\tau(s^{\tau-1}, s_\tau)}{\lambda_{\tau-1}(s^{\tau-1})} \times \cdots \times \frac{\lambda_{t+2}(s^{t+1}, s_{t+2})}{\lambda_{t+1}(s^{t+1})} \times \frac{\lambda_{t+1}(s^t, s_{t+1})}{\lambda_t(s^t)} \\
&= \frac{\lambda_\tau(s^{\tau-1}, s_\tau)}{\lambda_t(s^t)} \\
&= \beta^{\tau-t}\frac{U_{c,\tau}(s^\tau)}{U_{c,t}(s^t)}\frac{\bar{P}_t(s^t)}{\bar{P}_\tau(s^\tau)} f(s^\tau \mid s_t), \qquad \tau > t
\end{aligned}$$

where the last equality follows by using (2) to eliminate the Lagrange multipliers. [There was a typo in the problem set. The transitional probabilities $f(s^\tau \mid s_t)$ were missing from the expression for $Q_{\tau,t}(s^\tau|s^t)$].

6. The equilibrium conditions

$$\begin{aligned}
k_t(s^{t-1}) &= \int_0^1 k_t(i, s^t)di \\
n_t(s^t) &= \int_0^1 n_t(i, s^t)di \\
c_t(s^t) + k_{t+1}(s^t) &= y_t(s^t) + (1-\delta)k_t(s^{t-1}) \\
\mathcal{M}_{t+1}(s^t) &= M_{t+1}(s^t) \\
B_{t+1}(s^t, s') &= 0
\end{aligned}$$

require that i) the supply of capital equals the demand for capital by intermediate firms, ii) the supply of labor equal the demand for labor by intermediate firms, iii) the final goods market clears, iv) that money supply equal money demand, v) that bonds are in zero net supply.

7. Integrate the intermediate demand curves over i to get

$$\begin{aligned}
\int_0^1 y_t(i, s^t)di &= \int_0^1 \left(\frac{\bar{P}_t(s^t)}{P_t(i, s^{t-1})}\right)^{1/(1-\theta)} y_t(s^t)di \\
&= \bar{P}_t(s^t)^{1/(1-\theta)} \left(\int_0^1 P_t(i, s^{t-1})^{1/(\theta-1)}di\right) y_t(s^t)
\end{aligned}$$

Now the production function for an intermediate is

$$(7) \quad y_t(i,s^t) = \left(\frac{k_t(i,s^t)}{n_t(i,s^t)}\right)^\alpha n_t(i,s^t)$$
$$= \left(\frac{k_t(0,s^t)}{n_t(0,s^t)}\right)^\alpha n_t(i,s^t)$$

where the second equality follows because all intermediates use the same capital/labor ratio. Hence integrating over i gives

$$(8) \quad \int_0^1 y_t(i,s^t)di = \left(\frac{k_t(0,s^t)}{n_t(0,s^t)}\right)^\alpha \int_0^1 n_t(i,s^t)di$$
$$= \left(\frac{k_t(0,s^t)}{n_t(0,s^t)}\right)^\alpha n_t(s^t)$$

where the second equality makes use of labor market clearing. Now if the intermediate firms all use the same capital labor ratio — κ, say — then

$$\kappa = \frac{k_t(i,s^t)}{n_t(i,s^t)} = \frac{k_t(0,s^t)}{n_t(0,s^t)} \qquad \text{all } i \in [0,1]$$

hence

$$k_t(i,s^t) = \kappa n_t(i,s^t) \qquad \text{all } i \in [0,1]$$

and so on integrating both sides

$$\int_0^1 k_t(i,s^t)di = \kappa \int_0^1 n_t(i,s^t)di$$

meaning we can conclude

$$(9) \quad \left(\frac{k_t(0,s^t)}{n_t(0,s^t)}\right)^\alpha = \kappa^\alpha = \left(\frac{\int_0^1 k_t(i,s^t)di}{\int_0^1 n_t(i,s^t)di}\right)^\alpha = \left(\frac{k_t(s^{t-1})}{n_t(s^t)}\right)^\alpha$$

where the last equality follows from capital and labor market clearing. Putting (9) into (8) and then combining this with (7) gives

$$(10) \quad \int_0^1 y_t(i,s^t)di = \left(\frac{k_t(s^{t-1})}{n_t(s^t)}\right)^\alpha n_t(s^t) = \bar{P}_t(s^t)^{1/(1-\theta)} \left(\int_0^1 P_t(i,s^{t-1})^{1/(\theta-1)}di\right) y_t(s^t)$$

Hence we conclude the relationship between aggregate output and the aggregate inputs is given by

$$y_t(s^t) = A_t(s^t)k_t(s^{t-1})^\alpha n_t(s^t)^{1-\alpha}$$

$$A_t(s^t) \equiv \frac{\bar{P}_t(s^t)^{1/(\theta-1)}}{\int_0^1 P_t(i,s^{t-1})^{1/(\theta-1)}di}$$

The index $A_t(s^t)$ is a measure of the distortion created by monopolistic competition and price setting. For a given amount of aggregate capital and labor, this economy produces less output than would be produced under perfect competition. (Each monopolistic competitor has P greater than MR so as to extract some economic rents from the final goods firms that demand their product).

8. Let $N = 2$ and $\alpha = 0$ (only labor is used in production). With $N = 2$, the ideal price index is

$$\bar{P}_t(s^t) = \left(\frac{1}{2}P_t(s^{t-1})^{\theta/(\theta-1)} + \frac{1}{2}P_{t-1}(s^{t-2})^{\theta/(\theta-1)}\right)^{(\theta-1)/\theta}$$

(a) **Non-stochastic steady state**. We will need the following derivatives. Writing the period utility function as

$$\begin{aligned} U(c,m,\ell) &= \frac{1}{1-\sigma}\left\{\left[\omega c^{(\eta-1)/\eta} + (1-\omega)m^{(\eta-1)/\eta}\right]^{\eta/(\eta-1)} \ell^{\psi}\right\}^{1-\sigma} \\ &= \frac{1}{1-\sigma}\left\{\Phi(c,m)\ell^{\psi}\right\}^{1-\sigma} \end{aligned}$$

we have

$$\begin{aligned} U_c(c,m,1-n) &= \{\Phi(c,m)\}^{-\sigma}\left\{(1-n)^{\psi}\right\}^{1-\sigma}\Phi_c(c,m) \\ U_m(c,m,1-n) &= \{\Phi(c,m)\}^{-\sigma}\left\{(1-n)^{\psi}\right\}^{1-\sigma}\Phi_m(c,m) \\ U_\ell(c,m,1-n) &= \psi\{\Phi(c,m)\}^{1-\sigma}\{(1-n)^{\psi(1-\sigma)-1}\} \end{aligned}$$

where

$$\begin{aligned} \Phi_c(c,m) &= \omega\left[\omega c^{(\eta-1)/\eta} + (1-\omega)m^{(\eta-1)/\eta}\right]^{1/(\eta-1)} c^{-1/\eta} \\ &= \omega\Phi(c,m)^{1/\eta}c^{-1/\eta} \\ \Phi_m(c,m) &= (1-\omega)\left[\omega c^{(\eta-1)/\eta} + (1-\omega)m^{(\eta-1)/\eta}\right]^{1/(\eta-1)} m^{-1/\eta} \\ &= (1-\omega)\Phi(c,m)^{1/\eta}m^{-1/\eta} \end{aligned}$$

The symmetric non-stochastic steady state studied by CKM has zero inflation. Hence the nominal interest rate equals the real interest rate which is given by

$$\bar{\imath} = \bar{r} = \frac{1-\beta}{\beta}$$

In this zero inflation equilibrium, the nominal money supply and price levels are constant at M_0 and $\bar{P}$ respectively. There are two groups of intermediate producers ($N = 2$) which set prices P_0 and P_{-1} to satisfy

$$\bar{P} = \left(\frac{1}{2}P_0^{\theta/(\theta-1)} + \frac{1}{2}P_{-1}^{\theta/(\theta-1)}\right)^{(\theta-1)/\theta}$$

also, the price set today is

$$
\begin{aligned}
P_0 &= \frac{1}{\theta}\frac{\sum_{\tau=t}^{t+1} Q_{\tau,t-1}\bar{P}_\tau^{(2-\theta)/(1-\theta)} v_\tau y_\tau}{\sum_{\tau=t}^{t+1} Q_{\tau,t-1}\bar{P}_\tau^{1/(1-\theta)} y_\tau} \\
&= \frac{1}{\theta}\frac{Q_{t,t-1}\bar{P}_t^{(2-\theta)/(1-\theta)} v_t y_t + Q_{t+1,t-1}\bar{P}_{t+1}^{(2-\theta)/(1-\theta)} v_{t+1} y_{t+1}}{Q_{t,t-1}\bar{P}_t^{1/(1-\theta)} y_t + Q_{t+1,t-1}\bar{P}_{t+1}^{1/(1-\theta)} y_{t+1}}
\end{aligned}
$$

In steady state, we will have $v_t = v_{t+1} = \bar{v}$, $y_t = y_{t+1} = \bar{y}$ and $\bar{P}_t = \bar{P}_{t+1} = \bar{P}$ (this last because of zero inflation). Also, the intertemporal prices will satisfy

$$
Q_{\tau,t} = \beta^{\tau-t}, \qquad \tau > t
$$

since the marginal utility of consumption will also be constant. Hence

$$
\begin{aligned}
P_0 &= \frac{1}{\theta}\frac{\beta\bar{P}^{(2-\theta)/(1-\theta)}\bar{v}\bar{y} + \beta^2\bar{P}^{(2-\theta)/(1-\theta)}\bar{v}\bar{y}}{\beta\bar{P}^{1/(1-\theta)}\bar{y} + \beta^2\bar{P}^{1/(1-\theta)}\bar{y}} \\
&= \frac{1}{\theta}\frac{\bar{P}^{(2-\theta)/(1-\theta)}\bar{v}}{\bar{P}^{1/(1-\theta)}} \\
&= \frac{1}{\theta}\bar{P}\bar{v}
\end{aligned}
$$

A similar argument shows that P_{-1} also equals $\bar{P}\bar{v}/\theta$ and hence $\bar{P} = P_0 = P_{-1}$ and the steady state real marginal cost must therefore be $\bar{v} = \theta$. With no capital ($\alpha = 0$) the intermediate firms' technology is linear in labor. Hence the real wage must satisfy, from the intermediate firm's cost minimization problem,

$$
\bar{w} = \bar{v} = \theta
$$

(the level of technology is implicitly normalized to one). To solve for the real allocations $(\bar{c}, \bar{m}, \bar{n})$, we need to simultaneously solve

$$
\begin{aligned}
\frac{U_\ell(\bar{c}, \bar{m}, \bar{n})}{U_c(\bar{c}, \bar{m}, \bar{n})} &= \bar{w} = \theta \\
\bar{m} &= \bar{c} \\
\bar{c} &= \bar{A}\bar{y} = \bar{n}
\end{aligned}
$$

(Recall that CKM impose the ad hoc interest-inelastic money demand $m_t = c_t$; note also that in non-stochastic steady state with zero inflation $\bar{A} = 1$ since all the intermediate prices are the same and thus equal to the aggregate price level). Using the functional forms above:

$$
\theta = \frac{U_\ell(\bar{c}, \bar{c}, \bar{c})}{U_c(\bar{c}, \bar{c}, \bar{c})} = \psi\frac{\{\Phi(\bar{c}, \bar{c})\}^{1-\sigma}\{(1-\bar{c})^{\psi(1-\sigma)-1}\}}{\{\Phi(\bar{c}, \bar{c})\}^{-\sigma}\{(1-\bar{c})^{\psi(1-\sigma)}\}\Phi_c(\bar{c}, \bar{c})} = \psi\frac{\Phi(\bar{c}, \bar{c})}{\Phi_c(\bar{c}, \bar{c})}\frac{1}{1-\bar{c}}
$$

To simplify this expression, we need to use the facts that, at steady state,

$$
\begin{aligned}
\Phi(\bar{c}, \bar{c}) &= \left[\omega\bar{c}^{(\eta-1)/\eta} + (1-\omega)\bar{c}^{(\eta-1)/\eta}\right]^{\eta/(\eta-1)} = \bar{c} \\
\Phi_c(\bar{c}, \bar{c}) &= \omega\Phi(\bar{c}, \bar{c})^{1/\eta}\bar{c}^{-1/\eta} = \omega
\end{aligned}
$$

Hence

$$\theta = \frac{U_\ell(\bar{c},\bar{c},\bar{c})}{U_c(\bar{c},\bar{c},\bar{c})} = \frac{\psi}{\omega}\frac{\bar{c}}{1-\bar{c}}$$

Finally, then, we have steady state consumption

$$\bar{c} = \frac{\theta}{\theta + (\psi/\omega)}$$

This derivation is one of the key conclusions of their paper (see p. 1164), since their calibration of the parameters (ω, ψ, θ) will pin down the slope of the labor supply equation in their model — and this will drive the result that a reasonably calibrated version of Taylor overlapping contracts model cannot generate much endogenous price stickiness.

To complete the solution for the non-stochastic steady state, it only remains to note that since $\bar{m} = \bar{c}$, the price level is given by

$$\bar{P} = \frac{M_0}{\bar{m}} = \frac{\theta + (\psi/\omega)}{\theta} M_0$$

(b) **Log-linearization**. Most of this is simple and outlined well on pp 1163-1165. Here I will focus on the key derivation, namely the log-linear labor supply condition. Begin by noting that the money demand and goods market clearing conditions require

$$\hat{m}_t = \hat{c}_t$$

$$\hat{y}_t = \hat{c}_t = \hat{n}_t$$

The labor supply FONC can be written

$$\widehat{U_{\ell,t}} - \widehat{U_{c,t}} = \hat{w}_t$$

where

$$(11) \quad \widehat{U_{\ell,t}} = \left[\frac{U_{\ell c}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_\ell(\bar{c},\bar{c},\bar{c})} + \frac{U_{\ell m}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_\ell(\bar{c},\bar{c},\bar{c})} - \frac{U_{\ell\ell}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_\ell(\bar{c},\bar{c},\bar{c})}\right]\hat{c}_t$$

$$(12) \quad \widehat{U_{c,t}} = \left[\frac{U_{cc}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_c(\bar{c},\bar{c},\bar{c})} + \frac{U_{cm}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_c(\bar{c},\bar{c},\bar{c})} - \frac{U_{c\ell}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_\ell(\bar{c},\bar{c},\bar{c})}\right]\hat{c}_t$$

These two expressions have made use of $\bar{n}\hat{n}_t + (1-\bar{n})\hat{\ell}_t = 0$ to eliminate the log-deviation of leisure and also $\bar{c} = \bar{m} = \bar{n}$. The tedious bit is computing the second derivatives. It is made easier by noting that, for example,

$$\begin{aligned}
\left.\frac{U_{cc}(c,m,n)}{U_c(c,m,n)}\right|_{\bar{c}} &= \left.\frac{\partial}{\partial c}\log[U_c(c,m,n)]\right|_{\bar{c}} \\
&= -\sigma\frac{\Phi_c(\bar{c},\bar{c})}{\Phi(\bar{c},\bar{c})} + \frac{\Phi_{cc}(\bar{c},\bar{c})}{\Phi_c(\bar{c},\bar{c})} \\
&= -\sigma\frac{\Phi_c(\bar{c},\bar{c})}{\Phi(\bar{c},\bar{c})} + \frac{1}{\eta}\frac{\Phi_c(\bar{c},\bar{c})}{\Phi(\bar{c},\bar{c})} - \frac{1}{\eta}\frac{1}{\bar{c}} \\
&= -\sigma\frac{\omega}{\bar{c}} + \frac{1}{\eta}\frac{\omega}{\bar{c}} - \frac{1}{\eta}\frac{1}{\bar{c}}
\end{aligned}$$

Hence

$$\frac{U_{cc}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_c(\bar{c},\bar{c},\bar{c})} = -\sigma\omega - \frac{1-\omega}{\eta}$$

Similar calculations lead to

$$\begin{aligned}
\frac{U_{\ell c}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_\ell(\bar{c},\bar{c},\bar{c})} &= (1-\sigma)\omega \\
\frac{U_{\ell m}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_\ell(\bar{c},\bar{c},\bar{c})} &= (1-\sigma)(1-\omega) \\
\frac{U_{\ell\ell}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_\ell(\bar{c},\bar{c},\bar{c})} &= [(1-\sigma)\psi - 1]\frac{\bar{c}}{1-\bar{c}} \\
\frac{U_{cm}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_c(\bar{c},\bar{c},\bar{c})} &= -\sigma(1-\omega) + \frac{1-\omega}{\eta} \\
\frac{U_{c\ell}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_c(\bar{c},\bar{c},\bar{c})} &= (1-\sigma)\psi\frac{\bar{c}}{1-\bar{c}}
\end{aligned}$$

Hence subtracting off in pairs and simplifying,

$$\begin{aligned}
\frac{U_{\ell c}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_\ell(\bar{c},\bar{c},\bar{c})} - \frac{U_{cc}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_c(\bar{c},\bar{c},\bar{c})} &= \omega\frac{\eta-1}{\eta} + \frac{1}{\eta} \\
\frac{U_{\ell m}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_\ell(\bar{c},\bar{c},\bar{c})} - \frac{U_{cm}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_c(\bar{c},\bar{c},\bar{c})} &= (1-\omega)\frac{\eta-1}{\eta} \\
\frac{U_{\ell\ell}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_\ell(\bar{c},\bar{c},\bar{c})} - \frac{U_{c\ell}(\bar{c},\bar{c},\bar{c})\bar{c}}{U_c(\bar{c},\bar{c},\bar{c})} &= -\frac{\bar{c}}{1-\bar{c}}
\end{aligned}$$

but we know from the steady state calculations above that

$$\frac{\bar{c}}{1-\bar{c}} = \frac{\omega\theta}{\psi}$$

Putting all these pieces together, we use (11)-(12) to conclude (cf. equation 32 in

CKM's paper) that the log-linear approximation to labor supply is

$$
\begin{aligned}
\hat{w}_t &= \left[\left(\omega\frac{\eta-1}{\eta}+\frac{1}{\eta}\right)+\left((1-\omega)\frac{\eta-1}{\eta}\right)+\frac{\omega\theta}{\psi}\right]\hat{c}_t \\
&= \left(1+\frac{\omega\theta}{\psi}\right)\hat{c}_t \\
&= \gamma\hat{y}_t \\
&= \gamma\hat{n}_t
\end{aligned}
$$

where

$$\gamma = 1+\frac{\omega\theta}{\psi}$$

In Taylor's model, γ was a free parameter. Here it depends in a structural way on two preference parameters ($\omega \in (0,1)$ and $\psi > 0$) and the degree of imperfect competition as measured by $\theta \in (0,1]$. To get reasonable persistence, Taylor needed $\gamma \simeq 0.0003$. Since the three structural parameters are positive, this model generates a $\gamma > 1$. CKM go on to show that the implied log-linear price and output dynamics have no persistence if $\gamma > 1$.

Chris Edmond, 25 October 2003