316-406 ADVANCED MACROECONOMIC TECHNIQUES NOTE 1a

Chris Edmond ⟨cpedmond@unimelb.edu.au⟩

The Solow model (in continuous and discrete time)

This is a brief review of Robert Solow's (1956) model of economic growth. You should not find any of this material difficult, but you may not have seen it all before. For comparisons' sake, I present it both in continuous and discrete time. This material should be read in parallel with Note 1b, which has a brief introduction to simple linear differential and difference equations.

A. Continuous time

If time is continuous, $t \geq 0$, the fundamental object of the Solow model is the non-linear differential equation

$$\dot{k}_t = sf(k_t) - (\delta + n + g)k_t, \qquad k_0 \text{ given} \tag{1}$$

Let's recall how this is derived.

- National income accounting identity

$$C_t + I_t = Y_t$$

- Aggregate production function

$$Y_t = F(K_t, A_t N_t)$$

  Note labor-augmenting technical progress. Assume F has constant returns to scale and usual concavity properties.

- Constant physical depreciation rate δ

$$\dot{K}_t = I_t - \delta K_t, \qquad K_0 \text{ given} \tag{2}$$

  Here, a dot means the time derivative: $\dot{K}_t \equiv dK_t/dt$. Initial capital stock K_0 is a given parameter of the model.

  Constant savings rate s

$$S_t = I_t = sY_t$$

Putting this all together

$$\dot{K}_t = sF(K_t, A_t N_t) - \delta K_t \tag{3}$$

For pencil-and-paper convenience, let's assume exponential growth of population and technology

$$\begin{aligned}\dot{A}_t &= gA_t \Longleftrightarrow A_t = e^{gt}A_0, \qquad A_0 \text{ given}\\ \dot{N}_t &= nN_t \Longleftrightarrow N_t = e^{nt}N_0, \qquad N_0 \text{ given}\end{aligned}$$

The initial level of technology $A_0 > 0$ and population $N_0 > 0$ as well as the growth rates g and n are given parameters of the model. Both g and n may be either positive or negative.

- It is usually more convenient to work with **stationary variables** (variables that will not grow in the long-run). For example, define the capital stock per efficiency unit of labor

  $$k_t \equiv \frac{K_t}{A_t N_t}$$

  To go along with this change of variables, use constant returns and define the intensive version of the production function

  $$y_t \equiv \frac{Y_t}{A_t N_t} = \frac{F(K_t, A_t N_t)}{A_t N_t} = F(k_t, 1) \equiv f(k_t)$$

- Now we can write the differential equation (3) in terms of a single endogenous **state variable**, the capital stock per efficiency unit of labor k_t. To do this, first divide (3) on both sides by $A_t N_t$ to get

  $$\frac{\dot{K}_t}{A_t N_t} = sf(k_t) - \delta k_t \tag{4}$$

  and now take logs and differentiate the definition of k_t to get

  $$\begin{aligned}\frac{\dot{k}_t}{k_t} &= \frac{\dot{K}_t}{K_t} - \frac{\dot{A}_t}{A_t} - \frac{\dot{N}_t}{N_t}\\ &= \frac{\dot{K}_t}{K_t} - g - n\end{aligned}$$

  and multiplying both sides by k_t now gives

  $$\dot{k}_t = \frac{\dot{K}_t}{A_t N_t} - (g+n)k_t$$

  Plugging this back into (4) now gives the object we want to work with

  $$\dot{k}_t = sf(k_t) - (\delta + n + g)k_t \tag{5}$$

(with given initial condition $k_0 \equiv K_0/A_0N_0$).

B. Discrete time

If time is discrete, $t = 0, 1, 2, ...$, the analogous object is a non-linear difference equation, specifically

$$(1+n)(1+g)k_{t+1} = sf(k_t) + (1-\delta)k_t, \qquad k_0 \text{ given} \tag{6}$$

where everything is derived in the same way as before except that

	Continuous time	Discrete time
national income accounting	$C_t + I_t = Y_t$	$C_t + I_t = Y_t$
aggregate production function	$Y_t = F(K_t, A_tN_t)$	$Y_t = F(K_t, A_tN_t)$
accumulation of capital	$\dot{K}_t = I_t - \delta K_t$	$K_{t+1} - K_t = I_t - \delta K_t$
constant savings rate	$S_t = I_t = sY_t$	$S_t = I_t = sY_t$
growth of technology	$\dot{A}_t = gA_t$	$A_{t+1} - A_t = gA_t$
growth of population	$\dot{N}_t = nN_t$	$N_{t+1} - N_t = nN_t$

You should use this table to derive the non-linear difference equation (6).

C. Closed-form solution with Cobb-Douglas production function

In the continuous time version of the model, one important special case permits an explicit solution of the model. Before we get into it, let me just say that mathematically, we will not have to do anything much harder than the following derivation in the entire course.

Suppose the aggregate production function is of the Cobb-Douglas form

$$Y_t = K_t^\alpha (A_tN_t)^{1-\alpha} \Longleftrightarrow y_t = k_t^\alpha$$

so that in this case, the differential equation (5) specializes to

$$\dot{k}_t = sk_t^\alpha - (\delta + n + g)k_t \tag{7}$$

For future reference, let's solve for the non-trivial steady state. This is given by the unique $\bar{k} > 0$ that solves

$$0 = s\bar{k}^\alpha - (\delta + n + g)\bar{k}$$

which is

$$\bar{k} = \left(\frac{s}{\delta + n + g}\right)^{\frac{1}{1-\alpha}}$$

The key trick is to recognize that we can write equation (7) as an equivalent differential equation in the **capital/output ratio**. Defining

$$x_t \equiv \frac{k_t}{y_t} = k_t^{1-\alpha}$$

Hence

$$\frac{\dot{x}_t}{x_t} = (1-\alpha)\frac{\dot{k}_t}{k_t} = (1-\alpha)\left[\frac{sk_t^{\alpha} - (\delta + n + g)k_t}{k_t}\right] = (1-\alpha)\left[s\frac{1}{x_t} - (\delta + n + g)\right]$$

Multiplying both sides by x_t then gives

$$\dot{x}_t = (1-\alpha)s - (1-\alpha)(\delta + n + g)x_t \tag{8}$$

This is a **linear differential equation** with constant coefficients and a constant forcing term. It is easy to solve. Here's the cookbook.

Step 1. *Compute the steady state*:

$$\bar{x} = \frac{s}{\delta + n + g}$$

Step 2. *Transform the DE into a homogeneous equation*: Use a change of variable to re-write the differential equation in the form $\dot{z}_t = -\lambda z_t$ for some variable z_t and some coefficient λ. Mechanically, define $z_t \equiv x_t - \bar{x}$ which is the difference between the capital/output ratio and its steady state value. This implies $\dot{z}_t = \dot{x}_t$ and a few lines of algebra gives

$$\begin{aligned}
\dot{z}_t = \dot{x}_t &= (1-\alpha)s - (1-\alpha)(\delta + n + g)x_t \\
&= (1-\alpha)s - (1-\alpha)(\delta + n + g)[z_t + \bar{x}] \\
&= (1-\alpha)s - (1-\alpha)(\delta + n + g)\left[z_t + \frac{s}{\delta + n + g}\right] \\
&= -(1-\alpha)(\delta + n + g)z_t
\end{aligned}$$

or

$$\dot{z}_t = -\lambda z_t, \qquad \lambda \equiv (1-\alpha)(\delta + n + g)$$

Step 3. *Solve it*: We have already used the fact that $\dot{z}_t = -\lambda z_t$ has the solution $z_t = e^{-\lambda t} z_0$. Plugging back in the definition $z_t \equiv x_t - \bar{x}$ and rearranging gives

$$x_t = (1 - e^{-\lambda t})\bar{x} + e^{-\lambda t} x_0$$

The capital/output ratio is a weighted average of its steady state value and its initial value with the weight on the initial value vanishing asymptotically as $t \to \infty$. The **speed of convergence** to the steady state is measured by the coefficient λ. The speed of convergence is faster the more concave in the production function (the closer α is to zero), or the faster capital wears out, or the faster population or technology grows.

The final solution for the capital stock is then

$$k_t = \left[(1 - e^{-\lambda t})\left(\frac{s}{\delta + g + n}\right)^{1-\alpha} + e^{-\lambda t} k_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}, \qquad \lambda \equiv (1-\alpha)(\delta + n + g)$$

which inherits all the stability properties of x_t.

Chris Edmond

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