316-406 ADVANCED MACROECONOMIC TECHNIQUES NOTE 5a

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Introduction to the stochastic growth model

Now that we have some tools for modeling uncertainty, we can turn back to economic models. To begin with, let's consider the stochastic analogue of the Ramsey-Cass-Koopmans optimal growth model. The **stochastic growth model** is sometimes known as the Brock-Mirman model, after their 1972 paper. Although not especially interesting in its own right, the stochastic growth model is the point of departure for a lot of applied macroeconomics. The real business cycle model, which we'll turn to next, is a slight variation on it.

Consider, then, a social planner that solves:

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \mathsf{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t U(c_t) \right\}, \qquad 0 < \beta < 1$$

subject to, for each $t = 0, 1, ...$

$$\begin{aligned} c_t + k_{t+1} &\leq z_t f(k_t) + (1-\delta)k_t \\ c_t, k_{t+1} &\geq 0 \\ \text{given } k_0, z_0 & \end{aligned}$$

and an exogenous stochastic process for the level of technology $\{Z_t\}$.

A. Event-tree formulation

We can formulate this problem slightly more concretely. Let there be countable dates, $t = 0, 1, 2...$ and for simplicity let there be a finite number Z of possible states of nature that may be realized at each date $t \geq 1$. Index the states by $z_t \in \mathcal{Z} = \{1, 2, ..., Z\}$. A **history** of realizations z^t is a vector $z^t = (z_0, z_1, ..., z_t) = (z^{t-1}, z_t)$. The unconditional probability of a history z^t being realized as of date zero is denoted $\pi_t(z^t)$. The initial state z_0 is known as of date zero. The expectation in the planner's objective function is a sum over random utility outcomes $U[c_t(z^t)]$ weighted by the relevant probabilities of particular histories. (In the case of a stochastic process with continuous support, we would have probability densities for alternative histories and the expectation would be an integral). Similarly, the choice variables would be sequences of functions of the history, namely $\{c_t(z^t)\}_{t=0}^{\infty}$ and $\{k_{t+1}(z^t)\}_{t=0}^{\infty}$. Notice that capital accumulation $k_{t+1}(z^t)$ is a function of the history up to date t, because it is chosen conditional on the information up to that point. This means that the capital that enters the production function at date t is $k_t(z^{t-1})$.

We can now write the planner's problem as

$$\max_{\{c_t(z^t),k_{t+1}(z^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\sum_{z^t} \beta^t U[c_t(z^t)]\pi_t(z^t), \qquad 0<\beta<1$$

subject to, for each $t = 0, 1, ...$

$$\begin{aligned} c_t(z^t) + k_{t+1}(z^t) &\leq z_t f[k_t(z^{t-1})] + (1-\delta)k_t(z^{t-1}) \\ c_t(z^t), k_{t+1}(z^t) &\geq 0 \\ \text{given } k_0, z_0 & \end{aligned}$$

We can characterize solutions to this problem by forming a Lagrangian. Let $\lambda_t(z^t) \geq 0$ denote the multiplier associated with the date t history z^t resource constraint. Then the Lagrangian facing the planner is

$$\mathcal{L} = \sum_{t=0}^{\infty}\sum_{z^t} \beta^t U[c_t(z^t)]\pi_t(z^t) + \sum_{t=0}^{\infty}\sum_{z^t} \lambda_t(z^t)\{z_t f[k_t(z^{t-1})] + (1-\delta)k_t(z^{t-1}) - c_t(z^t) - k_{t+1}(z^t)\}$$

The interesting first order conditions associated with the problem include

$$\frac{\partial \mathcal{L}}{\partial c_t(z^t)} = 0 \Longleftrightarrow \beta^t U'[c_t(z^t)]\pi_t(z^t) = \lambda_t(z^t)$$

and

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}(z^t)} = 0 \Longleftrightarrow \lambda_t(z^t) = \sum_{z_{t+1}|z^t} \lambda_{t+1}(z^t, z_{t+1})\{1 + z_{t+1} f'[k_{t+1}(z^t)] - \delta\}$$

Notice that capital that is set aside following history z^t is worth something in each state z_{t+1} that follows. How much it is worth depends in part on the realization of the productivity shock. We can combine these conditions to write

$$U'[c_t(z^t)]\pi_t(z^t) = \beta \sum_{z_{t+1}|z^t} U'[c_{t+1}(z^t, z_{t+1})]\pi_{t+1}(z^t, z_{t+1})\{1 + z_{t+1} f'[k_{t+1}(z^t)] - \delta\}$$

and dividing both sides by $\pi_t(z^t)$ gives

$$U'[c_t(z^t)] = \beta \sum_{z_{t+1}|z^t} U'[c_{t+1}(z^t, z_{t+1})] \frac{\pi_{t+1}(z^t, z_{t+1})}{\pi_t(z^t)} \{1 + z_{t+1} f'[k_{t+1}(z^t)] - \delta\}$$

Notice that

$$\frac{\pi_{t+1}(z^t, z_{t+1})}{\pi_t(z^t)}$$

is the **transition probability** to state z_{t+1} given history z^t. With stationary Markov shocks, this simplifies to something like

$$\frac{\pi_{t+1}(z^t, z_{t+1})}{\pi_t(z^t)} = \Pr(Z_{t+1} = z_{t+1}|Z_t = z_t) \equiv \pi(z_{t+1}|z_t)$$

We will often use this Markov formulation in our applications. In any case, we can write the optimality condition as

$$U'(c_t) = \beta \mathsf{E}_t\{U'(c_{t+1})[1 + z_{t+1} f'(k_{t+1}) - \delta]\}$$

This is sometimes known as a **stochastic Euler equation**. It equates the marginal cost of consumption foregone to the expected marginal benefit, which depends on the marginal utility of consumption in the future (which is random) and on the gross return to capital (which is also random).

In addition to the stochastic Euler equation, optimal plans also depend on the resource constraint. As with the deterministic model, a transversality condition rules out capital accumulation paths that grow too quickly.

Given a specification of the law of motion for the exogenous shocks, a simple way to solve this model is to log-linearize the stochastic Euler equation and the resource constraint around some point and to then solve the resulting system of stochastic difference equations. The point around which the log-linearization is often performed is the non-stochastic steady state.

B. Non-stochastic steady state

Suppose that we shut off the shocks by setting $z_t = \bar{z}$ for all t and we then look for a steady state characterized by $k_{t+1} = k_t = \bar{k}$. Given the level $\bar{z}$, the steady state capital stock is found from the familiar equation

$$1 = \beta[1 + \bar{z} f'(\bar{k}) - \delta]$$

and then steady state consumption is backed out of the resource constraint

$$\bar{c} = \bar{z} f(\bar{k}) - \delta \bar{k}$$

C. Log-linear model

Log-linearizing each side of the resource constraint, we have

$$c_t + k_{t+1} \approx \bar{c} + \bar{k} + \bar{c}\hat{c}_t + \bar{k}\hat{k}_{t+1}$$

and

$$z_t f(k_t) + (1-\delta)k_t \approx \bar{z} f(\bar{k}) + (1-\delta)\bar{k} + f(\bar{k})\bar{z}\hat{z}_t + [\bar{z} f'(\bar{k}) + (1-\delta)]\bar{k}\hat{k}_t$$

Equating both sides and simplifying

$$\bar{c}\hat{c}_t + \bar{k}\hat{k}_{t+1} = f(\bar{k})\bar{z}\hat{z}_t + [\bar{z} f'(\bar{k}) + (1-\delta)]\bar{k}\hat{k}_t$$

More generally, we will write this in terms of coefficients that are known functions of the parameters of the model,

$$0 = A\hat{k}_{t+1} + B\hat{k}_t + C\hat{c}_t + D\hat{z}_t \tag{1}$$

for known coefficients A, B, C, D.

Similarly, log-linearizing the left hand side of the stochastic Euler equation

$$U'(c_t) \approx U'(\bar{c}) + U''(\bar{c})\bar{c}\hat{c}_t$$

and log-linearizing inside the conditional expectations

$$\begin{aligned} & \beta \mathsf{E}_t\{U'(c_{t+1})[1 + z_{t+1} f'(k_{t+1}) - \delta]\} \\ \approx \quad & \beta \mathsf{E}_t\{U'(\bar{c})[1 + \bar{z} f'(\bar{k}) - \delta] + U''(\bar{c})[1 + \bar{z} f'(\bar{k}) - \delta]\bar{c}\hat{c}_{t+1} + U'(\bar{c}) f'(\bar{k})\bar{z}\hat{z}_{t+1} + U'(\bar{c})\bar{z} f''(\bar{k})\bar{k}\hat{k}_{t+1}\} \end{aligned}$$

Equating both sides and simplifying

$$\frac{U''(\bar{c})\bar{c}}{U'(\bar{c})}\hat{c}_t = \mathsf{E}_t\left\{\frac{U''(\bar{c})\bar{c}}{U'(\bar{c})}\hat{c}_{t+1} + \beta f'(\bar{k})\bar{z}\hat{z}_{t+1} + \beta\bar{z} f''(\bar{k})\bar{k}\hat{k}_{t+1}\right\}$$

Recall that $\mathcal{R}(c) = -\frac{U''(\bar{c})\bar{c}}{U'(\bar{c})}$ is the Arrow/Pratt measure of relative risk aversion.

Again, writing this in terms of coefficients that are known functions of the parameters of the model,

$$0 = \mathsf{E}_t \left\{ F\hat{k}_{t+1} + J\hat{c}_{t+1} + K\hat{c}_t + L\hat{z}_{t+1} \right\} \tag{2}$$

for known F, J, K, L.

D. Method of undetermined coefficients

For simplicity, suppose that the exogenous shocks can be written

$$\hat{z}_{t+1} = N\hat{z}_t + \varepsilon_{t+1}$$

where $\{\varepsilon_{t+1}\}$ is mean zero white noise. Then we have the log-linear model

$$\begin{aligned} 0 &= A\hat{k}_{t+1} + B\hat{k}_t + C\hat{c}_t + D\hat{z}_t \\ 0 &= \mathsf{E}_t \left\{ F\hat{k}_{t+1} + J\hat{c}_{t+1} + K\hat{c}_t + L\hat{z}_{t+1} \right\} \\ \hat{z}_{t+1} &= N\hat{z}_t + \varepsilon_{t+1} \end{aligned}$$

The first equation contains the static resource constraint, the second equation contains the "forward-looking" Euler equation, while the last is the exogenous law of motion for the productivity shocks.

We can again try to solve this model with the method of undetermined coefficients. To do so, guess that the solutions are laws of motion

$$\begin{aligned} \hat{k}_{t+1} &= P\hat{k}_t + Q\hat{z}_t \\ \hat{c}_t &= R\hat{k}_t + S\hat{z}_t \end{aligned}$$

for unknown coefficients P, Q, R, S. If we could solve for these coefficients, then together with the initial conditions $\hat{k}_0, \hat{z}_0$ and the exogenous law of motion for the shocks, $\hat{z}_{t+1} = N\hat{z}_t + \varepsilon_{t+1}$, we would have a complete solution to the linear model. We would then be in a position to do things like simulate the model, compute impulse response functions and the like.

So, plug the guesses into the first equation to get

$$\begin{aligned} 0 &= A(P\hat{k}_t + Q\hat{z}_t) + B\hat{k}_t + C(R\hat{k}_t + S\hat{z}_t) + D\hat{z}_t \\ &= (AP + B + CR)\hat{k}_t + (AQ + D + CS)\hat{z}_t \end{aligned}$$

Since these must hold for all $\hat{k}_t$ and $\hat{z}_t$, our first set of restrictions is

$$
\begin{aligned}
0 &= AP + B + CR \Longrightarrow R = -C^{-1}(AP+B) \qquad (3)\\
0 &= AQ + D + CS \Longrightarrow S = -C^{-1}(AQ+D) \qquad (4)
\end{aligned}
$$

So once we know P, Q we also have solutions for R, S. Clearly, the main difficulty lies in solving for the law of motion for the endogenous state variable $\hat{k}_t$.

Again, plugging our guesses into the forward-looking equation,

$$
\begin{aligned}
0 &= \mathsf{E}_t\left\{F(P\hat{k}_t + Q\hat{z}_t) + J(R\hat{k}_{t+1} + S\hat{z}_{t+1}) + K(R\hat{k}_t + S\hat{z}_t) + L\hat{z}_{t+1}\right\}\\
&= \mathsf{E}_t\left\{FP\hat{k}_t + FQ\hat{z}_t + JR\hat{k}_{t+1} + JS\hat{z}_{t+1} + KR\hat{k}_t + KS\hat{z}_t + L\hat{z}_{t+1}\right\}\\
&= \mathsf{E}_t\left\{FP\hat{k}_t + FQ\hat{z}_t + JRP\hat{k}_t + JRQ\hat{z}_t + JS\hat{z}_{t+1} + KR\hat{k}_t + KS\hat{z}_t + L\hat{z}_{t+1}\right\}
\end{aligned}
$$

Now taking the conditional expectations and then grouping terms

$$
0 = (FP + JRP + KR)\hat{k}_t + (FQ + JRQ + JSN + KS + LN)\hat{z}_t
$$

Since these must hold for all $\hat{k}_t$ and $\hat{z}_t$, our second set of restrictions is

$$
\begin{aligned}
0 &= FP + JRP + KR \qquad (5)\\
0 &= FQ + JRQ + JSN + KS + LN \qquad (6)
\end{aligned}
$$

Now plugging in the result $R = -C^{-1}(AP+B)$ from equation (3) we get a quadratic equation in P

$$
\begin{aligned}
0 &= FP - J(C^{-1}(AP+B))P - KC^{-1}(AP+B)\\
&= -JC^{-1}AP^2 + (F - JC^{-1}B - KC^{-1}A)P - KC^{-1}B
\end{aligned}
$$

Because this model has a saddle-path, this quadratic equation will have one stable and one unstable root. Since only the stable root will give a solution that satisfies the transversality condition, we pick that one. Then with P solved for, we know $R = -C^{-1}(AP+B)$. With P and R solved for, we can get use equation (6) to get

$$
0 = FQ + JRQ + JSN + KS + LN
$$

$$= \quad FQ + JRQ - JC^{-1}(AQ + D)N - KC^{-1}(AQ + D) + LN$$

or

$$Q = \frac{JC^{-1}DN + KC^{-1}D - LN}{F + JR - JC^{-1}AN - KC^{-1}A}$$

which we can compute because we have already solved for P and R and all the other coefficients were known to begin with. Finally, we can recover the remaining coefficient via $S = -C^{-1}(AQ+D)$.

So, after all that, we have a method for solving for the unknown coefficients. To recap: we solve for the non-stochastic steady state and construct the known matrices of coefficients. We then solve a quadratic equation for the critical P. We then back out the associated R so that we have all the coefficients on the endogenous state variables. Using P and R we solve for the coefficients on the shocks, Q and S. And then we're done and ready to do interesting things like simulating the model by iterating on the linear laws of motion.

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