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Introduction to stochastic dynamic programming

If the shocks have the Markov property, we do not need to keep track of the entire history and we can form a recursive version of the stochastic growth model by considering a value function $V(k, z)$ in two state variables, the endogenous capital stock k and the exogenous technology shock z .

The **Bellman equation** for this problem is

$$V(k, z) = \max_{k' \geq 0} \{U(c) + \beta \mathbf{E}[V(k', z')|z]\}$$

where the maximization is subject to the resource constraint

$$c + k' \leq zf(k) + (1 - \delta)k$$

and a given law of motion for the exogenous technology shocks. (For example, the technology shocks might obey a stochastic difference equation or a Markov chain). Notice that $\mathbf{E}[V(k', z')|z]$ is the expectation of the value function evaluated at next period's state **conditional** on this period's realization of the technology shock, z . As with deterministic dynamic programming, the right hand side of the Bellman equation defines an **operator**. The **value function** V is a **fixed point** of this operator. Given a value function, we can easily solve for a **policy function** $k' = g(k, z)$ that maps the state into choices for capital accumulation. From the resource constraint, this will then give a policy function for consumption, namely $c = zf(k) + (1 - \delta)k - g(k, z)$.

The first order condition for the maximization problem on the right hand side of the Bellman equation is

$$U'(c) = \beta \mathbf{E} \left\{ \frac{\partial V(k', z')}{\partial k'} \Big| z \right\}$$

while the envelope condition is

$$\frac{\partial V(k, z)}{\partial k} = U'(c)[1 + zf'(k) - \delta]$$

so we again have the stochastic Euler equation

$$U'(c_t) = \beta \mathbf{E}_t \{U'(c_{t+1})[1 + z_{t+1}f'(k_{t+1}) - \delta]\}$$

Suppose that the technology shocks follow a stochastic difference equation. Then the capital stock also follows a difference equation, namely

$$k_{t+1} = g(k_t, z_t)$$

If we log-linearize this difference equation, we get

$$\hat{k}_{t+1} = \frac{g_k(\bar{k}, \bar{z}) \bar{k}}{g(\bar{k}, \bar{z})} \hat{k}_t + \frac{g_z(\bar{k}, \bar{z}) \bar{z}}{g(\bar{k}, \bar{z})} \hat{z}_t$$

The approximate elasticities of the policy function are exactly the coefficients (that we called P and Q) that we solve for with the method of undetermined coefficients.

A. Closed form example

One particular parametric example of the stochastic growth model can be solved in closed form (i.e., without any need for approximations). Let the period utility function be $U(c) = \log(c)$ and let the production function be $f(k) = k^\alpha$ with full depreciation, $\delta = 1$. Then the policy function is

$$k' = g(k, z) \equiv \alpha\beta z k^\alpha$$

Interestingly, this solution does not depend on the driving process for technology. To see that this solution works, notice that it satisfies the resource constraint so long as consumption is given by

$$c = (1 - \alpha\beta) z k^\alpha$$

Now the stochastic Euler equation requires

$$\frac{1}{c} = \beta \mathbf{E} \left\{ \frac{1}{c'} \alpha z' (k')^{\alpha-1} \middle| z \right\}$$

or

$$\begin{aligned} \frac{1}{(1 - \alpha\beta) z k^\alpha} &= \beta \mathbf{E} \left\{ \frac{1}{(1 - \alpha\beta) z' (k')^\alpha} \alpha z' (k')^{\alpha-1} \middle| z \right\} \\ &= \alpha\beta \mathbf{E} \left\{ \frac{1}{(1 - \alpha\beta) k'} \middle| z \right\} \\ &= \alpha\beta \mathbf{E} \left\{ \frac{1}{(1 - \alpha\beta) \alpha\beta z k^\alpha} \middle| z \right\} \end{aligned}$$

$$= \frac{1}{(1 - \alpha\beta)zk^\alpha}$$

so the proposed solution indeed satisfies the stochastic Euler equation.

B. Value function iteration

We now show that this guess can be obtained by value function iteration. We will be iterating on Bellman equations of the form

$$V_{j+1} = TV_j$$

where for $j = 0, 1, 2, \dots$

$$TV_j(k, z) = \max_{k' \geq 0} \{\log[zk^\alpha - k'] + \beta \mathbf{E}[V_j(k', z')|z]\}$$

Begin with the guess

$$V_0(k, z) = 0 \quad \text{all } k, z$$

Then applying the T operator gives

$$\begin{aligned} V_1(k, z) &= TV_0(k, z) \\ &= \max_{k' \geq 0} \{\log[zk^\alpha - k'] + \beta \mathbf{E}[V_0(k', z')|z]\} \\ &= \max_{k' \geq 0} \{\log[zk^\alpha - k'] + \beta 0\} \\ &= \log[zk^\alpha] \end{aligned}$$

(since the optimal policy if $V_0(k', z') = 0$ is to set $k' = 0$ too. If the value associated with next period's state is zero, there is no point saving).

Now our new estimate of the value function is

$$V_1(k, z) = TV_0(k, z) = \log(z) + \alpha \log(k)$$

And again applying the T operator

$$\begin{aligned} V_2(k, z) &= TV_1(k, z) \\ &= \max_{k' \geq 0} \{\log[zk^\alpha - k'] + \beta \mathbf{E}[\log(z') + \alpha \log(k')|z]\} \\ &= \max_{k' \geq 0} \{\log[zk^\alpha - k'] + \alpha\beta \log(k') + \beta \mathbf{E}[\log(z')|z]\} \end{aligned}$$

The maximization on the right hand side is maximization of a strictly concave function (notice that the conditional expectation simply defines an additive constant that will not affect the optimal policy, though it will affect the value function). The necessary and sufficient condition for a global maximum is

$$-\frac{1}{zk^\alpha - k'} + \frac{\alpha\beta}{k'} = 0$$

Solving this first order condition gives the policy associated with the estimate $V_1(k, z)$, namely

$$k' = \frac{\alpha\beta}{1 + \alpha\beta} zk^\alpha$$

Hence we have

$$\begin{aligned} V_2(k, z) &= TV_1(k, z) \\ &= \max_{k' \geq 0} \{ \log[zk^\alpha - k'] + \alpha\beta \log(k') + \beta \mathbf{E}[\log(z')|z] \} \\ &= \log \left[\frac{1}{1 + \alpha\beta} zk^\alpha \right] + \alpha\beta \log \left[\frac{\alpha\beta}{1 + \alpha\beta} zk^\alpha \right] + \beta \mathbf{E}[\log(z')|z] \end{aligned}$$

This is of the form

$$V_2(k, z) = A + B \log(k) + C \log(z)$$

for coefficients A, B, C . Now let's use this functional form to make a new estimate

$$\begin{aligned} V_3(k, z) &= TV_2(k, z) \\ &= \max_{k' \geq 0} \{ \log[zk^\alpha - k'] + \beta \mathbf{E}[A + B \log(k') + C \log(z')|z] \} \\ &= \max_{k' \geq 0} \{ \log[zk^\alpha - k'] + \beta A + \beta B \log(k') + \beta C \mathbf{E}[\log(z')|z] \} \end{aligned}$$

Again, computing the first order condition

$$-\frac{1}{zk^\alpha - k'} + \frac{\beta B}{k'} = 0$$

and solving for the optimal policy

$$k' = \frac{\beta B}{1 + \beta B} zk^\alpha$$

Hence we have

$$\begin{aligned}
V_3(k, z) &= TV_2(k, z) \\
&= \max_{k' \geq 0} \{ \log[zk^\alpha - k'] + \beta A + \beta B \log(k') + \beta C E[\log(z')|z] \} \\
&= \log \left[\frac{1}{1 + \beta B} zk^\alpha \right] + \beta A + \beta B \log \left[\frac{\beta B}{1 + \beta B} zk^\alpha \right] + \beta C E\{\log(z')|z\}
\end{aligned}$$

Just as with the last round of iterations, we are going to get a value function that is of the form

$$V_3(k, z) = A + B \log(k) + C \log(z)$$

All we have to do to complete our solution is to make sure that the unknown coefficients are consistent. In order to identify the coefficient on the policy function we only need to know the right value for B , though we need to know all of A, B, C to get the whole value function. In particular, B solves

$$B \log(k) = \alpha \log(k) + \beta B \alpha \log(k)$$

or

$$B = \frac{\alpha}{1 - \alpha\beta}$$

Hence

$$k' = \frac{\beta B}{1 + \beta B} zk^\alpha = \frac{\beta \frac{\alpha}{1 - \alpha\beta}}{1 + \beta \frac{\alpha}{1 - \alpha\beta}} zk^\alpha = \alpha\beta zk^\alpha$$

which was our original guess!

Some tedious algebra shows that the complete solution for the value function for this parametric example is

$$V(k, z) = A + B \log(k) + C \log(z)$$

where the coefficients are

$$\begin{aligned}
A &= \frac{1}{1 - \beta} \left[\log(1 - \alpha\beta) + \frac{\alpha\beta \log(\alpha\beta)}{1 - \alpha\beta} + \frac{\beta\mu}{1 - \alpha\beta} \right], & \mu &\equiv E_0\{\log(z)\} \\
B &= \frac{\alpha}{1 - \alpha\beta} \\
C &= \frac{1}{1 - \alpha\beta}
\end{aligned}$$

(See if you can show this yourself!)

C. Stochastic dynamics

Now let's look at the stochastic dynamics implied by this solution. In logs,

$$\log(k_{t+1}) = \log(\alpha\beta) + \alpha \log(k_t) + \log(z_t)$$

Notice that conditional on this period's technology shock, the capital stock for next period is **not random**. Nonetheless, output at $t + 1$ is random as indeed is the capital stock in two period's time. Notice that

$$\log(y_t) = \log(z_t) + \alpha \log(k_t)$$

so we can write

$$\begin{aligned}\log(k_{t+1}) &= \log(\alpha\beta) + [\log(y_t) - \log(z_t)] + \log(z_t) \\ &= \log(\alpha\beta) + \log(y_t)\end{aligned}$$

which gives us a relationship between log output at t and at $t + 1$, namely

$$\log(y_{t+1}) = \alpha \log(\alpha\beta) + \alpha \log(y_t) + \log(z_{t+1})$$

Although $\log(k_{t+1})$ is not random, $\log(y_{t+1})$ is and its statistical properties depend mostly on the capital intensity α and on the process for log technology. For example, if log technology is IID, then log output is an autoregression with persistence α .

We can solve for the statistical properties of various objects of interest by solving the law of motion for the capital stock. To do so, iterate recursively,

$$\log(k_1) = \log(\alpha\beta) + \alpha \log(k_0) + \log(z_0)$$

and for date $t = 2$

$$\begin{aligned}\log(k_2) &= \log(\alpha\beta) + \alpha \log(k_1) + \log(z_1) \\ &= \log(\alpha\beta) + \alpha \log(\alpha\beta) + \alpha^2 \log(k_0) + \alpha \log(z_0) + \log(z_1)\end{aligned}$$

and for date $t = 3$

$$\begin{aligned}\log(k_3) &= \log(\alpha\beta) + \alpha \log(k_2) + \log(z_2) \\ &= \log(\alpha\beta) + \alpha \log(\alpha\beta) + \alpha^2 \log(\alpha\beta) + \alpha^3 \log(k_0) + \alpha^2 \log(z_0) + \alpha \log(z_1) + \log(z_2)\end{aligned}$$

and indeed for $t \geq 1$,

$$\log(k_t) = \sum_{i=0}^{t-1} \alpha^i \log(\alpha\beta) + \alpha^t \log(k_0) + \sum_{i=0}^{t-1} \alpha^{t-1-i} \log(z_i)$$

It is often useful to break out the initial $\log(z_0)$ so that

$$\log(k_t) = \sum_{i=0}^{t-1} \alpha^i \log(\alpha\beta) + \alpha^t \log(k_0) + \alpha^{t-1} \log(z_0) + \sum_{i=1}^{t-1} \alpha^{t-1-i} \log(z_i)$$

Now suppose (as a very unrealistic example) that the technology shocks were Gaussian white noise with mean zero and standard deviation $\sigma > 0$. Then the distribution of the log capital stock at date $t \geq 1$ conditional only on the date $t = 0$ information is also normal with mean

$$\mathbb{E}_0\{\log(k_t)\} = \frac{1 - \alpha^t}{1 - \alpha} \log(\alpha\beta) + \alpha^t \log(k_0) + \alpha^{t-1} \log(z_0)$$

and variance

$$\mathbb{V}_0\{\log(k_t)\} = \sum_{i=1}^{t-1} \alpha^{2(t-1-i)} \sigma^2 = \sum_{i=0}^{t-2} \alpha^{2i} \sigma^2 = \frac{1 - \alpha^{2(t-1)}}{1 - \alpha^2} \sigma^2$$

Notice that these moments condition on the initial given $\log(z_0)$.

As $t \rightarrow \infty$, the moments converge to values independent of the initial conditions. The more capital intensive the production function, the more slowly this convergence takes place. The **stationary distribution** for the log capital stock is Gaussian with moments

$$\begin{aligned}\mathbb{E}\{\log(k_t)\} &= \frac{1}{1 - \alpha} \log(\alpha\beta) \\ \mathbb{V}\{\log(k_t)\} &= \frac{1}{1 - \alpha^2} \sigma^2\end{aligned}$$

Notice that the average log capital stock is higher the higher is the capital intensity of the production function α , and the higher is the time discount factor β . Because the capital intensity also governs the persistence of the capital stock, the long run variance is higher the higher is α and the higher is

the innovation variance σ^2 .

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