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Mehra and Prescott's equity premium puzzle

Consider an economy with risky trees ("**equity**"), denoted s, and sure claims ("**bonds**"), denoted B. We could price contingent claims and then back out the implications for bonds, but in this exercise we will go straight to the latter. Unlike our earlier asset pricing problems, this model will have a **non-stationary** environment. In particular, dividends (which in equilibrium will also equal consumption) will have a **stochastic trend** and so exhibit unit-root-like dynamics. To model this, let equity pay random dividends y each period. The (gross) **growth rate** of dividends $x' \equiv y'/y$ follows an n-state Markov chain with transition probabilities

$$\pi(x', x) = \Pr(x_{t+1} = x' | x_t = x)$$

and given initial conditions.

The Bellman equation for the household is then

$$V(w, x, y) = \max_{s' \geq 0, B' \geq 0} \left\{ U(c) + \beta \sum_{x'} V(w', x', y')\pi(x', x) \right\}$$

where w denotes household wealth and the maximization on the right hand side is subject to a budget constraint

$$c + p(x, y)s' + q(x, y)B' \leq w$$

and we have the laws of motion

$$\begin{aligned} w' &= [p(x', y') + y']s' + B' \\ y' &= x'y \end{aligned}$$

Definition. A **recursive competitive equilibrium** for this economy is a collection of functions: (i) a value function V, (ii) individual decision rules g_s and g_B, and (iii) pricing functions p and q such that:

1. Given the pricing functions p and q, the value function V and the individual decision rules g_s and g_B solve the household's dynamic programming problem, and

2. Markets clear

$$s' = g_s(w, x, y) = 1$$
$$B' = g_B(w, x, y) = 0$$

Combining these last two requirements with the individual budget constraint and the definition of wealth, we must also always have the goods market clearing condition

$$c = y$$

Notice once again that prices $p(x, y)$ and $q(x, y)$ are functions only of the aggregate state (x, y) while individual decisions and the value function are functions of both the aggregate state and the individual state, wealth w.

A. Solving the model

We can begin to solve the problem by finding the equilibrium price for equity. The key first order condition is

$$U'(c)p(x, y) = \beta \sum_{x'} \frac{\partial V(w', x', y')}{\partial w'} [p(x', x'y) + x'y]\pi(x', x)$$

(where we have used $y' = x'y$). The associated envelope condition is

$$\frac{\partial V(w, x, y)}{\partial w} = U'(c)$$

In equilibrium, with $c = y$, we have

$$p(x, y) = \beta \sum_{x'} \frac{U'(x'y)}{U'(y)} [p(x', x'y) + x'y]\pi(x', x)$$

We need to find a pricing function $p(x, y)$ that satisfies this functional equation. To solve this problem, we follow Mehra and Prescott (1985) and suppose that utility has the **constant relative risk aversion** form

$$U(c) = \frac{c^{1-\sigma}}{1 - \sigma}, \qquad \sigma > 0$$

Then the ratio of marginal utilities is

$$\beta\frac{U'(y')}{U'(y)} = \beta\left(\frac{y'}{y}\right)^{-\sigma} = \beta\left(x'\right)^{-\sigma}$$

Since dividend growth x' follows an n-state Markov chain with typical elements x_i and x_j we can write the functional equation problem as

$$p(x_i, y) = \beta\sum_{j=1}^{n} x_j^{-\sigma}[p(x_j, x_j y) + x_j y]\pi(x_j, x_i), \qquad i = 1, ..., n$$

Now notice that if we were to make the guess

$$p(x_i, y) = p_i y, \qquad i = 1, ..., n$$

for some as-yet unknown coefficients p_i, we would find that the coefficients p_i solve the equation

$$\begin{aligned} p_i y &= \beta\sum_{j=1}^{n} x_j^{-\sigma}(p_j x_j y + x_j y)\pi(x_j, x_i) \\ &= \beta y\sum_{j=1}^{n} x_j^{1-\sigma}(p_j + 1)\pi(x_j, x_i) \end{aligned}$$

or

$$p_i = \beta\sum_{j=1}^{n} x_j^{1-\sigma}(p_j + 1)\pi(x_j, x_i)$$

We have n linear equations in n unknown price coefficients p_i. These equations are fairly easy to solve. To do so, define an n-by-n matrix $\mathbf{A}$ with typical element

$$a_{ij} = \beta x_j^{1-\sigma}\pi(x_j, x_i)$$

and similarly define an n-by-1 vector $\mathbf{b}$ with typical element

$$b_i = \sum_{j=1}^{n} \beta x_j^{1-\sigma}\pi(x_j, x_i)$$

Then we can re-write the problem as one of solving

$$\mathbf{p} = \mathbf{A}\mathbf{p} + \mathbf{b}$$

where $\mathbf{p}$ is a n-vector with typical element p_i. This is equivalent to

$$(\mathbf{I}-\mathbf{A})\mathbf{p}=\mathbf{b}$$

Hence so long as $(\mathbf{I}-\mathbf{A})$ is invertible, our solution is just

$$\mathbf{p} = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}$$

(which is easily computed using Matlab). Once we've done this calculation to get the vector $\mathbf{p}$, we then obtain the actual equity prices $p(x, y)$ from $p(x_i, y) = p_i y$. Notice that this means that equity prices, like dividends y, are non-stationary. Equity prices inherit the unit-root-like dynamics of the level of dividends.

As a technical aside, notice that

$$(\mathbf{I}-\mathbf{A})^{-1} = \mathbf{I}+\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\cdots$$

A necessary and sufficient condition for this to be well defined is that $\lim_{m\to\infty}\mathbf{A}^m = 0$. In turn, this requires that all of the eigenvalues of $\mathbf{A}$ are less than one in absolute value. If one stares long enough at the elements $a_{ij} = \beta x_j^{1-\sigma}\pi(x_j, x_i)$, it's clear that this means that consumption cannot grow too fast relative to the discount factor. The reason for this requirement is that in a non-stationary economy with consumption growing over time, the requirement that $0 < \beta < 1$ is neither necessary nor sufficient for the infinite sum $\sum_{t=0}^{\infty}\beta^t U(c_t)$ to be well defined. A sufficient condition that is often imposed is

$$\sum_{i=1}^{n}\beta x_i^{1-\sigma}\bar{\pi}(x_i) = \mathsf{E}\{\beta x^{1-\sigma}\} < 1$$

(where $\bar{\pi}(x_i)$ is the stationary probability of being in state x_i). This guarantees that consumption does not grow so fast (relative to the discount factor β) that utility is unbounded. In practice, this means that $\lim_{m\to\infty}\mathbf{A}^m = 0$.

Now back to the main game. We can also find the price of the bond by using the first order condition

$$U'(c)q(x, y) = \beta\sum_{x'}\frac{\partial V(w', x', x'y)}{\partial w'}\pi(x', x)$$

The envelope condition is the same as for the price of equity, so we have in equilibrium

$$q(x, y) = \beta \sum_{x'} \frac{U'(x'y)}{U'(y)} \pi(x', x)$$

and with the CRRA assumption on preferences,

$$q(x_i, y) = \beta \sum_{j=1}^{n} x_j^{-\sigma} \pi(x_j, x_i)$$

Notice that everything on the right hand side of this expression is a primitive of the model, so this constitutes a bona-fide solution for the pricing function for bonds. Notice also that this solution does not in fact depend on the non-stationary level of endowments y. Without any loss of generality we can associate $q(x_i, y)$ with a vector of coefficients with typical element q_i. Notice that this means that bond prices — unlike equity prices — are stationary.

B. Rates of return

We're now in a position to compute the equity premium, the difference between the average rate of return on equity, $\bar{r}^e$, and the average rate of return on risk-free bonds, $\bar{r}^f$. First, note that the realized rate of return on equity is

$$\begin{aligned} \hat{r}^e(x_j, x_i) &= \frac{p(x_j, x_j y) + x_j y - p(x_i, y)}{p(x_i, y)} \\ &= \frac{p_j x_j y + x_j y - p_i y}{p_i y} \\ &= \frac{p_j x_j + x_j - p_i}{p_i} \end{aligned}$$

Therefore, the conditionally expected rate of return on equity if the current state is x_i is

$$r^e(x_i) = \sum_{j=1}^{n} \hat{r}^e(x_j, x_i) \pi(x_j, x_i)$$

The long run average rate of return on equity is then

$$\bar{r}^e = \sum_{i=1}^{n} r^e(x_i) \bar{\pi}(x_i) = \mathsf{E}\{r^e\}$$

Similarly, the return on a bond (which is risk free) is

$$r^f(x_i) = \frac{1 - q(x_i, y)}{q(x_i, y)} = \frac{1 - q_i}{q_i}$$

(this uses the fact that with CRRA utility, $q(x_i, y)$ does not depend on the level y). Then

$$\bar{r}^f = \sum_{i=1}^{n} r^f(x_i)\bar{\pi}(x_i) = \mathsf{E}\{r^f\}$$

The average equity premium is defined by

$$\bar{r}^e - \bar{r}^f$$

In Mehra and Prescott's data [actually — Grossman and Shiller's (1981) data!], the long run average return on equity $\bar{r}^e$ is a number like 0.07 (that is, equity return of 7%) over the full sample 1889-1978 while the long run average $\bar{r}^f$ is a number like 0.01 (bond return of 1%). The measured equity premium is then a number like $0.07 - 0.01 = 0.06$ or 6%.

C. Calibration

To calibrate this model, Mehra and Prescott (1985) set $n = 2$ and $x_1 = 1 + \mu - \delta$ (a "bad" state) and $x_2 = 1 + \mu + \delta$ (a "good" state). They further set μ to be the long-run average annual growth rate of per capita consumption and δ to the standard deviation of per capita consumption over the years 1889-1978. The first order autocorrelation coefficient of per capita consumption is governed by a single parameter ϕ such that if we define

$$\pi_{ij} = \pi(x_j, x_i) = \Pr(x_{t+1} = x_j | x_t = x_i)$$

then the transition matrix is symmetric with

$$\begin{pmatrix} \pi_{11} & 1 - \pi_{11} \\ 1 - \pi_{22} & \pi_{22} \end{pmatrix} = \begin{pmatrix} \phi & 1 - \phi \\ 1 - \phi & \phi \end{pmatrix}$$

This calibration leads to $\mu = 0.018$, annual average consumption growth of just less than 2%, $\delta = 0.036$, annual standard deviation of consumption growth of 3.6%, and $\phi = 0.43$, average first order autocorrelation of consumption growth of

$$2\phi - 1 = -0.14$$

Mehra and Prescott then experiment with different numbers for relative risk aversion $\sigma \in [0, 10]$ and $\beta \in (0, 1)$ and imposed the restriction that $\bar{r}^f \in (0, 0.04]$. The finding: the maximum equity premium that can be obtained from this calibration of a consumption-based asset pricing mode is 0.35 **of a percent** as opposed to 6 percent in the data. The model is out by a factor of almost twenty.

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