316-406 Advanced Macroeconomic Techniques

Problem Set #4 — *Due 18 October, 2004*

Question 1. (*Deterministic Dynamic Programming*). Consider the social planning problem of maximizing utility

$$\sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to a resource constraint

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t, \qquad k_0 \text{ given}$$

- Provide a dynamic programming representation of this problem. Using first order and envelope conditions, show how to characterize optimal consumption plans.

- Let the period utility function be

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma}, \qquad \sigma > 0$$

  and let the production function be

$$f(k) = k^\alpha, \qquad 0 < \alpha < 1$$

  and suppose the parameter values are

Symbol	Meaning	Value
β	time discount factor	0.95
α	capital's share in national output	0.33
δ	depreciation rate of physical capital	0.04
σ	inverse of intertemporal elasticity of substitution	2.00

  Solve for steady state consumption $\bar{c}$ and capital stock $\bar{k}$.

- Suppose capital stocks are constrained to belong to a grid

$$k \in \mathcal{K} \equiv [k_{\min} < \cdots < k_{\max}]$$

  In Matlab, construct a grid $\mathcal{K}$ with $k_{\min} = 0$ and $k_{\max} = 5\bar{k}$ with 1000 evenly spaced elements. Solve the dynamic programming problem on this discrete state space by value function iteration. Plot the value function that is a fixed point of the Bellman equation and plot the associated policy function. Hint: when de-bugging your program, you might find it easier to use a low value for the discount factor — say $\beta = 0.50$ — so that it takes fewer iterations to find a fixed point.

Question 2. (*Stochastic dynamic programming*). Consider the social planning problem of maximizing expected utility

$$\mathsf{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t U(c_t) \right\}$$

subject to a resource constraint

$$c_t + k_{t+1} = z_t f(k_t) + (1-\delta)k_t, \qquad k_0, z_0 \text{ given}$$

where z_t follows an exogenous stochastic process. In particular, suppose that z_t follows a 2-state Markov chain with support

$$z \in \mathcal{Z} = \{z_L, z_H\} \equiv \{0.97, 1.03\}$$

Let the transition matrix be

$$P = \begin{pmatrix} 0.95 & 0.05 \\ 0.05 & 0.95 \end{pmatrix}$$

- Provide a dynamic programming representation of this problem. Using first order and envelop conditions, show how to characterize optimal consumption plans.

- Let the period utility function $U(c)$ and production function $f(k)$ be as in Question 1 with the same parameter values. Suppose capital stocks are constrained to belong to a grid

  $$k \in \mathcal{K} \equiv [k_{\min} < \cdots < k_{\max}]$$

  as in Question 1 but with $k_{\max} = 5\bar{k}_H$ where $\bar{k}_H$ solves

  $$1 = \beta(1 + z_H f'(k) - \delta)$$

  Solve the dynamic programming problem on the discrete state space $\mathcal{K} \times \mathcal{Z}$ by value function iteration. Plot the value functions that are a fixed point of the Bellman equation for current $z = z_L$ and current $z = z_H$ and plot the associated policy functions.

- Show how to represent the stochastic dynamics of this model as a Markov chain on the state space $\mathcal{K} \times \mathcal{Z}$. Explain how you set up the transition matrix for this "big" Markov chain. That is, explain how you set up the probability of going from any (k, z) to any (k', z'). Use Matlab to compute this big transition matrix. Solve for the stationary distribution over the state space $\mathcal{K} \times \mathcal{Z}$. Plot the stationary distribution for current $z = z_L$ and current $z = z_H$.