**316-406 ADVANCED MACROECONOMIC TECHNIQUES** **Sample answers**

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The final will last 180 minutes and has two questions. The first question is worth 120 marks, while the second question is worth only 60 marks. Within each question there are a number of parts and the weight given to each part will also be indicated.

Here are two sample questions.

**Question 1.** *Term premia* (120 marks). Mehra and Presocтt study the equity premium, the average excess return on equity over bonds. In this question, you will study the term premium, the difference between the returns on bonds of differing maturity. Consider a representative agent consumption based asset pricing model where preferences are

$$\mathsf{E}_0\left\{\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\gamma}}{1-\gamma}\right\}, \qquad 0<\beta<1 \text{ and } \gamma\geq 0$$

There are two kinds of assets. First, there is a "Lucas tree" with dividends $\{y_t\}$ with gross growth rate that follows a Markov chain. That is, let

$$x_{t+1}=\frac{y_{t+1}}{y_t}$$

Then $\{x_t\}$ follows a Markov chain with transition probabilities

$$\pi(x',x)=\Pr(x_{t+1}=x'|x_t=x)$$

Suppose the representative agent can trade in shares in the tree (with constant exogenous supply normalized to 1) and can trade in one and two period bonds. A 1-period bond is a riskless claim to a unit of consumption to be delivered next period, while a 2-period bond is a riskless claim to one unit of consumption to be delivered in two period's time.

(a) (20 marks): Let $q_j(x,y)$ denote the price of a $j$-bond ($j=1,2$) if the current aggregate state is $(x,y)$ and let $p(x,y)$ denote the price of a claim to the Lucas tree. Let $V(w,x,y)$ denote the consumer's value function if their individual wealth is $w$ and the aggregate state is $(x,y)$. Write down a Bellman equation for the consumer's problem. Be careful to explain the Bellman equation and any constraints that you provide. [*Hint*: a 2-period bond bought this period can be re-sold as a 1-period bond next period].

**Solution:** The Bellman equation for this problem is

$$V(w,x,y) = \max_{s',B_1',B_2'} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \sum_{x'} V(w',x',y')\pi(x',x) \right\}$$

where the maximization is subject to the budget constraint

$$c + p(x,y)s' + q_1(x,y)B_1' + q_2(x,y)B_2' \leq w$$

and where next period's wealth is

$$w' = [p(x',y') + x']s' + B_1' + q_1(x',y')B_2'$$

and

$$y' = x'y$$

Notice that a two-period bond bought today at $q_2(x,y)$ can be sold next period as a one-period bond with price $q_1(x',y')$. All kinds of bonds are perfect substitutes at their maturity in the sense that a one-period bond bought yesterday and a two-period bond bought two days ago both deliver one unit of consumption today.

(b) (15 marks): Define a **recursive competitive equilibrium** for this economy.

**Solution:** A recursive competitive equilibrium consists of a value function $V(w,x,y)$, individual decision rules $g_s(w,x,y)$, $g_{B1}(w,x,y)$ and $g_{B2}(w,x,y)$ and pricing functions $p(x,y), q_1(x,y)$ and $q_2(x,y)$ such that given the prices the value function and individual decision rules solve the consumer's problem and markets clear, namely

$$\begin{aligned} s' &= g_s(w,x,y) = 1 \\ B_1' &= g_{B1}(w,x,y) = 0 \\ B_2' &= g_{B2}(w,x,y) = 0 \end{aligned}$$

These market clearing conditions imply

$$c = y$$

(c) (15 marks): Go as far as you can in solving for the price of the tree, $p(x, y)$. Carefully explain how you could implement this solution on a computer. In your answer, let the Markov chain have $i = 1, ..., n$ states.

**Solution:** The first order and envelope conditions include

$$c^{-\gamma} p(x, y) = \beta \sum_{x'} \frac{\partial V(w', x', y')}{\partial w'} [p(x', y') + x'] \pi(x', x)$$

and

$$\frac{\partial V(w, x, y)}{\partial w} = c^{-\gamma}$$

Also we know $y' = x'y$ and in equilibrium $c = y$ so we can put these together to write

$$p(x, y) = \beta \sum_{x'} (x')^{-\gamma} [p(x', x'y) + x'] \pi(x', x)$$

Since dividend growth $x'$ follows an $n$-state Markov chain with typical elements $x_i$ and $x_k$ we can write the functional equation problem as

$$p(x_i, y) = \beta \sum_{k=1}^{n} x_k^{-\gamma} [p(x_k, x_k y) + x_k y] \pi(x_k, x_i), \qquad i = 1, ..., n$$

Now guess that the price can be written in the linear form

$$p(x_i, y) \equiv \hat{p}_i y$$

where $\hat{p}_i$ are a set of as-yet unknown coefficients associated with the $n$ Markov states. Plugging in this guess and simplifying gives

$$\hat{p}_i = \beta \sum_{k=1}^{n} x_k^{1-\gamma} (\hat{p}_k + 1) \pi(x_k, x_i), \qquad i = 1, ..., n$$

In standard vector notation, this is just the linear algebra problem

$$\hat{\mathbf{p}} = \mathbf{A} \hat{\mathbf{p}} + \mathbf{b}$$

where

$$
\begin{aligned}
a_{ik} &\equiv \beta x_k^{1-\gamma}\pi(x_k, x_i) \\
b_i &\equiv \sum_{k=1}^{n} \beta x_k^{1-\gamma}\pi(x_k, x_i)
\end{aligned}
$$

As long as the average growth rate of dividends is not too high, this linear algebra problem has a unique solution given by

$$\hat{\mathbf{p}} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$$

We can then recover the equilibrium prices using the definition $p(x_i, y) \equiv \hat{p}_i y$.

(d) (15 marks): Use first order and envelope conditions to characterize the optimal decisions of the representative consumer. Using these and market clearing conditions, solve for the prices $q_j(x, y)$ for $j = 1, 2$. Give economic intuition for your solutions. Explain the difference (if any) between the price of a 2-period bond and the price of two 1-period bonds. [*Hint*: what would these prices be if the consumer was risk neutral ($\gamma = 0$)? How (if at all) does this change when the consumer is risk averse ($\gamma > 0$)?]

**Solution:** The first order conditions are

$$
\begin{aligned}
c^{-\gamma} q_1(x, y) &= \beta \sum_{x'} \frac{\partial V(w', x', y')}{\partial w'} \pi(x', x) \\
c^{-\gamma} q_2(x, y) &= \beta \sum_{x'} \frac{\partial V(w', x', y')}{\partial w'} q_1(x', y')\pi(x', x)
\end{aligned}
$$

Notice that the payoff of a two-period bond is not 1 but instead the price in the next period of a one-period bond. Again we have the envelope condition

$$\frac{\partial V(w, x, y)}{\partial w} = c^{-\gamma}$$

Also, $y' = x'y$ and in equilibrium $c = y$ so we can write

$$
\begin{aligned}
q_1(x, y) &= \beta \sum_{x'} (x')^{-\gamma} \pi(x', x) \\
q_2(x, y) &= \beta \sum_{x'} (x')^{-\gamma} q_1(x', x'y)\pi(x', x)
\end{aligned}
$$

This is a bona-fide solution for the price of a one period bond, and indirectly a solution for the price of a two-period bond. Notice that the price of the one period bond does not depend on $y$ (it does not appear on the right hand side) so we can write $\hat{q}_{1i} \equiv q_1(x_i, y)$. But this implies that the price of a two period bond also does not depend on $y$, since the only way it can depend on the level is through the price of the one period bond which we already know does not depend on $y$. So we can also write $\hat{q}_{2i} \equiv q_2(x_i, y)$ with

$$\begin{aligned}
\hat{q}_{1i} &= \beta \sum_{k=1}^{n} x_k^{-\gamma} \pi(x_k, x_i), \qquad i = 1, ..., n \\
\hat{q}_{2i} &= \beta \sum_{k=1}^{n} x_k^{-\gamma} \hat{q}_{1k} \pi(x_k, x_i), \qquad i = 1, ..., n
\end{aligned}$$

We can now solve for the price of a two-period bond by

$$\begin{aligned}
\hat{q}_{2i} &= \beta \sum_{k=1}^{n} x_k^{-\gamma} \left[ \beta \sum_{l=1}^{n} x_l^{-\gamma} \pi(x_l, x_k) \right] \pi(x_k, x_i), \qquad i = 1, ..., n \\
&= \beta^2 \sum_{k=1}^{n} x_k^{-\gamma} \sum_{l=1}^{n} x_l^{-\gamma} \pi(x_l, x_k) \pi(x_k, x_i) \\
&= \beta^2 \sum_{k=1}^{n} \sum_{l=1}^{n} (x_k x_l)^{-\gamma} \pi(x_l, x_k) \pi(x_k, x_i)
\end{aligned}$$

Just as with a one period bond, the price depends on time discount factor (squared because consumption is delivered in two period's time) and on the fluctuations in consumption growth between now and when the bond pays off. The risk neutral case is $q_1(x, y) = \beta$ and $q_2(x, y) = \beta^2 < q_1(x, y)$. If consumption growth is positive on average (average $x'$ is bigger than one), then the price of a bond tends to be **lower** than in the risk neutral case. Similarly, if consumption growth is negative on average (average $x'$ is less than one), then the price of a bond tends to be **higher** than in the risk neutral case. The sensitivity of the differential depends on $\gamma$, the coefficient of relative risk aversion.

(e) (30 marks): Define bond **returns** by the formula $R_j(x, y) \equiv [1/q_j(x, y)]^{1/j}$. Provide solutions for bond returns. For given state $(x, y)$, explain whether $R_1(x, y) \geq R_2(x, y)$ or not. Give as much economic intuition as possible. Again, it might be useful to consider the risk neutral case as a benchmark and then explain how (if at all) your answer changes when the consumer is risk averse.

**Solution:** Since the bond prices don't depend on the level $y$, neither do the returns. So I will introduce the notation

$$\hat{R}_{1i} = \frac{1}{\hat{q}_{1i}}$$

$$\hat{R}_{2i} = \left(\frac{1}{\hat{q}_{2i}}\right)^{1/2}$$

The net returns on a one period bond are therefore

$$\begin{aligned}\log \hat{R}_{1i} &= -\log \hat{q}_{1i} \\ &= -\log \beta - \log \left[\sum_{k=1}^{n} x_k^{-\gamma} \pi(x_k, x_i)\right]\end{aligned}$$

The term $-\log \beta > 0$ is the rate of time preference. Although the log of the sum is not the sum of the logs, it is still the case that net returns depend on the average and variance of the growth rate of consumption. [In fact, the difference between the log of the sum and the sum of the logs is a measure of the variance of the $x_k$]. Net returns are higher when the average growth rate of consumption is higher and are higher when the variance of consumption growth is higher. The The sensitivities are proportional to $\gamma$. Similarly,

$$\log \hat{R}_{2i} = -\frac{1}{2} \log \hat{q}_{2i} = -\log \beta - \log \left[\sum_{k=1}^{n} \sum_{l=1}^{n} (x_k x_l)^{-\gamma} \pi(x_l, x_k) \pi(x_k, x_i)\right]$$

Therefore, the two-period return is also the sum of the rate of time preference and a correction term that depends on the average and variance of the growth rate of consumption. To see which return is higher, compare

$$\log \hat{R}_{1i} - \log \hat{R}_{2i} = \log \left[\sum_{k=1}^{n} \sum_{l=1}^{n} (x_k x_l)^{-\gamma} \pi(x_l, x_k) \pi(x_k, x_i)\right] - \log \left[\sum_{k=1}^{n} x_k^{-\gamma} \pi(x_k, x_i)\right]$$

Without specifying any further information about the Markov states and transition probabilities, you cannot definitively say whether the two period or one period return is higher. If I gave you more information, you could. Obviously, in the risk neutral case $\gamma = 0$ and $\hat{R}_{1i} = \hat{R}_{2i} = \beta$ all $i$.

(f) (25 marks): The **forward price** $f$ of a 2-period bond (i.e., the price of a 2-period bond that can be locked in safely one period in advance) is given by

$$f(x,y) \equiv \frac{q_2(x,y)}{q_1(x,y)}$$

The **holding period return** $h$ on a 2-period bond that is bought at $q_2(x,y)$ and held for one period and then sold at $q_1(x',y')$ is

$$h(x',y',x,y) \equiv \frac{q_1(x',y')}{q_2(x,y)}$$

Using your answers from part (d), provide solutions for the forward price and holding period return. Go as far as you can in explaining the stochastic pattern you would expect to see in forward prices and holding period returns. Again, explain how your answer depends on the degree of risk aversion.

**Solution:** Since the bond prices don't depend on the level $y$, neither does the forward price or the holding period return. This justifies the notations

$$\hat{f}_i \equiv \frac{\hat{q}_{2i}}{\hat{q}_{1i}}, \qquad \hat{h}_{ik} \equiv \frac{\hat{q}_{1k}}{\hat{q}_{2i}}$$

Again, we can write

$$\begin{aligned}
\log \hat{f}_i &= \log \hat{q}_{2i} - \log \hat{q}_{1i} \\
&= -2\log\beta - \log\left[\sum_{k=1}^{n}\sum_{l=1}^{n}(x_k x_l)^{-\gamma}\pi(x_l,x_k)\pi(x_k,x_i)\right] + \log\beta + \log\left[\sum_{k=1}^{n} x_k^{-\gamma}\pi(x_k,x_i)\right] \\
&= -\log\beta - \log\left[\sum_{k=1}^{n}\sum_{l=1}^{n}(x_k x_l)^{-\gamma}\pi(x_l,x_k)\pi(x_k,x_i)\right] + \log\left[\sum_{k=1}^{n} x_k^{-\gamma}\pi(x_k,x_i)\right] \\
&= -\log\beta + \log\hat{R}_{2i} - \log\hat{R}_{1i}
\end{aligned}$$

The forward price has two components, the time discount rate $-\log\beta > 0$ plus the term premium $\log\hat{R}_{2i} - \log\hat{R}_{1i}$. Again, this premium can be positive or negative and is zero under risk neutrality. Put differently, if there is a positive term premium, the log forward price is higher than the price a risk neutral agent would pay. Similarly, the holding period return is

$$\begin{aligned}
\log \hat{h}_{ik} &= \log \hat{q}_{1k} - \log \hat{q}_{2i} \\
&= \log\beta + \log\left[\sum_{l=1}^{n} x_l^{-\gamma}\pi(x_l,x_k)\right] - 2\log\beta - \log\left[\sum_{k=1}^{n}\sum_{l=1}^{n}(x_k x_l)^{-\gamma}\pi(x_l,x_k)\pi(x_k,x_i)\right]
\end{aligned}$$

$$= \quad -\log\beta + \log\left[\sum_{l=1}^{n} x_l^{-\gamma}\pi(x_l, x_k)\right] - \log\left[\sum_{k=1}^{n}\sum_{l=1}^{n}(x_k x_l)^{-\gamma}\pi(x_l, x_k)\pi(x_k, x_i)\right]$$

[You could give some further discussion of the properties of the forward price and the holding period returns if you took some approximations, but I was not expecting you to do this].

**Question 2.** *Solving the stochastic growth model* (60 marks). Consider a planner with the problem of maximizing

$$\mathsf{E}_0\left\{\sum_{t=0}^{\infty}\beta^t U(c_t)\right\}, \qquad 0 < \beta < 1$$

subject to a resource constraint

$$c_t + k_{t+1} = z_t f(k_t) + (1-\delta)k_t, \qquad 0 < \delta < 1$$

and the non-negativity constraints

$$c_t \geq 0, \quad k_t \geq 0$$

where $c_t$ denotes consumption, $k_{t+1}$ denotes capital carried into the next period, $\delta$ denotes a constant depreciation rate, and $z_t$ is the level of technology, which follows a Markov chain on a discrete set $\mathcal{Z}$ with transitions given by

$$\pi(z', z) = \Pr(z_{t+1} = z' | z_t = z)$$

(a) (10 marks): Let $V(k, z)$ denote the value function. Set up a Bellman equation for this dynamic programming problem.

**Solution:** The Bellman equation for this problem can be written

$$V(k, z) = \max_{k' \geq 0}\left\{U[zf(k) + (1-\delta)k - k'] + \beta\sum_{z'} V(k', z')\pi(z', z)\right\}$$

(b) (10 marks): Use first order and envelope conditions to characterize the solution to this problem.

**Solution:** The first order condition is

$$U'(c) = \beta\sum_{z'}\frac{\partial V(k', z')}{\partial k'}\pi(z', z)$$

while the envelope condition is

$$\frac{\partial V(k,z)}{\partial k} = U'(c)[zf(k) + 1 - \delta]$$

Combining these gives

$$U'(c) = \beta \sum_{z'} U'(c')[z'f'(k') + 1 - \delta]\pi(z', z)$$

In more familiar time series notation, this is just the usual stochastic Euler equation

$$U'(c_t) = \beta \mathsf{E}_t \left\{ U'(c_t)[z_{t+1}f'(k_{t+1}) + 1 - \delta] \right\}$$

(of course in these expressions, consumption also has to satisfy the resource constraint).

(c) (10 marks): Give an algorithm that explains how you would find approximate solutions by value function iteration on a discrete state space. In your answer, let $\mathcal{K} \times \mathcal{Z}$ denote the discretized state space.

**Solution:** The Markov chain for technology shocks already takes values on a discrete space $\mathcal{Z}$. We could also discretize the domain of capital choices to a grid like

$$k' \in \mathcal{K} = [0 < \cdots < k_{\max}]$$

for some appropriately large (i.e., non-binding) choice of $k_{\max}$. For each possible $z \in \mathcal{Z}$, we can construct a return matrix, a square matrix with as many rows as there are points in $\mathcal{K}$ with typical elements

$$R_z(k, k') = U[zf(k) + (1 - \delta)k - k']$$

We then guess a value function (i.e., a matrix $V_0(k, z)$ with dimensions given by the size of $\mathcal{K} \times \mathcal{Z}$) and compute the solution to the maximization on the right hand side of the Bellman equation. We call the associated value $TV_0(k, z)$, namely

$$TV_0(k, z) = \max_{k' \geq 0} \left\{ R_z(k, k') + \beta \sum_{z'} V_0(k', z')\pi(z', z) \right\}$$

If this is the same as our initial guess, we're done. If not, we update our guess and compute, say after $j$ rounds,

$$TV_j(k,z) = \max_{k' \geq 0} \left\{ R_z(k,k') + \beta \sum_{z'} V_j(k',z')\pi(z',z) \right\}$$

And we keep iterating until

$$\max\{|TV_j(k,z) - V_j(k,z)|\} < \mathsf{tol}$$

for some small tolerance criterion.

(d) (20 marks): Suppose that $k' = g(k,z)$ denotes the policy function that you obtain from solving your dynamic programming problem. Let $\mu_t(k,z)$ denote the unconditional distribution of $(k,z)$ pairs on $\mathcal{K} \times \mathcal{Z}$. That is,

$$\mu_t(k,z) = \Pr(k_t = k, z_t = z)$$

Explain how you can use the policy function $g(k,z)$ and the transitions $\pi(z',z)$ to create a law of motion that maps $\mu_t(k,z)$ to $\mu_{t+1}(k',z')$. Give an algorithm that explains how you could solve for a stationary distribution [i.e., a time-invariant $\mu(k,z)$].

**Solution:** The unconditional distribution $\mu_t(k,z)$ has law of motion given by

$$\begin{aligned} & \Pr(k_{t+1} = k', z_{t+1} = z') \\ = \; & \sum_{k_t}\sum_{z_t} \Pr(k_{t+1} = k' | k_t = k, z_t = z) \Pr(z_{t+1} = z' | z_t = z) \Pr(k_t = k, z_t = z) \end{aligned}$$

or

$$\mu_{t+1}(k',z') = \sum_{k_t}\sum_{z_t} \Pr(k_{t+1} = k' | k_t = k, z_t = z)\pi(z',z)\mu_t(k,z)$$

But the probability $\Pr(k_{t+1} = k' | k_t = k, z_t = z)$ is either 1 if $k' = g(k,z)$ or 0 otherwise. So if we write an indicator function

$$I_g(k',k,z) = \begin{cases} 1, & \text{if } k' = g(k,z) \text{ and} \\ 0, & \text{otherwise} \end{cases}$$

we can write the law of motion as

$$\mu_{t+1}(k', z') = \sum_k \sum_z I_g(k', k, z)\pi(z', z)\mu_t(k, z)$$

(I use the subscript $g$ to emphasize the dependence on the policy function). A stationary distribution is a time-invariant $\mu(k, z)$ that is a fixed-point of this law of motion, i.e., a distribution that satisfies

$$\mu(k', z') = \sum_k \sum_z I_g(k', k, z)\pi(z', z)\mu(k, z)$$

Let $x = (k, z)$. Then the law of motion for $\mu_{t+1}(k', z')$ implicitly defines a Markov chain on the state $x$. The matrix of transition probabilities has typical element given by

$$\begin{aligned} P(x', x) &= \Pr(k_{t+1} = k', z_{t+1} = z' | k_t = k, z_t = z) \\ &= \Pr(k_{t+1} = k' | k_t = k, z_t = z) \Pr(z_{t+1} = z' | z_t = z) \\ &= I_g(k', k, z)\pi(z', z) \end{aligned}$$

One can then find the stationary distribution for $x$ by solving for the eigenvector associated with a unit eigenvalue of the transition matrix $P$.

(e) (10 marks): Suppose that $\mathcal{Z} = \{z_L, z_H\}$ with $z_L < z_H$. If the utility and production functions have the usual properties (strictly increasing, strictly concave, etc) sketch the policy functions $k' = g(k, z_L)$ and $k' = g(k, z_H)$ on a "45-degree" phase diagram. Explain how you could determine which subset $[\underline{k}, \overline{k}]$ of $\mathcal{K}$ has positive probability in the stationary distribution.

**Solution:** Under the usual regularity conditions, the policy functions $g_L(k) \equiv g(k, z_L)$ and $g_H(k) \equiv g(k, z_H)$ are both continuous, strictly increasing and strictly concave with

$$g_L(k) < g_H(k) \qquad \text{for all } k > 0$$

Each has a single crossing point with the 45-degree line. These are the solutions $\underline{k}, \overline{k}$ to the independent fixed point problems

$$\begin{aligned} \overline{k} &= g_H(\overline{k}) \\ \underline{k} &= g_L(\underline{k}) \end{aligned}$$

The only capital stocks that have positive probability in the stationary distribution are those that lie in the closed interval $[\underline{k}, \overline{k}]$. Because each of the fixed points $\underline{k}, \overline{k}$ is (locally) stable and the policy functions are monotone increasing, once it becomes optimal to choose a capital stock that lies inside $[\underline{k}, \overline{k}]$, it will never be optimal to choose a subsequent capital stock that lies outside $[\underline{k}, \overline{k}]$. To convince yourself of this, you may find it helpful to draw a sketch of the deterministic dynamics associated with two difference equations $k_{t+1} = g_H(k_t)$ and $k_{t+1} = g_L(k_t)$.

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