316-406/671 ADVANCED MACRO TECHNIQUES **Final Solutions**

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This exam lasts **180 minutes** and has two questions. The first question is worth 120 marks, while the second question is worth only 60 marks. Allocate your time accordingly. Within each question there are a number of parts and the weight given to each part is also indicated. Even if you cannot complete one part of a question, you should be able to move on an answer other parts, so do not spend too much time. If you feel like you are getting stuck, move on to the next part. You also have **15 minutes perusal** before you can start writing answers.

Question 1. *Term Structure of Interest Rates* (120 marks): Consider a representative agent asset pricing model where preferences are

$$\mathsf{E}_0\left\{\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\gamma}}{1-\gamma}\right\},\qquad 0<\beta<1 \text{ and } \gamma>0$$

There are two kinds of assets. First, there is a "Lucas tree" with dividends $\{x_t\}$ that follow an autoregression of the form

$$x_{t+1}=\bar{x}^{1-\phi}x_t^{\phi}\varepsilon_{t+1},\qquad 0<\phi<1 \text{ and } \bar{x}>0 \tag{1}$$

where $\log(\varepsilon_{t+1})$ are IID normal with mean 1 and variance σ^2. Second, there are also bonds **of various maturities**. A j-period bond ($j\geq 1$) is a riskless claim to one unit of consumption to be delivered in j-period's time.

(a) (15 marks): Suppose we let bonds up-to maturity $J=2$ be traded. That is, one-period ($j=1$) and two-period ($j=2$) bonds are traded. Let $q_j(x)$ denote the price of a j-bond if the current aggregate state is x and let $p(x)$ denote the price of a claim to the Lucas tree. Let $V(w,x)$ denote the consumer's value function if their individual wealth is w and the aggregate state is x. Write down a Bellman equation for the consumer's problem. Be careful to explain the Bellman equation and any constraints that you provide.

Solution: The Bellman equation for this problem is

$$V(w,x)=\max_{s',B_1',B_2'}\{U(c)+\beta\mathsf{E}[V(w',x')|x]\}$$

where the maximization is subject to the budget constraint

$$c + p(x)s' + q_1(x)B_1' + q_2(x)B_2' \leq w$$

and where next period's wealth is

$$w' = [p(x') + x']s' + B_1' + q_1(x')B_2'$$

Notice that a two-period bond bought today at $q_2(x)$ can be sold next period as a one-period bond with price $q_1(x')$. All kinds of bonds are perfect substitutes at their maturity in the sense that a one-period bond bought yesterday and a two-period bond bought two days ago both deliver one unit of consumption today.

(b) (15 marks): Define a **recursive competitive equilibrium** for this economy.

Solution: A recursive competitive equilibrium consists of a value function $V(w,x)$, individual decision rules $g_s(w,x)$, $g_{B1}(w,x)$ and $g_{B2}(w,x)$ and pricing functions $p(x), q_1(x)$ and $q_2(x)$ such that given the prices the value function and individual decision rules solve the consumer's problem and markets clear, namely

$$\begin{aligned} s' &= g_s(w,x) = 1 \\ B_1' &= g_{B1}(w,x) = 0 \\ B_2' &= g_{B2}(w,x) = 0 \end{aligned}$$

These market clearing conditions imply

$$c = x$$

(c) (15 marks): Suppose we let bonds with a maturity up to an arbitrary J be traded. Explain why **in equilibrium** the price of a j-bond satisfies the relationship

$$q_j(x_t) = \mathsf{E}_t \left\{ \beta^j \left(\frac{x_{t+j}}{x_t} \right)^{-\gamma} \right\}$$

where $\mathsf{E}_t\{\}$ denotes expectations conditional on the current aggregate state x_t. [*Hint*: begin by using first order and envelope conditions to derive bond prices for the $j = 1$ and $j = 2$

cases, then explain why this generalizes to arbitrary j].

Solution: Consider the $J = 2$ case. The first order and envelope conditions are

$$\begin{aligned} U'(c)q_1(x) &= \beta \mathsf{E}\left\{ \frac{\partial V(w',x')}{\partial w'} \middle| x \right\} \\ U'(c)q_2(x) &= \beta \mathsf{E}\left\{ \frac{\partial V(w',x')}{\partial w'} q_1(x') \middle| x \right\} \end{aligned}$$

and

$$\frac{\partial V(w,x)}{\partial w} = U'(c)$$

In equilibrium, $c = x$ and so we have

$$\begin{aligned} q_1(x) &= \mathsf{E}\left\{ \beta \frac{U'(x')}{U'(x)} \middle| x \right\} \\ q_2(x) &= \mathsf{E}\left\{ \beta \frac{U'(x')}{U'(x)} q_1(x') \middle| x \right\} \end{aligned}$$

But we can substitute the former into the latter to give

$$\begin{aligned} q_2(x) &= \mathsf{E}\left\{ \beta \frac{U'(x')}{U'(x)} \mathsf{E}\left\{ \beta \frac{U'(x'')}{U'(x')} \middle| x' \right\} \middle| x \right\} \\ &= \mathsf{E}\left\{ \beta^2 \frac{U'(x')}{U'(x)} \frac{U'(x'')}{U'(x')} \middle| x \right\} \\ &= \mathsf{E}\left\{ \beta^2 \frac{U'(x'')}{U'(x)} \middle| x \right\} \end{aligned}$$

(the second equality follows using the law of iterated expectations). Rewriting these expressions using the time-series notation gives

$$\begin{aligned} q_1(x_t) &= \mathsf{E}_t \left\{ \beta \frac{U'(x_{t+1})}{U'(x_t)} \right\} = \mathsf{E}_t \left\{ \beta \left(\frac{x_{t+1}}{x_t} \right)^{-\gamma} \right\} \\ q_2(x_t) &= \mathsf{E}_t \left\{ \beta \frac{U'(x_{t+2})}{U'(x_t)} \right\} = \mathsf{E}_t \left\{ \beta^2 \left(\frac{x_{t+2}}{x_t} \right)^{-\gamma} \right\} \end{aligned}$$

A similar argument establishes the general result for $j \geq 1$.

(d) (15 marks): By iterating on equation (1), show that log-dividends in j periods time satisfy

$$\log\left(\frac{x_{t+j}}{\bar{x}}\right) = \phi^j \log\left(\frac{x_t}{\bar{x}}\right) + \sum_{k=0}^{j-1} \phi^k \log(\varepsilon_{t+j-k})$$

Use this to derive an expression for dividend growth between t and $t+j$, namely

$$\log\left(\frac{x_{t+j}}{x_t}\right)$$

Solution: Taking logs of the law of motion for dividends, we have

$$\log(x_{t+1}) = (1-\phi)\log(\bar{x}) + \phi\log(x_t) + \log(\varepsilon_{t+1})$$

Subtracting $\log(\bar{x})$ we have

$$\log\left(\frac{x_{t+1}}{\bar{x}}\right) = \phi\log\left(\frac{x_t}{\bar{x}}\right) + \log(\varepsilon_{t+1})$$

Iterating once gives

$$\begin{aligned}\log\left(\frac{x_{t+2}}{\bar{x}}\right) &= \phi\log\left(\frac{x_{t+1}}{\bar{x}}\right) + \log(\varepsilon_{t+2}) \\ &= \phi^2\log\left(\frac{x_t}{\bar{x}}\right) + \phi\log(\varepsilon_{t+1}) + \log(\varepsilon_{t+2})\end{aligned}$$

And for any $j \geq 1$ we have

$$\begin{aligned}\log\left(\frac{x_{t+j}}{\bar{x}}\right) &= \phi\log\left(\frac{x_{t+j-1}}{\bar{x}}\right) + \log(\varepsilon_{t+j}) \\ &= \phi^j\log\left(\frac{x_t}{\bar{x}}\right) + \sum_{k=0}^{j-1}\phi^k\log(\varepsilon_{t+j-k})\end{aligned}$$

Subtracting $\log(x_t)$ from both sides and rearranging gives

$$\log\left(\frac{x_{t+j}}{x_t}\right) = (\phi^j - 1)\log\left(\frac{x_t}{\bar{x}}\right) + \sum_{k=0}^{j-1}\phi^k\log(\varepsilon_{t+j-k})$$

(e) (30 marks): Now use your solutions from part (c) to solve for bond prices $q_j(x_t)$. You should make use of the following three pieces of information: First, you can write

$$q_j(x_t) = \mathsf{E}_t\left\{\beta^j\left(\frac{x_{t+j}}{x_t}\right)^{-\gamma}\right\} = \beta^j\mathsf{E}_t\left\{\exp\left[-\gamma\log\left(\frac{x_{t+j}}{x_t}\right)\right]\right\}$$

Second, if $\log(\varepsilon_{t+j-k})$ is normal so is the linear combination $\sum_{k=0}^{j-1}\phi^k\log(\varepsilon_{t+j-k})$. Third, if some random variable X is normal with mean μ_X and variance σ_X^2 then $\mathsf{E}\{\exp(X)\} =$

$\exp(\mu_X + \frac{1}{2}\sigma_X^2)$.

Solution: Since $\log(\varepsilon_{t+j-k})$ is normal, so is $\sum_{k=0}^{j-1}\phi^k \log(\varepsilon_{t+j-k})$ and so is $\log\left(\frac{x_{t+j}}{x_t}\right)$. In particular, dividend growth over j periods is normal with mean

$$\begin{aligned}
\mathsf{E}_t\left\{\log\left(\frac{x_{t+j}}{x_t}\right)\right\} &= (\phi^j - 1)\log\left(\frac{x_t}{\bar{x}}\right) + \mathsf{E}_t\left\{\sum_{k=0}^{j-1}\phi^k \log(\varepsilon_{t+j-k})\right\} \\
&= (\phi^j - 1)\log\left(\frac{x_t}{\bar{x}}\right) + \sum_{k=0}^{j-1}\phi^k \\
&= (\phi^j - 1)\log\left(\frac{x_t}{\bar{x}}\right) + \frac{1-\phi^j}{1-\phi}
\end{aligned}$$

and variance

$$\mathsf{Var}_t\left\{\log\left(\frac{x_{t+j}}{\bar{x}}\right)\right\} = \mathsf{Var}_t\left\{\sum_{k=0}^{j-1}\phi^k \log(\varepsilon_{t+j-k})\right\} = \sum_{k=0}^{j-1}\phi^{2k}\sigma_\varepsilon^2 = \frac{1-\phi^{2j}}{1-\phi}\sigma_\varepsilon^2$$

Hence we can write

$$q_j(x_t) = \beta^j \mathsf{E}_t\left\{\exp(X_{t+j})\right\}$$

where X_{t+j} is normal with mean

$$\mu_{X_{t+j}} = -\gamma\left\{(\phi^j - 1)\log\left(\frac{x_t}{\bar{x}}\right) + \frac{1-\phi^j}{1-\phi}\right\}$$

and variance

$$\sigma_{X_{t+j}}^2 = \gamma^2\frac{1-\phi^{2j}}{1-\phi}\sigma_\varepsilon^2$$

Bond prices are therefore given by

$$\begin{aligned}
q_j(x_t) &= \beta^j \exp\left(\mu_{X_{t+j}} + \frac{1}{2}\sigma_{X_{t+j}}^2\right) \\
&= \beta^j \exp\left(-\gamma\left\{(\phi^j - 1)\log\left(\frac{x_t}{\bar{x}}\right) + \frac{1-\phi^j}{1-\phi}\right\} + \frac{1}{2}\gamma^2\frac{1-\phi^{2j}}{1-\phi}\sigma_\varepsilon^2\right)
\end{aligned}$$

(f) (30 marks): Define j-period bond returns by the formula $R_j(x_t) \equiv [1/q_j(x_t)]^{1/j}$. Now use your solution from part (e) to explain why j-period returns can be written

$$\log[R_j(x_t)] = a_j + b_j \log\left(\frac{x_t}{\bar{x}}\right)$$

for some coefficients a_j and b_j. Provide an explicit solution for b_j in terms of the parameters of the model. Explain whether b_j and $|b_j|$ are increasing or decreasing in j. Is the variance of short term interest rates higher or lower than long term interest rates? Are short term interest rates more or less sensitive to the state of the economy (i.e., to x_t) than are long term interest rates? What happens to the slope of the term structure as $\phi \to 1$ (as dividends become very persistent)? Give economic intuition wherever you can.

Solution: Since $R_j(x_t) \equiv [1/q_j(x_t)]^{1/j}$ we have

$$\begin{aligned}\log[R_j(x_t)] &= -\frac{1}{j}\log[q_j(x_t)] \\ &= -\frac{1}{j}\log\left[\beta^j \exp\left(-\gamma\left\{(\phi^j-1)\log\left(\frac{x_t}{\bar{x}}\right)+\frac{1-\phi^j}{1-\phi}\right\}+\frac{1}{2}\gamma^2\frac{1-\phi^{2j}}{1-\phi}\sigma_\varepsilon^2\right)\right]\end{aligned}$$

and on simplifying

$$\log[R_j(x_t)] = -\log(\beta) - \frac{1}{j}\left[-\gamma\left\{(\phi^j-1)\log\left(\frac{x_t}{\bar{x}}\right)+\frac{1-\phi^j}{1-\phi}\right\}+\frac{1}{2}\gamma^2\frac{1-\phi^{2j}}{1-\phi}\sigma_\varepsilon^2\right]$$

which is of the affine form

$$\log[R_j(x_t)] = a_j + b_j \log\left(\frac{x_t}{\bar{x}}\right)$$

with intercepts

$$a_j \equiv -\log(\beta) + \frac{\gamma}{j}\frac{1-\phi^j}{1-\phi} - \frac{1}{2j}\frac{1-\phi^{2j}}{1-\phi}(\gamma\sigma_\varepsilon)^2$$

and slope coefficients

$$b_j \equiv \gamma\frac{\phi^j-1}{j}$$

Notice that $b_j < 0$ for all j and b_j is decreasing in j for all j. Therefore, the volatility of interest rates decreases with maturity and the sensitivity of interest rates to current conditions also decreases with maturity. As $\phi \to 1$, the term structure becomes flat with $b_j \to 0$ for all j. As dividends become very persistent, current conditions are no longer good forecasts of future dividend growth and so are no longer good forecasts of future interest rates. If so, current interest rates will not depend on the current state of the economy.

Question 2. *Incomplete Markets* (60 marks). Consider a household with the problem of maximizing

$$\mathsf{E}_0\left\{\sum_{t=0}^{\infty}\beta^t U(c_t)\right\}, \qquad 0<\beta<1$$

subject to a flow budget constraint

$$c_t + a_{t+1} = (1+r)a_t + w_t$$

where a_{t+1} denotes assets carried into the next period, r denotes a constant interest rate, and w_t is the real wage rate, which follows a Markov chain on a discrete set $\mathcal{W}$ with transition matrix P and typical elements

$$P(w', w) = \Pr(w_{t+1} = w' | w_t = w)$$

Households also face the borrowing constraint

$$a_{t+1} \geq -\phi$$

(a) (10 marks): Let $V(a, w)$ denote the value function of a household with current assets a facing real wage w. Set up a Bellman equation for the household's problem.

Solution: The Bellman equation for the household's problem can be written

$$V(a,w) = \max_{a' \geq -\phi}\left\{U[(1+r)a + w - a'] + \beta\sum_{w'} V(a', w')P(w', w)\right\}$$

(b) (10 marks): Give an algorithm that explains how you would find approximate solutions by value function iteration on a discrete state space. In your answer, let $\mathcal{A}\times\mathcal{W}$ denote the discretized state space.

Solution: The Markov chain for wages already takes values on a discrete space $\mathcal{W}$. We could also discretize the domain of asset choices to a grid like

$$a' \in \mathcal{A} = [-\phi < \cdots < a_{\max}]$$

for some appropriately large choice of $a_{\max}$. The lower limit of the grid enforces the borrowing constraint $a' \geq -\phi$. For each possible $w \in \mathcal{W}$, we can construct a return matrix, a square matrix with as many rows as there are points in $\mathcal{A}$ with typical elements

$$R_w(a, a') = U[(1+r)a + w - a']$$

We then guess a value function (i.e., a matrix $V_0(a, w)$ with dimensions given by the size of $\mathcal{A} \times \mathcal{W}$) and compute the solution to the maximization on the right hand side of the Bellman equation. We call the associated value $TV_0(a, w)$, namely

$$TV_0(a, w) = \max_{a' \geq -\phi} \left\{ R_w(a, a') + \beta \sum_{w'} V_0(a', w') P(w', w) \right\}$$

If this is the same as our initial guess, we're done. If not, we update our guess and compute, say after k rounds,

$$TV_k(a, w) = \max_{a' \geq -\phi} \left\{ R_w(a, a') + \beta \sum_{w'} V_k(a', w') P(w', w) \right\}$$

And we keep iterating until

$$\max\{|TV_k(a, w) - V_k(a, w)|\} < \mathsf{tol}$$

for some small tolerance criterion.

(c) (20 marks): Suppose that $a' = g(a, w)$ denotes the policy function that you obtain from solving your dynamic programming problem. Let $\mu_t(a, w)$ denote the unconditional distribution of (a, w) pairs on $\mathcal{A} \times \mathcal{W}$. That is,

$$\mu_t(a, w) = \Pr(a_t = a, w_t = w)$$

Explain how you can use the policy function $g(a, w)$ and the transition matrix $P(w', w)$ to create a law of motion that maps $\mu_t(a, w)$ to $\mu_{t+1}(a', w')$. Give an algorithm that explains how you could solve for a stationary distribution [i.e., a time-invariant $\mu(a, w)$]. Explain why this distribution has **both** a time-series and a cross-sectional interpretation.

Solution: The unconditional distribution $\mu_t(a, w)$ has law of motion given by

$$\begin{aligned} & \Pr(a_{t+1} = a', w_{t+1} = w') \\ = & \sum_{a_t} \sum_{w_t} \Pr(a_{t+1} = a' | a_t = a, w_t = w) \Pr(w_{t+1} = w' | w_t = w) \Pr(a_t = a, w_t = w) \end{aligned}$$

or

$$\mu_{t+1}(a', w') = \sum_{a_t} \sum_{w_t} \Pr(a_{t+1} = a' | a_t = a, w_t = w) P(w', w) \mu_t(a, w)$$

But the probability $\Pr(a_{t+1} = a' | a_t = a, w_t = w)$ is either 1 if $a' = g(a, w)$ or 0 otherwise. So if we write an indicator function

$$I_g(a', a, w) = \begin{cases} 1, & \text{if } a' = g(a, w) \text{ and} \\ 0, & \text{otherwise} \end{cases}$$

we can write the law of motion as

$$\mu_{t+1}(a', w') = \sum_{a} \sum_{w} I_g(a', a, w) P(w', w) \mu_t(a, w)$$

(I use the subscript g to emphasize the dependence on the policy function). A stationary distribution is a time-invariant $\mu(a, w)$ that is a fixed-point of this law of motion, i.e., a distribution that satisfies

$$\mu(a', w') = \sum_{a} \sum_{w} I_g(a', a, w) P(w', w) \mu(a, w)$$

Let $x = (a, w)$. Then the law of motion for $\mu_{t+1}(a', w')$ implicitly defines a Markov chain on the state x. The transition probabilities are given by

$$\begin{aligned} P_X(x', x) &= \Pr(a_{t+1} = a', w_{t+1} = w' | a_t = a, w_t = w) \\ &= \Pr(a_{t+1} = a' | a_t = a, w_t = w) \Pr(w_{t+1} = w' | w_t = w) \\ &= I_g(a', a, w) P(w', w) \end{aligned}$$

One can then find the stationary distribution for x by solving for the eigenvector associated with a unit eigenvalue of the transition matrix P_X.

(d) (10 marks): Define a **stationary competitive equilibrium** for this model. Your definition should involve the policy function $g(a, w)$, the stationary distribution $\mu(a, w)$, and the real interest rate r. What is the market clearing condition that characterizes r?

Solution: A stationary competitive equilibrium is (i) a value function V and policy function g, (ii) a stationary distribution μ, and (iii) a real interest rate r (a number) such that: given the real interest rate, the value function and policy function solve the household's dynamic programming problem; the stationary distribution is induced via the policy function and the exogenous Markov chain for wages; and the asset market clears

$$\sum_a \sum_w g(a, w)\mu(a, w) = 0$$

(e) (10 marks): Give an algorithm that explains how you could compute a stationary competitive equilibrium as you've defined it.

Solution: One can begin by fixing a guess for the real interest rate, say r_0, then solve the dynamic programming problem for the value and policy functions V_0 and g_0, then use these to compute the stationary distribution of assets μ_0. Then compute the aggregate asset position

$$\sum_a \sum_w g_0(a, w)\mu_0(a, w) = A_0$$

If $A_0 = 0$ stop. Otherwise, either increase the real interest rate to $r_1 > r_0$ if $A_0 < 0$, or lower the real interest rate to $r_1 < r_0$ if $A_0 > 0$. In either case, one has a new real interest rate and can recompute value functions and policy functions to get V_1 and g_1, then use these to compute the stationary distribution of assets μ_1 and then compute the aggregate asset position A_1. The asset market clearing condition is again checked and the real interest rate adjusted again as needed. The whole procedure continues until an $|A_k| < \texttt{tol}$ is found.