316-406 Advanced Macroeconomic Techniques

Problem Set #1 *Selected answers*

Question 1. If the resource constraint and Euler equation are

$$\begin{aligned} c_t + k_{t+1} &= f(k_t) + (1-\delta)k_t \\ U'(c_t) &= \beta U'(c_{t+1})[1 + f'(k_{t+1}) - \delta] \end{aligned}$$

then the steady state $(\bar{c}, \bar{k})$ satisfies

$$\begin{aligned} \bar{c} + \bar{k} &= f(\bar{k}) + (1-\delta)\bar{k} \\ 1 &= \beta[1 + f'(\bar{k}) - \delta] \end{aligned}$$

and the log-linear approximations around the steady state are

$$\bar{c}\hat{c}_t + \bar{k}\hat{k}_{t+1} = [f'(\bar{k}) + 1 - \delta]\bar{k}\hat{k}_t = \frac{1}{\beta}\bar{k}\hat{k}_t$$

and

$$\frac{U''(\bar{c})}{U'(\bar{c})}\bar{c}\hat{c}_t = \frac{U''(\bar{c})}{U'(\bar{c})}\bar{c}\hat{c}_{t+1} + \beta f''(\bar{k})\bar{k}\hat{k}_{t+1}$$

Now with the assumed utility function, we have

$$\begin{aligned} U'(c) &= c^{-\sigma} > 0 \\ U''(c) &= -\sigma c^{-\sigma-1} < 0 \end{aligned}$$

so that the coefficient of relative risk aversion is

$$\frac{U''(c)c}{U'(c)} = \frac{-\sigma c^{-\sigma-1}c}{c^{-\sigma}} = -\sigma < 0$$

Similarly, with the assumed production function

$$\begin{aligned} f'(k) &= \alpha z k^{\alpha-1} = \alpha\frac{f(k)}{k} > 0 \\ f''(k) &= (\alpha-1)\alpha z k^{\alpha-2} = -(1-\alpha)\frac{f'(k)}{k} < 0 \end{aligned}$$

At steady state, but not generally, this last relationship implies

$$f''(\bar{k})\bar{k} = -(1-\alpha)f'(\bar{k}) = -(1-\alpha)\frac{1-\beta+\delta\beta}{\beta}$$

Using these functional forms, we can simplify our log-linear model to

$$\begin{aligned} 0 &= \hat{k}_{t+1} - \frac{1}{\beta}\hat{k}_t + \frac{\bar{c}}{\bar{k}}\hat{c}_t \\ 0 &= \frac{(1-\alpha)(1-\beta+\delta\beta)}{\sigma}\hat{k}_{t+1} + \hat{c}_{t+1} - \hat{c}_t \end{aligned}$$

So our log-linear model can be written in the "standard form"

$$\begin{aligned} 0 &= A\hat{k}_{t+1} + B\hat{k}_t + C\hat{c}_t \\ 0 &= F\hat{k}_{t+1} + G\hat{k}_t + J\hat{c}_{t+1} + K\hat{c}_t \end{aligned}$$

with known coefficients

$$\begin{aligned} A &= 1, \\ B &= -\frac{1}{\beta} \\ C &= \frac{\bar{c}}{\bar{k}} \\ F &= \frac{(1-\alpha)(1-\beta+\delta\beta)}{\sigma} \\ G &= 0 \\ J &= 1 \\ K &= -1 \end{aligned}$$

This is all that you needed to do to answer Question 1, but for future reference, note that

$$\frac{\bar{c}}{\bar{k}} = \frac{f(\bar{k}) - \delta\bar{k}}{\bar{k}} = \frac{f(\bar{k})}{\bar{k}} - \delta$$

Now with our assumed functional form, the steady state capital stock is

$$\bar{k} = \left(\frac{\alpha z\beta}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}}$$

so steady state output is

$$f(\bar{k}) = z\left(\frac{\alpha z\beta}{1-\beta(1-\delta)}\right)^{\frac{\alpha}{1-\alpha}}$$

and the output/capital ratio is in turn

$$\frac{f(\bar{k})}{\bar{k}} = z\left(\frac{\alpha z\beta}{1-\beta(1-\delta)}\right)^{\frac{\alpha-1}{1-\alpha}} = z\left(\frac{\alpha z\beta}{1-\beta(1-\delta)}\right)^{-1} = \frac{1-\beta(1-\delta)}{\alpha\beta}$$

Hence

$$\frac{\bar{c}}{\bar{k}} = \frac{f(\bar{k})}{\bar{k}} - \delta = \frac{1-\beta(1-\delta)}{\alpha\beta} - \delta$$

which does not depend on z. This will be important when we answer Question 4.

Question 2. Our guesses are

$$\begin{aligned} \hat{k}_{t+1} &= P\hat{k}_t \\ \hat{c}_t &= R\hat{k}_t \end{aligned}$$

Plug these into the system of log-linear equations to get

$$\begin{aligned} 0 &= (AP + B + CR)\hat{k}_t \\ 0 &= (FP + G + JRP + KR)\hat{k}_t \end{aligned}$$

But these must hold for all values of $\hat{k}_t$, which is only possible if

$$
\begin{aligned}
0 &= AP + B + CR \\
0 &= FP + G + JRP + KR
\end{aligned}
$$

Solving the first equation for R in terms of P gives

$$R = -C^{-1}(AP + B)$$

And plugging this into the second equation gives a quadratic equation in P, namely

$$
\begin{aligned}
0 &= FP + G - JC^{-1}(AP + B)P - KC^{-1}(AP + B) \\
&= (-JC^{-1}A)P^2 + (F - JC^{-1}B - KC^{-1}A)P + (G - KC^{-1}B)
\end{aligned}
$$

Using the given parameter values, we find

$$
\begin{aligned}
A &= 1 \\
B &= -1.0309 \\
C &= 0.1749 \\
F &= 0.0230 \\
G &= 0 \\
J &= 1 \\
K &= -1
\end{aligned}
$$

And then solving the quadratic equation, we get the two roots

$$P_1 = 1.0833, \quad P_2 = 0.9516$$

which are real, positive, and on either side of 1 — as promised. Picking the stable root

$$P = 0.9516$$

gives

$$R = -C^{-1}(AP + B) = 0.4534$$

Question 3. (*Recovery from a war*). Approximately, $\hat{k}_0$ has the value -0.10. The time plots of consumption, capital, investment, output, etc are attached. With capital temporarily below its long run level, output and consumption both fall. Investment is positive along the transition back to the steady state. Notice that in log-deviations

$$
\begin{aligned}
\bar{k}\hat{k}_{t+1} &= (1 - \delta)\bar{k}\hat{k}_t + \bar{\imath}\hat{\imath}_t \\
&= (1 - \delta)\bar{k}\hat{k}_t + \delta\bar{k}\hat{\imath}_t
\end{aligned}
$$

so investment is given by

$$\hat{\imath}_t = \frac{1}{\delta}[\hat{k}_{t+1} - (1 - \delta)\hat{k}_t] = \frac{1}{\delta}[P - (1 - \delta)]\hat{k}_t$$

while output is

$$\bar{y}\hat{y}_t = f'(\bar{k})\bar{k}\hat{k}_t = \alpha\frac{f(\bar{k})}{\bar{k}}\bar{k}\hat{k}_t = \alpha\bar{y}\hat{k}_t$$

or

$$\hat{y}_t = \alpha\hat{k}_t$$

It takes about 14 or 15 years until $\hat{k}_t = 0.5\hat{k}_0 = -0.05$.

Question 4. (*Technological breakthrough*). The new steady state values are

$$\begin{aligned}\bar{c}' &= 4.8835\\ \bar{k}' &= 27.9165\end{aligned}$$

One can quickly check that the consumption/capital ratio is unchanged by the increase in z. Namely,

$$\begin{aligned}\frac{\bar{c}}{\bar{k}} &= \frac{1.7355}{9.9212} = 0.1749\\ \frac{\bar{c}'}{\bar{k}'} &= \frac{4.8835}{27.9165} = 0.1749\end{aligned}$$

Given this, all of our known coefficients are unchanged by the shift in z. This means that the solutions P and R are unaffected too. But both the locus of points given by $\Delta k_{t+1} = 0$ and the locus of points given by Δc_{t+1} have shifted, and so we have to work out the transitional dynamics to the new steady state.

The capital stock is an historically given initial condition. It cannot jump. The implied deviation relative to the new steady state is

$$\hat{k}_0^{\text{new}} = \log\left(\frac{\bar{k}}{\bar{k}'}\right) = \log\left(\frac{9.9212}{27.9165}\right) = -1.0345$$

For the purposes of drawing pictures, and so as not to get the economics confused, it's a bit nicer to measure each variable relative to its old steady state level. So, for example,

$$\begin{aligned}\hat{k}_t^{\text{old}} &\equiv \log\left(\frac{k_t}{\bar{k}}\right) = \log\left(\frac{k_t}{\bar{k}'}\frac{\bar{k}'}{\bar{k}}\right) = \hat{k}_t^{\text{new}} + \log\left(\frac{\bar{k}'}{\bar{k}}\right)\\ \hat{c}_t^{\text{old}} &\equiv \log\left(\frac{c_t}{\bar{c}}\right) = \log\left(\frac{c_t}{\bar{c}'}\frac{\bar{c}'}{\bar{c}}\right) = \hat{c}_t^{\text{new}} + \log\left(\frac{\bar{c}'}{\bar{c}}\right)\end{aligned}$$

and so on for the other variables. Notice that in each case, this just involves adding a constant to each time path.

The time paths of consumption, capital, investment, output, etc are attached. While the capital stock cannot jump, consumption investment and output can and do jump to new higher levels. After an initial jump, each of these variables then follows a slow transition to its new long-run level.

Question 5. (*Factor prices*). The wage rate and rental rate of capital are given by

$$\begin{aligned}w_t &= f(k_t) - f'(k_t)k_t = (1-\alpha)zk_t^{\alpha}\\ r_t &= f'(k_t) = \alpha z k_t^{\alpha-1}\end{aligned}$$

In log-linear terms, these are

$$\begin{aligned}\bar{w}\hat{w}_t &= (1-\alpha)z\alpha\bar{k}^{\alpha-1}\bar{k}\hat{k}_t \\ \bar{r}\hat{r}_t &= \alpha z(\alpha-1)\bar{k}^{\alpha-2}\bar{k}\hat{k}_t\end{aligned}$$

and simplifying gives

$$\begin{aligned}\hat{w}_t &= \alpha\frac{(1-\alpha)z\bar{k}^{\alpha}}{\bar{w}}\hat{k}_t = \alpha\hat{k}_t \\ \hat{r}_t &= -(1-\alpha)\frac{\alpha z\bar{k}^{\alpha-1}}{\bar{r}}\hat{k}_t = -(1-\alpha)\hat{k}_t\end{aligned}$$

In the "recovery from a war" scenario, the low initial capital stock implies a high initial marginal product of capital and drives low real wage rate. Both factor prices return to their long run values as investment and hence capital accumulation slowly drive down the returns to capital and drive up the productivity of labor and the real wage. As capital recovers, so does output.

In the "technological breakthrough" scenario, the breakthrough leads to an immediate jump up in the marginal product of both factors, since $w = (1-\alpha)zk^{\alpha}$ and $r = \alpha zk^{\alpha-1}$ and the initial capital stock is historically given. But notice that the rental rate at steady state, $\bar{r} = f'(\bar{k})$ is determined exclusively by the time discount rate and the depreciation rate via

$$\bar{r} = f'(\bar{k}) = \frac{1-\beta+\delta\beta}{\beta}$$

so in the long run, as the capital stock moves to its new higher level, the short run spike in the returns to capital are dissipated while the productivity of labor and hence the real wage keeps moving higher.

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