Question 1. (*Real Business Cycles*). By substituting the constraints into the objective function, the problem can be reduced to one of choosing plans for labor and capital accumulation to maximize

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [\log(z_t k_t^\alpha n_t^{1-\alpha} + (1-\delta)k_t - k_{t+1}) + \log(1-n_t)] \right\}$$

The first order condition for labor is

$$\beta^t \frac{1}{c_t}(1-\alpha)z_t k_t^\alpha n_t^{-\alpha} = \beta^t \frac{1}{\ell_t}$$

which simplifies to

$$\frac{c_t}{\ell_t} = (1-\alpha)z_t k_t^\alpha n_t^{-\alpha}$$

This is the standard marginal rate of substitution condition for labor/leisure choice when the real wage is $w_t \equiv (1-\alpha)z_t k_t^\alpha n_t^{-\alpha}$. The first order condition for capital accumulation is

$$\beta^t \frac{1}{c_t} = E_t \left\{ \beta^{t+1} \frac{1}{c_{t+1}}(z_{t+1}\alpha k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + 1 - \delta) \right\}$$

which simplifies to

$$1 = E_t \left\{ \beta \frac{c_t}{c_{t+1}}(z_{t+1}\alpha k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + 1 - \delta) \right\}$$

This is the standard consumption Euler equation for capital accumulation when the gross return on capital is $R_{t+1} \equiv 1 + z_{t+1}\alpha k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} - \delta$. To solve for the non-stochastic steady state, let $\bar{z} = 1$ and let $k_t = k_{t+1} = \bar{k}$. Then the gross return on capital satisfies

$$1 = \beta \bar{R}$$

and so the capital/labor ratio is

$$1 + \alpha \left(\frac{\bar{k}}{\bar{n}}\right)^{\alpha-1} - \delta = \frac{1}{\beta}$$

Solving this out gives

$$\frac{\bar{k}}{\bar{n}} = \left(\frac{\alpha\beta}{1 - \beta + \delta\beta}\right)^{\frac{1}{1-\alpha}}$$

(Note: we still need to solve for capital and labor separately). Now the first order condition for labor supply requires

$$\frac{\bar{c}}{\bar{\ell}} = (1-\alpha)\left(\frac{\bar{k}}{\bar{n}}\right)^\alpha = (1-\alpha)\left(\frac{\alpha\beta}{1 - \beta + \delta\beta}\right)^{\frac{\alpha}{1-\alpha}}$$

or

$$\bar{c} = (1-\alpha)\left(\frac{\alpha\beta}{1 - \beta + \delta\beta}\right)^{\frac{\alpha}{1-\alpha}}(1-\bar{n}) \tag{1}$$

Finally, we have the resource constraint

$$\bar{c} + \delta\bar{k} = \bar{k}^\alpha \bar{n}^{1-\alpha}$$

Dividing both sides by $\bar{n}$ and rearranging gives

$$\frac{\bar{c}}{\bar{n}} = \left(\frac{\alpha\beta}{1 - \beta + \delta\beta}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{\alpha\beta}{1 - \beta + \delta\beta}\right)^{\frac{1}{1-\alpha}} \tag{2}$$

We can now solve equations (1)-(2) as two linear equations in two unknowns, namely $\bar{c}$ and $\bar{n}$. When I do this in Matlab and use the given parameter values, I get

$$\bar{c} = 0.8879$$
$$\bar{n} = 0.4763$$

I can then use these to back out other interesting variables, for example

$$\bar{k} = 7.9399$$
$$\bar{\ell} = 0.5237$$
$$\bar{R} = 1.0101$$
$$\bar{\imath} = 0.3176$$
$$\bar{y} = 1.2055$$

- The log-linear first order condition for labor supply can be written

$$\hat{c}_t - \hat{\ell}_t = \hat{z}_t + \alpha(\hat{k}_t - \hat{n}_t)$$

  with

$$\hat{\ell}_t = -\frac{\bar{n}}{1 - \bar{n}}\hat{n}_t$$

  so we have

$$\hat{c}_t + \frac{\bar{n}}{1 - \bar{n}}\hat{n}_t = \hat{z}_t + \alpha(\hat{k}_t - \hat{n}_t)$$

  Also, the log-linear resource constraint is

$$\bar{c}\hat{c}_t + \bar{k}\hat{k}_{t+1} = \hat{z}_t + [\alpha\bar{y} + (1 - \delta)\bar{k}]\hat{k}_t + (1 - \alpha)\bar{y}\hat{n}_t$$

  And the consumption Euler equation can be written

$$0 = \mathsf{E}_t\left\{\hat{c}_t - \hat{c}_{t+1} + \beta\bar{r}[\hat{z}_{t+1} - (1 - \alpha)(\hat{k}_{t+1} - \hat{n}_{t+1})]\right\}$$

  where $\bar{r} = \bar{R} - 1 + \delta$. In Uhlig's notation, this collection of equations can be written with

$$X_t \equiv \hat{k}_{t+1}$$
$$Y_t \equiv \begin{pmatrix} \hat{c}_t \\ \hat{n}_t \end{pmatrix}$$
$$Z_t \equiv \hat{z}_t$$

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My system of **static equations** is made up of the labor supply condition and the resource constraint, written as

$$
\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\bar{k} \end{pmatrix} \hat{k}_{t+1} + \begin{pmatrix} \alpha \\ \alpha\bar{y} + (1-\delta)\bar{k} \end{pmatrix} \hat{k}_t + \begin{pmatrix} -1 & -\left(\alpha + \frac{\bar{n}}{1-\bar{n}}\right) \\ -\bar{c} & (1-\alpha)\bar{y} \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{n}_t \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \hat{z}_t
$$

My single **forward-looking** equation is

$$
\begin{aligned}
0 = \ & \mathsf{E}_t \Bigg\{ (0)\,\hat{k}_{t+2} + (-\beta\bar{r}(1-\alpha))\,\hat{k}_{t+1} + (0)\,\hat{k}_t + \begin{pmatrix} -1 & \beta\bar{r}(1-\alpha) \end{pmatrix} \begin{pmatrix} \hat{c}_{t+1} \\ \hat{n}_{t+1} \end{pmatrix} \\
& + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{n}_t \end{pmatrix} + (\beta\bar{r})\,\hat{z}_{t+1} + (0)\hat{z}_t \Bigg\}
\end{aligned}
$$

(And I also have the exogenous law of motion for log technology).

Question 2. (*Uhlig's Toolkit*). We are looking for coefficients P, Q, R, S for the system

$$
\begin{aligned}
\hat{k}_{t+1} &= P\hat{k}_t + Q\hat{z}_t \\
\begin{pmatrix} \hat{c}_t \\ \hat{n}_t \end{pmatrix} &= R\hat{k}_t + S\hat{z}_t
\end{aligned}
$$

Using Harald Uhlig's toolkit (see the attached code for details), I obtain

$$P = 0.9328$$

which implies

$$Q = 0.1405$$

and

$$
R = \begin{pmatrix} 0.5386 \\ -0.1683 \end{pmatrix}, \qquad S = \begin{pmatrix} 0.3481 \\ 0.5258 \end{pmatrix}
$$

Question 3. (*Impulse Responses*). See the attached plot. There is a persistent but ultimately transient change in the level of technology. While technology is high, both the real wage and the return to capital are high. Because of this, both capital accumulation (investment) and labor supply increase and so does consumption. As the technology shock slowly fades away, all these variables return to their long run levels.

Question 4. (*Simulations*). See the attached plot for a typical simulation. We drop a large number of initial observations to free our calculations of long run moments from any dependence on an arbitrary set of initial conditions. After doing so, I obtain typical standard deviations like

$$
\begin{aligned}
\frac{\mathsf{Std}(\hat{z}_t)}{\mathsf{Std}(\hat{y}_t)} &= 0.6778 \\
\frac{\mathsf{Std}(\hat{c}_t)}{\mathsf{Std}(\hat{y}_t)} &= 0.7713 \\
\frac{\mathsf{Std}(\hat{n}_t)}{\mathsf{Std}(\hat{y}_t)} &= 0.2497 \\
\frac{\mathsf{Std}(\hat{\imath}_t)}{\mathsf{Std}(\hat{y}_t)} &= 1.9550
\end{aligned}
$$

So productivity, consumption and employment are all less volatile than output while investment is nearly twice as volatile as output. One could compute many other interesting statistics, such as the contemporaneous and lagged correlations across variables and autocorrelations for each variable.