316-632 INTERNATIONAL MONETARY ECONOMICS NOTE 3a

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Two-period model with complete markets

We now turn to a multi-country model that allow us to endogenize trade in financial assets.

Let there be two dates, $t = 0, 1$ and let there be S possible states of nature that may be realized at date $t = 1$. Index the states by $s \in \mathcal{S} = \{1, 2, ..., S\}$. Let there be I countries with $i \in \mathcal{I} = \{1, 2, ..., I\}$.

- *Preferences*: For simplicity, assume that each country has a representative consumer with identical preferences of the form

$$u(c^i) = U(c_0^i) + \beta \sum_s U[c_1^i(s)]\pi(s)$$

with common constant time discount factor $0 < \beta < 1$ and subjective probabilities $\pi(s)$.

- *Endowments*: Although consumers have the same preferences, they have different endowments and this gives rise to an incentive to trade. Let each country have endowments y_0^i and $y_1^i(s)$ and denote world output by the sums

$$\begin{aligned} Y_0 &\equiv \sum_i y_0^i \\ Y_1(s) &\equiv \sum_i y_1^i(s) \end{aligned}$$

Since $y_1^i(s)$ depends on s, there is generally **idiosyncratic** (i.e., country-specific) risk. If it so happens that Y_1 does not depend on s, so that the risk averages out across countries, we say that there is **no aggregate risk**. For example, if there are two countries and $y_1^1(s)+y_1^2(s) = 1$, there is no aggregate risk. Otherwise, if the sum $Y_1(s)$ genuinely varies with the state s, there is aggregate (or "macro") risk.

A. Social planner's problem

Before turning to the decentralized problem with complete asset markets, we first study an equivalent social planner's problem. Suppose that the social planner attaches welfare weights $\omega_i \geq 0$ to country

i. Then the planner's problem is to choose consumption allocations c^i to maximize

$$\sum_i \omega_i u(c^i)$$

subject to $S+1$ resource constraints

$$\begin{aligned}\sum_i c_0^i &\leq Y_0 \\ \sum_i c_1^i(s) &\leq Y_1(s)\end{aligned}$$

Let the Lagrange multipliers associated with these resource constraints be $q_0 \geq 0$ and $q_1(s) \geq 0$. Then the Lagrangian for this problem can be written

$$\mathcal{L} = \sum_i \omega_i \left(U(c_0^i) + \beta \sum_s U[c_1^i(s)]\pi(s) \right) + q_0 \left(Y_0 - \sum_i c_0^i \right) + \sum_s q_1(s) \left(Y_1(s) - \sum_i c_1^i(s) \right)$$

and the first order conditions for this problem are, for each country i

$$\omega_i U'(c_0^i) = q_0 \tag{1}$$

and for each country i and each state s

$$\omega_i \beta U'[c_1^i(s)]\pi(s) = q_1(s) \tag{2}$$

Constant relative risk aversion

A simple solution to the planning problem is possible when $U(c)$ has the CRRA functional form

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

with marginal utility $U'(c) = c^{-\sigma}$ for $\sigma > 0$. In this case, the first order conditions can be written

$$c_0^i = \left(\frac{\omega_i}{q_0}\right)^{1/\sigma} \tag{3}$$

$$c_1^i(s) = \left(\frac{\omega_i \beta \pi(s)}{q_1(s)}\right)^{1/\sigma} \tag{4}$$

Summing over i and using the resource constraints gives solutions for the Lagrange multipliers in terms of the exogenous preference weights ω_i and the other parameters of the model

$$Y_0 = \sum_i \left(\frac{\omega_i}{q_0}\right)^{1/\sigma} \Longrightarrow q_0 = \left(\sum_i \omega_i^{1/\sigma}\right)^{\sigma} Y_0^{-\sigma}$$

and

$$Y_1(s) = \sum_i \left(\frac{\omega_i \beta\pi(s)}{q_1(s)}\right)^{1/\sigma} \Longrightarrow q_1(s) = \beta\pi(s)\left(\sum_i \omega_i^{1/\sigma}\right)^{\sigma} Y_1(s)^{-\sigma}$$

Plugging the solutions for the shadow prices back into the first order conditions (3)-(4) gives the consumption allocations for each country i at different dates and in different states of nature

$$\begin{aligned} c_0^i &= \left(\frac{\omega_i^{1/\sigma}}{\sum_i \omega_i^{1/\sigma}}\right) Y_0 \\ c_1^i(s) &= \left(\frac{\omega_i^{1/\sigma}}{\sum_i \omega_i^{1/\sigma}}\right) Y_1(s) \end{aligned}$$

That is, each country gets a **fixed share** (proportional to its relative welfare weight) of the world endowment at that date or state. Notice that if there is no aggregate risk (say, $Y_1(s) = \bar{Y}_1$ independent of s) then there is no volatility in an individual country's consumption. In this case, individual countries are perfectly insured against the idiosyncratic component of any fluctuations in their endowment. Equivalently, the ratio of consumption in country i to some other country j is a time and state independent constant that depends only on the relative welfare weights

$$\frac{c_0^i}{c_0^j} = \frac{c_1^i(s)}{c_1^j(s)} = \left(\frac{\omega_i}{\omega_j}\right)^{1/\sigma} \qquad \text{each } s \in \mathcal{S}$$

Consumption is **perfectly correlated** across countries: e.g., if the **world** endowment is unusually high, every country's consumption shifts up in parallel.

B. Complete markets equilibrium

Now let's turn to the decentralized model. We will assume that the representative consumers in each country can freely borrow or lend in a complete set of asset markets. Specifically, we assume the existence of S securities that pay one unit of consumption if and only if state s is realized at date $t = 1$. Let the price of these securities at date zero be $q_1(s)$ (Look at the choice of notation! These asset prices will correspond to the planner's Lagrange multipliers for a particular set of welfare weights). Similarly, let the price of consumption at date $t = 0$ be q_0. Later we will choose a normalization of the prices such that $q_0 = 1$, but for now let's just write q_0 so as to entrench the parallel with the social planner's Lagrange multipliers.

The budget constraints of each consumer are therefore

$$q_0 c_0^i + \sum_s q_1(s) c_1^i(s) = q_0 y_0^i + \sum_s q_1(s) y_1^i(s)$$

Of course, all countries face the same financial prices $q_0, q_1(s)$. The real interest rate r on a bond that pays one unit of consumption for sure (regardless of the state that realizes) is given by

$$\frac{1}{1+r} = \sum_s q_1(s)$$

Let the Lagrange multiplier of consumer i be $\lambda_i \geq 0$. Then the Lagrangian is

$$\mathcal{L}_i = U(c_0^i) + \beta \sum_s U[c_1^i(s)]\pi(s) + \lambda_i \left[q_0(y_0^i - c_0^i) + \sum_s q_1(s)[y_1^i(s) - c_1^i(s)] \right]$$

Each consumer has first order conditions

$$U'(c_0^i) = \lambda_i q_0$$

and for each s

$$\beta U'[c_1^i(s)]\pi(s) = \lambda_i q_1(s)$$

Now compare these first order conditions to (1)-(2) from the planner's problem. We see that if we identify $\lambda_i = \omega_i^{-1} \geq 0$, then the consumption allocations under the planner's problem will coincide with the market allocations. Notice that in the planner's problem, the welfare weights were exogenously given. With markets, however, the Lagrange multipliers λ_i are endogenous and are related to the **marketable value** of a country's intertemporal wealth.

To see this more explicitly, let utility again have the CRRA functional form. Then

$$c_1^i(s) = \left(\beta\pi(s) \frac{q_0}{q_1(s)} \right)^{1/\sigma} c_0^i$$

Summing across countries gives

$$Y_1(s) = \left(\beta\pi(s)\frac{q_0}{q_1(s)}\right)^{1/\sigma} Y_0$$

This implies a solution for the **relative** security prices $q_1(s)/q_0$ in terms of the primitives of the model, the discount factor, the subjective probabilities, and the supplies of the consumption good in different states of nature, specifically

$$\frac{q_1(s)}{q_0} = \beta\pi(s)\left(\frac{Y_1(s)}{Y_0}\right)^{-\sigma}$$

Since we can only determine relative prices, we are free to pick a normalization. It's convenient to measure everything in terms of consumption at date $t = 0$ so we set $q_0 = 1$. Not surprisingly, the price $q_1(s)$ of a unit of consumption in state s tomorrow is higher the more consumption in the future is valued (the higher is β), the more likely is state s, and the scarcer is the world supply $Y_1(s)$ of goods in state s.

We can also solve for the world real interest rate

$$\frac{1}{1+r} = \sum_s q_1(s) = \beta\sum_s\left(\frac{Y_1(s)}{Y_0}\right)^{-\sigma}\pi(s) = \beta\mathsf{E}_0\left\{\left(\frac{Y_1}{Y_0}\right)^{-\sigma}\right\}$$

Now let's find the consumption allocations. First use the solution for securities prices (with $q_0 = 1$) to write

$$\frac{c_1^i(s)}{c_0^i} = \left(\beta\frac{\pi(s)}{q_1(s)}\right)^{1/\sigma} = \frac{Y_1(s)}{Y_0}$$

or for each country i and each state s

$$\frac{c_1^i(s)}{Y_1(s)} = \frac{c_0^i}{Y_0} = \mu_i$$

Hence consumption allocations take the form

$$\begin{aligned} c_0^i &= \mu_i Y_0 \\ c_1^i(s) &= \mu_i Y_1(s) \end{aligned}$$

for some time and state independent coefficients $\mu_i \geq 0$ such that $\sum_i \mu_i = 1$. Notice that as with the planner's problem individual consumptions fluctuate only because of fluctuation in the **world** (i.e., aggregate) supply of goods. If the world supply was a constant (say $\bar{Y}_1$ independent of s), then each country would also have constant consumption. International trade in financial assets allows each country to eliminate the idiosyncratic (country-specific) component of any fluctuations in their endowment, but aggregate shocks that affect the entire world cannot be insured against in this fashion.

To solve for these μ_i, use the individual intertemporal budget constraints to get

$$\mu_i = \left(\frac{1 + \sum_s \beta\pi(s) \left(\frac{Y_1(s)}{Y_0}\right)^{-\sigma} \frac{y_1^i(s)}{y_0^i}}{1 + \sum_s \beta\pi(s) \left(\frac{Y_1(s)}{Y_0}\right)^{-\sigma} \frac{Y_1(s)}{Y_0}} \right) \left(\frac{y_0^i}{Y_0}\right) \tag{5}$$

Each individual's constant consumption share μ_i is directly proportional to the discounted present value of their endowments (i.e., their "intertemporal wealth"). Finally, the Lagrange multiplier's for each individual are

$$\lambda_i = \mu_i^{-\sigma} Y_0^{-\sigma}$$

and are inversely proportional to their intertemporal wealth. Recall that this complete markets solution is the same as the solution to the planner's problem if the weights are $\omega_i = \lambda_i^{-1} \geq 0$ so that the consumption shares μ_i given by (5) are in fact the same as the normalized weights $\frac{\omega_i^{1/\sigma}}{\sum_i \omega_i^{1/\sigma}}$. To reiterate, once we endow countries with "property rights" over individual endowments, market forces pick out a particular set of consumption allocations. These complete market consumption allocations correspond to the solution of a planner's problem when the planner weighs each individual **directly proportional to their intertemporal wealth**.

Trade in financial assets

Once we have determined equilibrium consumption, c_0^i and $c_1^i(s)$, it's simple to back out the implied pattern of trade across countries. Specifically, each country i has a trade balance $y_0^i - c_0^i = y_0^i - \mu_i Y_0$ at date zero. If this is positive, then the country is lending to the rest of the world (and if it is negative, the country is borrowing from the rest of the world). Similarly, we have trade balances $y_1^i(s) - c_1^i(s) = y_1^i(s) - \mu_i Y_1(s)$ in each state s at date $t = 1$. These state-contingent trade balances satisfy the budget constraint

$$\begin{aligned} c_1^i(s) &= y_1^i(s) + B_1^i(s) \\ \Longleftrightarrow B_1^i(s) &= -[y_1^i(s) - c_1^i(s)] \end{aligned}$$

$$= \mu_i Y_1(s) - y_1^i(s)$$

so we can figure out the state contingent net asset position $B_1^i(s)$ for each country i and each state s and in so doing figure out the pattern of cross country trade in financial assets. Of course, in this simple two-period model countries that run a trade deficit at date $t = 0$ must on average run a trade surplus at date $t = 1$. In genuine multi-period models, there is no such simple pattern of net trades.

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