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Aside on quantity and price indices with CES utility

Consider a static 2-good utility maximization problem of the form: maximize utility

$$U(c_1, c_2) = V(C(c_1, c_2))$$

where

$$C(c_1, c_2) \equiv \left[\alpha c_1^{\frac{\theta-1}{\theta}} + (1-\alpha)c_2^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}}, \qquad 0 < \alpha < 1, \text{ and } \theta > 0$$

subject to the budget constraint

$$p_1 c_1 + p_2 c_2 \leq y$$

The function C is a constant elasticity of substitution **aggregator** and overall utility U is some monotonic increasing transformation V of C. The parameter $\theta > 0$ is the **elasticity of substitution** between c_1 and c_2. When $\theta \to \infty$ the two goods are **perfect substitutes**, when $\theta \to 0$ the two goods are **perfect complements**, and when $\theta \to 1$ the utility function is Cobb-Douglas. (You need to use l'Hôpital's rule to show this last claim). The parameter α will turn out to measure an expenditure share.

Let's solve this problem. The first order conditions are

$$\begin{aligned} V'(C(c_1, c_2))\frac{\partial C(c_1, c_2)}{\partial c_1} &= \lambda p_1 \\ V'(C(c_1, c_2))\frac{\partial C(c_1, c_2)}{\partial c_2} &= \lambda p_2 \end{aligned}$$

for some unknown Lagrange multiplier λ. Computing the marginal utilities on the left hand side

$$\begin{aligned} \frac{\partial C(c_1, c_2)}{\partial c_1} &= \frac{\theta}{\theta-1}\left[\alpha c_1^{\frac{\theta-1}{\theta}} + (1-\alpha)c_2^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}-1}\frac{\theta-1}{\theta}\alpha c_1^{\frac{\theta-1}{\theta}-1} \\ \frac{\partial C(c_1, c_2)}{\partial c_2} &= \frac{\theta}{\theta-1}\left[\alpha c_1^{\frac{\theta-1}{\theta}} + (1-\alpha)c_2^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}-1}\frac{\theta-1}{\theta}(1-\alpha) c_2^{\frac{\theta-1}{\theta}-1} \end{aligned}$$

The marginal rate of substitution at an optimum is therefore

$$\frac{\alpha}{1-\alpha}\frac{c_1^{\frac{\theta-1}{\theta}-1}}{c_2^{\frac{\theta-1}{\theta}-1}} = \frac{\alpha}{1-\alpha}\frac{c_1^{\frac{-1}{\theta}}}{c_2^{\frac{-1}{\theta}}} = \frac{\alpha}{1-\alpha}\left(\frac{c_1}{c_2}\right)^{-\frac{1}{\theta}} = \left(\frac{p_1}{p_2}\right)$$

or

$$\left(\frac{c_1}{c_2}\right) = \left(\frac{1-\alpha}{\alpha}\right)^{\theta} \left(\frac{p_1}{p_2}\right)^{-\theta}$$

Notice that this implies

$$\frac{d\log(\frac{c_1}{c_2})}{d\log(\frac{p_1}{p_2})} = -\theta$$

which justifies the name given to the aggregator.

Now we can solve for the demand functions by combining this tangency condition with the budget constraint. That is,

$$\begin{aligned} y &= p_1 c_1 + p_2 c_2 \\ &= \left[p_1 \left(\frac{1-\alpha}{\alpha}\right)^{\theta} \left(\frac{p_1}{p_2}\right)^{-\theta} + p_2\right] c_2 \end{aligned}$$

so

$$\begin{aligned} \hat{c}_2 &= \frac{1}{1+\left(\frac{1-\alpha}{\alpha}\right)^{\theta}\left(\frac{p_1}{p_2}\right)^{1-\theta}} \left(\frac{y}{p_2}\right) = \frac{\alpha^{\theta} p_2^{1-\theta}}{\alpha^{\theta} p_2^{1-\theta} + (1-\alpha)^{\theta} p_1^{1-\theta}} \left(\frac{y}{p_2}\right) \\ \hat{c}_1 &= \frac{\left(\frac{1-\alpha}{\alpha}\right)^{\theta}\left(\frac{p_1}{p_2}\right)^{-\theta}}{1+\left(\frac{1-\alpha}{\alpha}\right)^{\theta}\left(\frac{p_1}{p_2}\right)^{1-\theta}} \left(\frac{y}{p_2}\right) = \frac{(1-\alpha)^{\theta} p_1^{1-\theta}}{\alpha^{\theta} p_2^{1-\theta} + (1-\alpha)^{\theta} p_1^{1-\theta}} \left(\frac{y}{p_1}\right) \end{aligned}$$

Computing the price index

We now want to find functions $C(\hat{c}_1, \hat{c}_2)$ and $P(p_1, p_2)$ such that

$$p_1\hat{c}_1 + p_2\hat{c}_2 = P(p_1, p_2)C(\hat{c}_1, \hat{c}_2)$$

and

$$U(\hat{c}_1, \hat{c}_2) = V(C(\hat{c}_1, \hat{c}_2))$$

at the utility maximizing demands $\hat{c}_1, \hat{c}_2$. Mechanically, we do this by minimizing expenditure PC subject to the constraint that $C = 1$.

Now $C = 1$ if and only if

$$U(\hat{c}_1, \hat{c}_2) = \left[\alpha \hat{c}_1^{\frac{\theta-1}{\theta}} + (1-\alpha)\hat{c}_2^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} = 1$$

Plugging in the demand functions with $y = PC = P$ gives

$$1 = \left\{ \alpha \left[\frac{\alpha^\theta p_2^{1-\theta}}{\alpha^\theta p_2^{1-\theta} + (1-\alpha)^\theta p_1^{1-\theta}} \left(\frac{P}{p_2}\right) \right]^{\frac{\theta-1}{\theta}} + (1-\alpha) \left[\frac{(1-\alpha)^\theta p_1^{1-\theta}}{\alpha^\theta p_2^{1-\theta} + (1-\alpha)^\theta p_1^{1-\theta}} \left(\frac{P}{p_1}\right) \right]^{\frac{\theta-1}{\theta}} \right\}^{\frac{\theta}{\theta-1}}$$

We need to solve this expression for P as a function of p_1 and p_2. Write

$$\begin{aligned}
1 &= \left\{ \alpha^\theta \left[\frac{p_2^{1-\theta}}{\alpha^\theta p_2^{1-\theta} + (1-\alpha)^\theta p_1^{1-\theta}} \left(\frac{P}{p_2}\right) \right]^{\frac{\theta-1}{\theta}} + (1-\alpha)^\theta \left[\frac{p_1^{1-\theta}}{\alpha^\theta p_2^{1-\theta} + (1-\alpha)^\theta p_1^{1-\theta}} \left(\frac{P}{p_1}\right) \right]^{\frac{\theta-1}{\theta}} \right\}^{\frac{\theta}{\theta-1}} \\
&= \left\{ \alpha^\theta \left[\frac{P p_2^{-\theta}}{\alpha^\theta p_2^{1-\theta} + (1-\alpha)^\theta p_1^{1-\theta}} \right]^{\frac{\theta-1}{\theta}} + (1-\alpha)^\theta \left[\frac{P p_1^{-\theta}}{\alpha^\theta p_2^{1-\theta} + (1-\alpha)^\theta p_1^{1-\theta}} \right]^{\frac{\theta-1}{\theta}} \right\}^{\frac{\theta}{\theta-1}} \\
&= \left\{ \left[\frac{P}{\alpha^\theta p_2^{1-\theta} + (1-\alpha)^\theta p_1^{1-\theta}} \right]^{\frac{\theta-1}{\theta}} \left[\alpha^\theta p_2^{1-\theta} + (1-\alpha)^\theta p_1^{1-\theta} \right] \right\}^{\frac{\theta}{\theta-1}}
\end{aligned}$$

Hence

$$\frac{P^{\frac{\theta-1}{\theta}}}{\left[\alpha^\theta p_2^{1-\theta} + (1-\alpha)^\theta p_1^{1-\theta} \right]^{\frac{\theta-1}{\theta}}} = \left[\alpha^\theta p_2^{1-\theta} + (1-\alpha)^\theta p_1^{1-\theta} \right]^{-1}$$

or

$$P = \left[\alpha^\theta p_2^{1-\theta} + (1-\alpha)^\theta p_1^{1-\theta} \right]^{\frac{1}{1-\theta}}$$

After all that algebra, we see that the price index is itself a CES aggregate of the individual prices p_1 and p_2. We will use this result in our model of complete markets with non-traded goods (see Note 4a).

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