**316-632 International Monetary Economics**
*Problem Set #1* *Selected answers*

**Question 1.** The key first order condition for a consumer's problem is

$$\beta U'[c_1(s)]\pi(s) = U'(c_0^i)q_1(s)$$

**a)** If period utility is $U(c) = \log(c)$, then the first order condition can be re-written

$$c_0^i\beta\pi(s) = q_1(s)c_1^i(s)$$

Summing over the $i$ countries and rearranging gives solutions for the state prices

$$q_1(s) = \beta\pi(s)\frac{Y_0}{Y_1(s)}$$

Now plugging this expression for $q_1(s)$ back into the first order conditions gives

$$\frac{c_0^i}{c_1^i(s)} = \frac{Y_0}{Y_1(s)} = \mu^i$$

where $\mu^i$ is a time and state independent constant. We solve for $\mu^i$ from the intertemporal budget constraint, namely

$$\mu^i\left[Y_0 + \sum_s q_1(s)Y_1(s)\right] = y_0^i + \sum_s q_1(s)y_1^i(s)$$

or

$$\begin{aligned}\mu^i &= \frac{y_0^i + \sum_s q_1(s)y_1^i(s)}{Y_0 + \sum_s q_1(s)Y_1(s)} = \frac{y_0^i + \sum_s \beta\pi(s)\frac{Y_0}{Y_1(s)}y_1^i(s)}{Y_0 + \sum_s \beta\pi(s)\frac{Y_0}{Y_1(s)}Y_1(s)}\\ &= \frac{1}{1+\beta}\frac{y_0^i}{Y_0} + \frac{\beta}{1+\beta}\sum_s \pi(s)\frac{y_1^i(s)}{Y_1(s)}\end{aligned}$$

This is a weighted average of country $i$'s income share in period $t = 0$ and its **expected** income share in period $t = 1$. The weights correspond to the relative importance of each period, as measured by $\beta$. Finally, we can solve for the world interest rate via the relationship

$$\frac{1}{1+r} = \sum_s q_1(s) = \beta\sum_s \pi(s)\frac{Y_0}{Y_1(s)} = \beta\mathsf{E}_0\left\{\frac{Y_0}{Y_1}\right\}$$

so that the real interest rate is higher when the expected growth rate of the world endowment is higher.

**b)** Similarly, if $U(c) = -\gamma\exp(-\gamma c)$, then the key first order condition can be written

$$\gamma\beta\exp[-\gamma c_1^i(s)]\pi(s) = \gamma\exp(-\gamma c_0^i)q_1(s)$$

Taking logs of both sides and simplifying

$$\log[\beta\pi(s)] - \gamma c_1^i(s) = -\gamma c_0^i + \log[q_1(s)]$$

Summing over the $I$ countries and using the market clearing conditions

$$I\log[\beta\pi(s)] - \gamma Y_1(s) = -\gamma Y_0 + I\log[q_1(s)]$$

Hence

$$q_1(s) = \beta\pi(s)\exp\left\{\frac{\gamma[Y_0 - Y_1(s)]}{I}\right\}$$

The world real interest rate is then given by

$$\frac{1}{1+r} = \sum_s q_1(s) = \beta\sum_s \pi(s)\exp\left\{\frac{\gamma[Y_0 - Y_1(s)]}{I}\right\} = \beta\mathsf{E}_0\left\{\frac{\gamma(Y_0 - Y_1)}{I}\right\}$$

Using the expressions for $q_1(s)$, consumption allocations are now given by

$$c_0^i - c_1^i(s) = \frac{Y_0 - Y_1(s)}{I}$$

which is independent of $\gamma$. Guess that

$$\begin{aligned} c_0^i &= \frac{Y_0}{I} - \mu^i \\ c_1^i(s) &= \frac{Y_1(s)}{I} - \mu^i \end{aligned}$$

for some time and state independent constants $\mu^i$. Again, we can solve for this constant using the intertemporal budget constraint

$$Y_0 - I\mu^i + \sum_s q_1(s)[Y_1(s) - I\mu^i] = Iy_0^i + \sum_s q_1(s)Iy_1^i(s)$$

Rearranging

$$\mu^i = \frac{(Y_0/I) - y_0^i + \sum_s q_1(s)[(Y_1(s)/I) - y_1^i(s)]}{1 + \sum_s q_1(s)}$$

where the state prices are given as above.

**c)** Now we have $I = 2$ and first order conditions

$$\begin{aligned} \gamma_1\beta\exp[-\gamma_1 c_1^1(s)]\pi(s) &= \gamma_1\exp(-\gamma_1 c_0^1)q_1(s) \\ \gamma_2\beta\exp[-\gamma_2 c_1^2(s)]\pi(s) &= \gamma_2\exp(-\gamma_2 c_0^2)q_1(s) \end{aligned}$$

Taking logs of both sides and simplifying

$$\begin{aligned} \log[\beta\pi(s)] - \gamma_1 c_1^1(s) &= -\gamma_1 c_0^1 + \log[q_1(s)] \\ \log[\beta\pi(s)] - \gamma_2 c_1^2(s) &= -\gamma_2 c_0^2 + \log[q_1(s)] \end{aligned}$$

or

$$\begin{aligned} c_0^1 - c_1^1(s) &= \frac{1}{\gamma_1}\{\log[q_1(s)] - \log[\beta\pi(s)]\} \\ c_0^2 - c_1^2(s) &= \frac{1}{\gamma_2}\{\log[q_1(s)] - \log[\beta\pi(s)]\} \end{aligned}$$

Now summing over countries

$$Y_0 - Y_1(s) = \frac{\gamma_1 + \gamma_2}{\gamma_1 \gamma_2} \{\log[q_1(s)] - \log[\beta\pi(s)]\}$$

and solving for the state prices

$$q_1(s) = \beta\pi(s) \exp\left\{\frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2}[Y_0 - Y_1(s)]\right\}$$

Notice that if $\gamma_1 = \gamma_2 = \gamma$, the term $\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2} = \frac{\gamma}{2}$ which is what we would have from part b), but with $I = 2$. Now using this expression for the state prices to solve for relative consumption

$$\begin{aligned} c_0^1 - c_1^1(s) &= \frac{\gamma_2}{\gamma_1 + \gamma_2}[Y_0 - Y_1(s)] \\ c_0^2 - c_1^2(s) &= \frac{\gamma_1}{\gamma_1 + \gamma_2}[Y_0 - Y_1(s)] \end{aligned}$$

Hence it seems that consumption allocations are given by

$$\begin{aligned} c_0^1 &= \frac{\gamma_2}{\gamma_1 + \gamma_2} Y_0 - \mu^1 \\ c_1^1(s) &= \frac{\gamma_2}{\gamma_1 + \gamma_2} Y_1(s) - \mu^1 \end{aligned}$$

and similarly for country 2. Again, we could solve for the time and state independent constants $\mu^i$ from each country's intertemporal budget constraint. Notice that if country 1 is more risk averse, $\gamma_1 > \gamma_2$, then country 2 gets a larger share of world output.

**Question 2.** The equivalent planning problem is to choose allocations $a_t^i(z^t), b_t^i(z^t)$ for $i = 1, 2$ to maximize the social welfare function

$$\sum_i \sum_{t=0}^{\infty} \sum_{z^t} \omega_i \beta^t \left[\frac{a_t^i(z^t)^{1-\alpha} + b_t^i(z^t)^{1-\alpha}}{1-\alpha}\right] \pi_t(z^t)$$

subject to the resource constraints

$$\begin{aligned} \sum_i a_t^i(z^t) &\leq x_t(z^t) \\ \sum_i b_t^i(z^t) &\leq y_t(z^t) \end{aligned}$$

Let $Q_t^x(z^t) \equiv \beta^t \pi_t(z^t) q_t^x(z^t)$ and $Q_t^y(z^t) \equiv \beta^t \pi_t(z^t) q_t^y(z^t)$ denote the planner's Lagrange multipliers. Then the problem breaks down into a sequence of static maximization problems, one for each date and state.

**a)** The key first order conditions are

$$\begin{aligned} \omega_i a_t^i(z^t)^{-\alpha} &= q_t^x(z^t), \qquad i = 1,2 \\ \omega_i b_t^i(z^t)^{-\alpha} &= q_t^y(z^t), \qquad i = 1,2 \end{aligned}$$

along with the resource constraints.

**b)** Rewrite the first order conditions as

$$\begin{aligned} a_t^i(z^t) &= \left(\frac{\omega_i}{q_t^x(z^t)}\right)^{1/\alpha}, \qquad i = 1,2 \\ b_t^i(z^t) &= \left(\frac{\omega_i}{q_t^y(z^t)}\right)^{1/\alpha}, \qquad i = 1,2 \end{aligned}$$

and now sum over $i = 1,2$ to get

$$\begin{aligned} x_t(z^t) q_t^x(z^t)^{1/\alpha} &= (\omega_1^{1/\alpha} + \omega_2^{1/\alpha}) \\ y_t(z^t) q_t^y(z^t)^{1/\alpha} &= (\omega_1^{1/\alpha} + \omega_2^{1/\alpha}) \end{aligned}$$

Hence the Lagrange multipliers are

$$\begin{aligned} q_t^x(z^t) &= (\omega_1^{1/\alpha} + \omega_2^{1/\alpha})^{\alpha} x_t(z^t)^{-\alpha} \\ q_t^y(z^t) &= (\omega_1^{1/\alpha} + \omega_2^{1/\alpha})^{\alpha} y_t(z^t)^{-\alpha} \end{aligned}$$

which vary inversely with the supply of each good with elasticity $-\alpha$. Now using these to solve for the consumption allocations

$$\begin{aligned} a_t^i(z^t) &= \frac{\omega_i^{1/\alpha}}{\omega_1^{1/\alpha} + \omega_2^{1/\alpha}} x_t(z^t) \\ b_t^i(z^t) &= \frac{\omega_i^{1/\alpha}}{\omega_1^{1/\alpha} + \omega_2^{1/\alpha}} y_t(z^t) \end{aligned}$$

Finally, the terms of trade are given by

$$\mathsf{tot}_t(z^t) = \frac{q_t^y(z^t)}{q_t^x(z^t)} = \left(\frac{x_t(z^t)}{y_t(z^t)}\right)^{\alpha}$$

So when the world supply of $x$ rises, its (shadow) price declines and country 1's terms of trade worsen (i.e., $\mathsf{tot}_t(z^t)$ **declines**).

**c)** The implied welfare weights are found by considering a market economy. Let $\lambda_i$ denote the Lagrange multiplier on a country's intertemporal budget constraint. Then if $\lambda_i = 1/\omega_i$, the market allocations will be the same as the planner's allocations. Now the intertemporal budget constraint for country $i = 1$ is

$$\sum_{t=0}^{\infty} \sum_{z^t} [Q_t^x(z^t) a_t^1(z^t) a_t^1(z^t) + Q_t^y(z^t) b_t^1(z^t)] = \sum_{t=0}^{\infty} \sum_{z^t} Q_t^x(z^t) x_t(z^t)$$

Using the planners Lagrange multipliers

$$\omega_1^{1/\alpha}\sum_{t=0}^{\infty}\sum_{z^t}\{\beta^t\pi_t(z^t)[x_t(z^t)^{1-\alpha}+y_t(z^t)^{1-\alpha}]\} = (\omega_1^{1/\alpha}+\omega_2^{1/\alpha})\sum_{t=0}^{\infty}\sum_{z^t}\beta^t\pi_t(z^t)x_t(z^t)^{1-\alpha}$$

Imposing the normalization $(\omega_1^{1/\alpha}+\omega_2^{1/\alpha}) = 1$, which is implicitly a normalization of the price system, we have

$$\omega_1^{1/\alpha} = \frac{\sum_{t=0}^{\infty}\sum_{z^t}\beta^t\pi_t(z^t)x_t(z^t)^{1-\alpha}}{\sum_{t=0}^{\infty}\sum_{z^t}\{\beta^t\pi_t(z^t)[x_t(z^t)^{1-\alpha}+y_t(z^t)^{1-\alpha}]\}}$$

So the implicit weights correspond to relative shares of intertemporal wealth.

**d)** The trade balance for country 1 is

$$\mathsf{tb}_t(z^t) = x_t(z^t) - a_t^1(z^t) - \frac{q_t^y(z^t)}{q_t^x(z^t)}b_t^1(z^t)$$

Plugging in our previous solutions

$$\mathsf{tb}_t(z^t) = x_t(z^t) - \frac{\omega_1^{1/\alpha}}{\omega_1^{1/\alpha}+\omega_2^{1/\alpha}}x_t(z^t) - \left(\frac{x_t(z^t)}{y_t(z^t)}\right)^{\alpha}\frac{\omega_1^{1/\alpha}}{\omega_1^{1/\alpha}+\omega_2^{1/\alpha}}y_t(z^t)$$

Again using the normalization $(\omega_1^{1/\alpha}+\omega_2^{1/\alpha}) = 1$, this simplifies to

$$\mathsf{tb}_t(z^t) = (1-\omega_1^{1/\alpha})x_t(z^t) - \omega_1^{1/\alpha}x_t(z^t)^{\alpha}y_t(z^t)^{1-\alpha}$$

With the normalization $(\omega_1^{1/\alpha}+\omega_2^{1/\alpha}) = 1$, it must be the case that $0 < \omega_1^{1/\alpha} < 1$. So one effect of an increase in the supply of the domestic $x$ good is to improve the trade balance. But there is also an offsetting terms of trade effect where the increase in the supply of $x$ reduces the world price for that commodity and so worsens the domestic terms of trade. An increase in the supply of the foreign good $y$ worsens the trade balance if $1-\alpha > 0$ and improves the trade balance otherwise. Put differently, if $\alpha < 1$, then the increase in the supply of $y$ is sufficient to improve the domestic terms of trade so much that the domestic trade balance improves. So if $\alpha < 1$, the terms of trade and the trade balance tend to covary in a positive manner.

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