BASIC

PORTFOLIO

ANALYSIS

Fall 2002

Mean and Standard Deviation of Individual Securities

Define:

(1) R_{ij} **jth return on stock i**

(2) $\bar{R}_i$ **expected return stock on i**

(3) σ_i **standard deviation of return stock i**

(4) M **number of periods**

(5) N **number of assets**

$$\overline{R}_i = \sum_{j=1}^{M} \frac{R_{ij}}{M}$$

$$\sigma_i^2 = \sum_{j=1}^{M} \frac{\left(R_{ij} - \overline{R}_i\right)^2}{M} = E\left(R_{ij} - \overline{R}_i\right)^2$$

Note some use M-1.

Example:

MONTH	**Return**
Dec	**5%**
Nov	**-2%**
Oct	**3%**
Sept	**2%**
Aug	**-1%**
July	**-1%**
	6%

$$\overline{R}_i = \Sigma \frac{R_{ij}}{6} = \frac{6\%}{6} = 1\%$$

$(5-1)^2 = 16$

$(-2-1)^2 = 9$

$(3-1)^2 = 4$

$(2-1)^2 = 1$

$(-1-1)^2 = 4$

$(-1-1)^2 = 4$

$$\sigma_i^2 = 38/6 = 6\,2/3$$

$$\sigma_i = \sqrt{6\frac{2}{3}}$$

MEAN AND VARIANCE OF PORTFOLIOS

Two General Rules:

1. $$E\left(R_{1j}+R_{2j}\right)=E\left(R_{1}\right)+E\left(R_{2}\right)=\overline{R}_{1}+\overline{R}_{2}$$

2. $$E\left(CR_{1j}\right)=C\overline{R}_{1}$$

Two Asset Case (both risky)

Define:

X_i **as the proportion in security i.**

(1) Return on portfolio

$$R_{pj}=\left[X_{1}R_{1j}+X_{2}R_{2j}\right]=\Sigma X_{i}R_{ij}$$

(2) **Mean return on portfolio**

$$\overline{R}_P = E\left[X_1 R_{1j} + X_2 R_{2j}\right]$$

$$= E\left[X_1 R_{1j}\right] + E\left(X_2 R_{2j}\right)$$

$$= X_1 \overline{R}_1 + X_2 \overline{R}_2 = \Sigma X_i \overline{R}_i$$

(3) Variance = $\sigma_p^2 = E\left[R_p - \overline{R}_p\right]^2$

$$\sigma_p^2 = E\left[(X_1 R_{1j} + X_2 R_{2j}) - (X_1 \overline{R}_1 + X_2 \overline{R}_2)\right]^2$$

$$= E\left[X_1 (R_{1j} - \overline{R}_1) + X_2 (R_{2j} - \overline{R}_2)\right]^2$$

$$= E\left[X_1^2 (R_{1j} - \overline{R}_1)^2 + X_2^2 (R_{2j} - \overline{R}_2)^2 + 2X_1 X_2 (R_{1j} - \overline{R}_1)(R_{2j} - \overline{R}_2)\right]$$

$$= X_1^2 E(R_{1j} - \overline{R}_1)^2 + X_2^2 E(R_{2j} - \overline{R}_2)^2 + 2X_1 X_2 E\left[(R_{1j} - \overline{R}_1)(R_{2j} - \overline{R}_2)\right]$$

$$= X_1^2 \sigma_1^2 + X_2^2 \sigma_2^2 + 2X_1 X_2 \sigma_{12}$$

$$\sigma_{12} = E\left[(R_{1j} - \overline{R}_1)(R_{2j} - \overline{R}_2)\right]$$

Note:

(1) **Measures joint movement**

(2) **Unrestricted to sign**

<u>Example</u> (assume equally likely) **6A**

		<u>Return</u>			
Condition	**A**	**B**	**C**	**Rainfall**	**D**
Good	**12**	**7**	**8**	**Heavy**	**8**
Average	**10**	**9**	**6**	**Average**	**6**
Poor	**8**	**11**	**4**	**Light**	**4**
$\bar{r}$	**10**	**9**	**6**		**6**
σ	**8/3**	**8/3**	**8/3**		**8/3**

Useful

$$\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$$

$$-1 \le \rho_{ij} \le +1$$

Calculating σ_{AB}

(12 - 10) (7 - 9) = -4

(10 - 10) (9 - 9) = 0

(8 - 10) (11 - 9) = -4

$$\sigma_{AB} = -\frac{8}{3}$$

$$-\frac{8}{3} = \rho_{AB}\sqrt{\frac{8}{3}}\sqrt{\frac{8}{3}}$$

$$\rho_{AB} = -1$$

Calculating σ_{AC} **6C**

(12 - 10) (8 - 6) = 4

(10 - 10) (6 - 6) = 0

(8 - 10) (4 - 6) = <u>4</u>
8

$$\sigma_{AB} = \frac{8}{3}$$

$$\frac{8}{3} = \rho_{AB}\sqrt{\frac{8}{3}}\sqrt{\frac{8}{3}}$$

$$\rho_{AB} = 1$$

Calculating σ_{AD} **6D**

(12 - 10) (8 - 6) = +4

(12 - 10) (6 - 6) = 0

(12 - 10) (4 - 6) = -4

(10 - 10) (8 - 6) = 0

(10 - 10) (6 - 6) = 0

(10 - 10) (4 - 6) = 0

(8 - 10) (8 - 6) = -4

(8 - 10) (6 - 6) = 0

(8 - 10) (4 - 6) = +4

$$\sigma_{AD} = 0$$

$$\rho_{AD} = 0$$

Three Security Case

1. **Return on portfolio**

$$R_{Pj} = X_1 R_{1j} + X_2 R_{2j} + X_3 R_{3j} = \Sigma X_i R_{ij}$$

2. **Mean return on portfolio**

$$\overline{R}_P = E(X_1 R_{1j} + X_2 R_{2j} + X_3 R_{3j})$$

$$\overline{R}_p = X_1 \overline{R}_1 + X_2 \overline{R}_2 + X_3 \overline{R}_3$$

3. **Variance of return**

$$\sigma_P^2 = E\left[(X_1 R_{1j} + X_2 R_{2j} + X_3 R_{3j}) - (X_1 \overline{R}_1 + X_2 \overline{R}_2 + X_3 \overline{R}_3)\right]^2$$

$$\sigma_P^2 = E\left[X_1 (R_{1j} - \overline{R}_1) + X_2 (R_{2j} - \overline{R}_2) + X_3 (R_{3j} - \overline{R}_3)\right]^2$$

Terms Variance

$$X_1^2 E(R_{1j} - \overline{R}_1)^2$$

$$X_2^2 E(R_{2j} - \overline{R}_2)^2$$

$$X_3^2 E(R_{3j} - \overline{R}_3)^2$$

Terms Covariance

$$2X_1X_2E\left[(R_{1j} - \overline{R}_1)(R_{2j} - \overline{R}_2)\right]$$

$$2X_1X_3E\left[(R_{1j} - \overline{R}_1)(R_{3j} - \overline{R}_3)\right]$$

$$2X_2X_3E\left[(R_{2j} - \overline{R}_2)(R_{3j} - \overline{R}_{3)}\right]$$

General Formulas:

Mean Return on Portfolio:

$$\bar{R}_p = \Sigma X_i \bar{R}_i$$

Variance of Return on Portfolio

$$\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{k=1 \\ k \neq i}}^{N} X_i X_k \sigma_{ik}$$

The Effect of Diversification

Assume random selection and equal amount in each security.

$$X_i = \frac{1}{N}$$

$$\sigma_P^2 = \sum_{i=1}^{N} \left(\frac{1}{N}\right)^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{k=1 \\ k \neq i}}^{N} \left(\frac{1}{N}\right)^2 \sigma_{ik}$$

$$= \frac{1}{N}\left[\sum_{i=1}^{N} \frac{\sigma_i^2}{N}\right] + \left(\frac{N-1}{N}\right)\left[\sum_{i=1}^{N} \sum_{\substack{k=1 \\ k \neq i}}^{N} \left(\frac{1}{N}\right)\left(\frac{1}{N-1}\right)\sigma_{ik}\right]$$

$$= \frac{1}{N}\bar{\sigma}_i^2 + \left(\frac{N-1}{N}\right)\bar{\sigma}_{ik}$$

$$=\frac{1}{N}\overline{\sigma}^{2}+\left(1-\frac{1}{N}\right)\overline{\sigma}_{ik}$$

$$=\frac{1}{N}\left(\overline{\sigma}_{i}^{2}-\overline{\sigma}_{ik}\right)+\overline{\sigma}_{ik}$$

Efficient Set Theorem

(1). Holding $\overline{R}_P$ constant minimize σ_P

(2). Holding σ_P constant maximize $\overline{R}_P$

Plotting Efficient Frontier

(two risky assets)

	$\bar{R}$	σ	Proportion
A	**14**	**4**	X_A
B	**8**	**2**	$X_B = (1 - X_A)$

Perfectly Positively Correlated

Expected Return:

$$\overline{R}_p = X_A R_A + (1 - X_A)\overline{R}_B$$

$$= \overline{R}_B + X_A(\overline{R}_A - \overline{R}_B)$$

$$\sigma_p^2 = X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2X_A(1 - X_A)\rho_{AB}\sigma_A\sigma_B$$

IF

$$\rho = +1$$

$$\sigma_p^2 = \left(X_A^2 \sigma_A^2 + 2X_A (1 - X_A) \sigma_A \sigma_B + (1 - X_A)^2 \sigma_B^2 \right)$$

$$= \left(X_A \sigma_A + (1 - X_A) \sigma_B \right)^2$$

or

$$\sigma_p = X_A\sigma_A + (1 - X_A)\sigma_B$$

$$\sigma_p = \sigma_B + X_A(\sigma_A - \sigma_B)$$

or

$$X_A = \frac{\sigma_p - \sigma_B}{\sigma_A - \sigma_B}$$

Substituting into expected return equation:

$$\overline{R}_p = \overline{R}_B + \frac{\sigma_p - \sigma_B}{\sigma_A - \sigma_B}\left(\overline{R}_A - \overline{R}_B\right)$$

$$= \left[\overline{R}_B - \sigma_B\left(\frac{\overline{R}_A - \overline{R}_B}{\sigma_A - \sigma_B}\right)\right] + \sigma_p\left(\frac{\overline{R}_A - \overline{R}_B}{\sigma_A - \sigma_B}\right)$$

This is, of course, a straight line. With the example:

$$\overline{R}_p = \left[\left[8 - 2\left(\frac{14-8}{4-2}\right)\right] + \sigma_p\left(\frac{14-8}{4-2}\right)\right]$$

$$\overline{R}_p = 2 + 3\sigma_p$$

Perfect Negative Correlation

If $\rho = -1$

$$\sigma_p^2 = \left(X_A^2 \sigma_A^2 - 2X_A \left(1 - X_A\right) \sigma_A \sigma_B + (1 - X_A)^2 \sigma_B^2 \right)$$

This can come from either

$$\sigma_p = \left(X_A \sigma_A - (1 - X_A) \sigma_B \right)$$

or

$$\sigma_p = \left(-X_A \sigma_A + (1 - X_A) \sigma_B \right)$$

and

$$\sigma_p = -\sigma_B + X_A(\sigma_A + \sigma_B)$$

$$= +\sigma_B - X_A(\sigma_A + \sigma_B)$$

$$X_A = \frac{\sigma_p + \sigma_B}{\sigma_A + \sigma_B} \text{ or } \frac{-\sigma_p + \sigma_B}{\sigma_A + \sigma_B}$$

Substituting into expected return:

$$\overline{R}_p = \overline{R}_B + \frac{\sigma_p + \sigma_B}{\sigma_A + \sigma_B}(\overline{R}_A - \overline{R}_B)$$

or

$$\overline{R}_p = \overline{R}_B + \frac{-\sigma_p + \sigma_B}{\sigma_A + \sigma_B}(\overline{R}_A - \overline{R}_B)$$

$$\overline{R}_p = \left[\overline{R}_B + \sigma_B\left(\frac{\overline{R}_A - \overline{R}_B}{\sigma_A + \sigma_B}\right)\right] \pm \sigma_p\left(\frac{\overline{R}_A - \overline{R}_B}{\sigma_A + \sigma_B}\right)$$

$$= 8 + 2\left(\frac{14-8}{4+2}\right) \pm \sigma_p\left(\frac{14-8}{4+2}\right)$$

$$= 10 \pm \sigma_p$$

with other ρ's not a straight line

In standard definition proceeds full usable

X_1	X_2	$\overline{R}_p$
+2	-1	20
+3	-2	26
+4	-3	32

Efficient Frontier with Riskless Asset

$$\bar{R}_C = (1 - X)R_F + X\bar{R}_A = R_F + X(\bar{R}_A - R_F)$$

where X is fraction in risky portfolio A

$$\sigma_C^2 = \left[(1 - X)^2\sigma_F^2 + X^2\sigma_A^2 + 2X(1 - X)\rho_{AF}\sigma_A\sigma_F\right]$$

$$\sigma_C^2 = X^2\sigma_A^2 \Rightarrow X = \frac{\sigma_C}{\sigma_A}$$

$$\bar{R}_C = R_F + \frac{\sigma_C}{\sigma_A}(\bar{R}_A - R_F)$$

$$\bar{R}_C = R_F + \left(\frac{\bar{R}_A - R_F}{\sigma_A}\right)\sigma_C$$

(1). Separation Theorem:

Investors optimum choice of a risky portfolio is separate from his or her preferences.

(2). Two Fund Theorem:

An investor is not hurt by restriction to a choice of two funds.

(3). Unambiguous objective function.