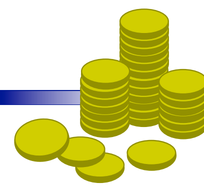
Yield to Maturity



Outline

- Bond yield to maturity (internal rate of return)
- Bond yields and zero rates (L.O.O.P in terms of yields)
- Coupon effects
- Annuity yields
- Par bonds, premium bonds, and discount bonds
- Par rates

Reading

- Tuckman and Serrat, Chapters 2 and 3

Definition of Yield

Suppose a bond (or portfolio of bonds) has price P and positive fixed cash flows $K_1, K_2, \ldots, K_n$ at times $t_1, t_2, \ldots, t_n$. Its yield to maturity is the single rate y that solves:

$$\frac{K_1}{(1+y/2)^{2t_1}} + \frac{K_2}{(1+y/2)^{2t_2}} + \ldots + \frac{K_n}{(1+y/2)^{2t_n}} = P$$

or

$$\sum_{j=1}^{n} \frac{K_j}{(1+y/2)^{2t_j}} = P$$

Note that the higher the price, the lower the yield.

Example

- Recall the 1.5-year, 8.5%-coupon bond.
- Using the zero rates 5.54%, 5.45%, and 5.47% from last lecture, the bond price is 1.043066 per dollar par value.
- That implies a yield of 5.4704%:

$$\frac{0.0425}{(1+0.0554/2)^1} + \frac{0.0425}{(1+0.0545/2)^2} + \frac{1.0425}{(1+0.0547/2)^3}$$
$$= 1.043066$$
$$= \frac{0.0425}{(1+0.054704/2)^1} + \frac{0.0425}{(1+0.054704/2)^2} + \frac{1.0425}{(1+0.054704/2)^3}$$

Bond Yields and Zero Rates

- Recall that we can construct coupon bonds from zeroes, and we can construct zeroes from coupon bonds. So zero prices imply coupon bond prices and coupon bond prices imply zero prices. Therefore, zero rates imply coupon bond yields and coupon bond yields imply zero yields.
- I.e., by the **law of one price**,
  $P(c,T) = (c/2)d_{0.5} + (c/2)d_1 + \ldots + (1+c/2)d_T$
  $= (c/2)/(1+r_{0.5}/2) + (c/2)/(1+r_1/2)^2 + \ldots(1+c/2)/(1+r_T/2)^{2T}$
- At the same time, by definition of bond yield y,
  $P(c,T) = (c/2)/(1+y/2) + (c/2)/(1+y/2)^2 + \ldots(1+c/2)/(1+y/2)^{2T}$
- So the **law of one price in terms of yields** is:
  $(c/2)/(1+r_{0.5}/2) + (c/2)/(1+r_1/2)^2 + \ldots(1+c/2)/(1+r_T/2)^{2T}$
  $= (c/2)/(1+y/2) + (c/2)/(1+y/2)^2 + \ldots(1+c/2)/(1+y/2)^{2T}$
- I.e., a bond's yield is a kind of average of the zero rates that correspond to its cash flows.
- This leads to the so-called "coupon effect" in bond yields.

Example of a Coupon Bond Yield Implied by Zero Rates

Compare the two formulas for the 1.5-year 8.5%-coupon bond:

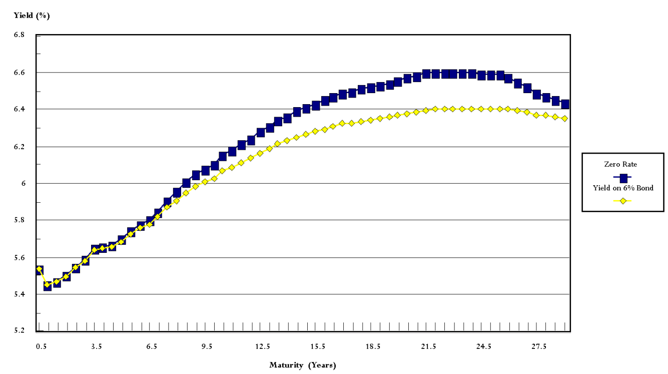
$$1.043066 = \frac{0.0425}{(1+0.0554/2)^1} + \frac{0.0425}{(1+0.0545/2)^2} + \frac{1.0425}{(1+0.0547/2)^3}$$

$$1.043066 = \frac{0.0425}{(1+0.054704/2)^1} + \frac{0.0425}{(1+0.054704/2)^2} + \frac{1.0425}{(1+0.054704/2)^3}$$

- The yield of 5.4704% is a kind of average of the zero rates 5.54%, 5.45%, and 5.47%.
- Math result: To be more precise, the yield of a portfolio is approximately the *dollar-duration-weighted average yield* of its pieces, as we shall see later.

Yield Curves for Zeroes and 6% Bonds

Here are the yield curves based on the historical STRIPS rates used for most of the course examples.

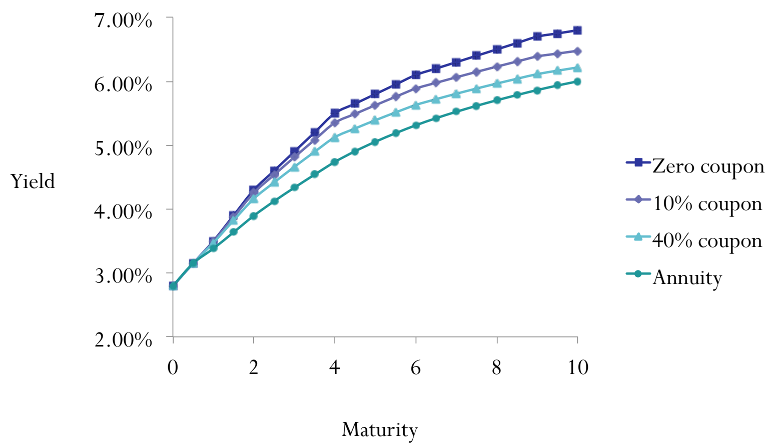


Why does the coupon bond yield curve lie below the zero curve?

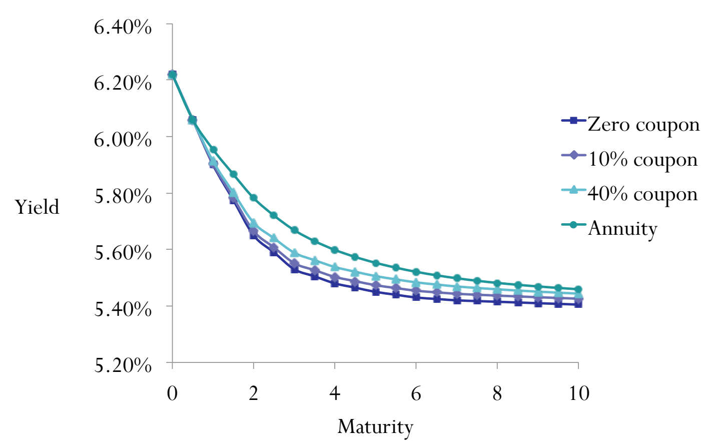
Caution Using Yields! The Coupon Effect

- Yields vary across bonds for different reasons – credit, liquidity, maturity. It turns out they also vary with coupon.
- **Proposition 1** If the yield curve is not flat, then bonds with the same maturity but different coupons will have different yields.
- **Proposition 2** If the yield curve is upward-sloping, then for any given maturity, higher coupon bonds will have lower yields.
- **Proposition 3** If the yield curve is downward-sloping, then for any given maturity, higher coupon bonds will have higher yields.

Upward Sloping Yield Curve (10/30/1992)



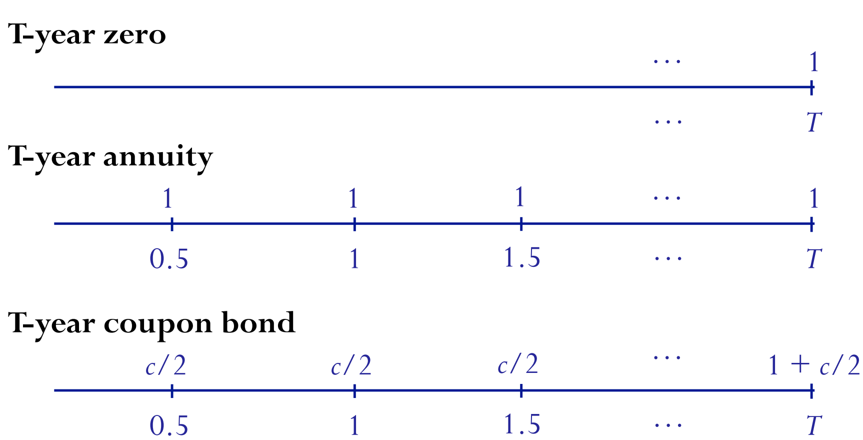
Downward Sloping Yield Curve (11/30/2000)



Why the coupon effect? Start by comparing zero rates and annuity yields

- An annuity for a given maturity pays \$1 each period until maturity, let's say every six months.
- The annuity yield is an average of the zero rates associated with each of its cash flows.
- If the zero yield curve is upward sloping,
  - the annuity yield curve will be upward sloping too, because each time we extend the annuity maturity, we introduce another, higher, zero rate into the average.
  - Also, the annuity yield for a given maturity will be lower than the zero rate for that maturity, because it is the average of the zero rates associated with its cash flows. So it's lower than maximum, which is the zero rate for that maturity.

Zeroes, Annuities, and Coupon Bonds



T-year coupon bond = 1 T-year zero + $c/2$ T-year annuities

Example of Zero Rates and Annuity Rates

For nice round numbers, use the zero rates and zero prices below for an example. What are the corresponding annuity prices and rates?

Maturity	Zero rate	Zero price	Annuity price	Annuity yield
0.5	2%	0.9901	0.9901	2%
1.0	3%	0.9707	1.9608	2.66%
1.5	4%	0.9423		

- A 0.5-year annuity pays \$1 at time 0.5, so it is the same as the 0.5-year zero.
- A 1-year annuity pays \$1 at time 0.5 and \$1 at time 1, so it is the sum of the 0.5-year zero and the 1-year zero.
  - Its price is 0.9901 + 0.9707 = 1.9608.
  - Its yield is y such that $\frac{1}{1+y/2}+\frac{1}{(1+y/2)^2}=1.9608 \Rightarrow y=2.66\%$
- The annuity yield is a kind of average of the zero rates corresponding to its cash flows, in this case, a kind of average of the 2% and the 3%.

Class Problem: Zero and Annuity Rates

Take the zero rates and zero prices below as given.
What is the 1.5-year annuity rate?

Maturity	Zero rate	Zero price	Annuity price	Annuity yield
0.5	2%	0.9901	0.9901	2%
1.0	3%	0.9707	1.9608	2.66%
1.5	4%	0.9423	?	?

Example of Zero and Annuity Yield Curves

Maturity	Zero rate	Zero price	Annuity price	Annuity yield
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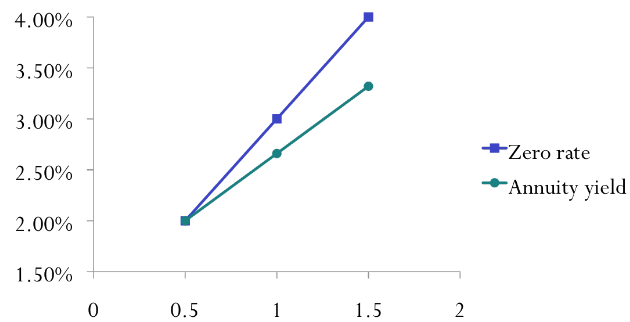
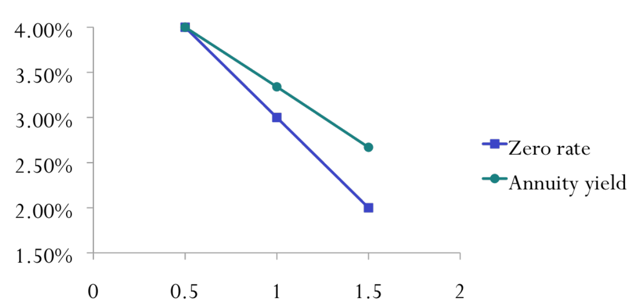


Zero rates and annuity yields: Downward-sloping yield curve

- By the same logic, if the zero yield curve is downward sloping,
  - the annuity curve will be downward sloping
  - and the annuity yield will be higher than the zero rate of the same maturity, because the average is greater than the minimum.



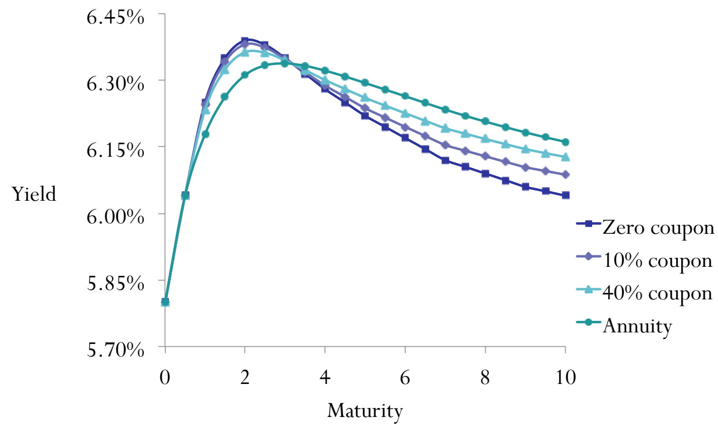
Now coupon bonds and the coupon effect..

- Every coupon bond consists of a coupon stream and a par payment.
- So a coupon bond of a given maturity is a combination of an annuity and a zero with that same maturity.
- So the yield on the coupon bond of a given maturity is an average of the annuity yield and the zero rate for that same maturity.
  - The higher the coupon, the closer the bond's yield is to the annuity rate.
  - The lower the coupon, the closer the bond's yield is to the zero rate.

The coupon effect in upward or downward sloping yield curves…

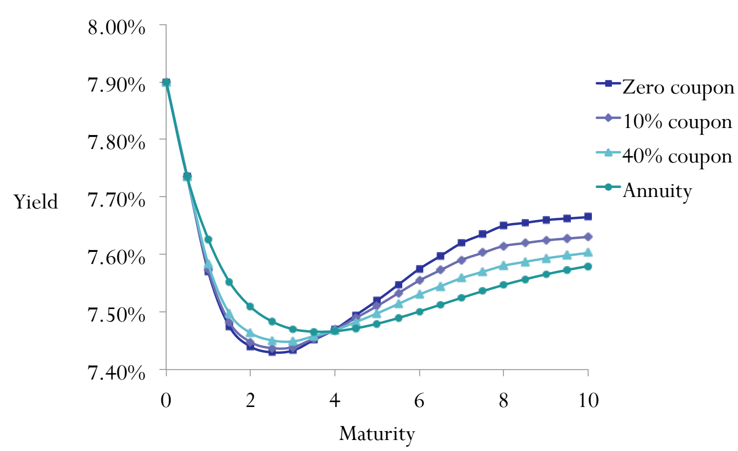
- In an upward-sloping yield curve, zero rates are higher than annuity rates for the same maturity, so lower coupon bonds have higher yields.
- In a downward-sloping yield curve, zero rates are lower than annuity rates, so lower coupon bonds have lower yields.

Humped Yield Curve (3/31/2000)



Must all curves intersect at the same point?

Inverted Humped Yield Curve (7/31/1989)



Yield of a Bond on a Coupon Date

For an ordinary semi-annual coupon bond on a coupon date, the yield formula is

$$P = \frac{c}{2}\sum_{s=1}^{2T}\frac{1}{(1+y/2)^s} + \frac{1}{(1+y/2)^{2T}}$$

where c is the coupon rate and T is the maturity of the bond in years.

Annuity Formula

Math result: $$\sum_{s=1}^{2T}\frac{1}{(1+y/2)^s} = \frac{1}{y/2}\left(1-\frac{1}{(1+y/2)^{2T}}\right)$$

Finance application:

This formula gives the present value of an annuity of $1 to be received every period for n periods at a simply compounded rate of *r* per period.

Yield-to-Price Formula for a Coupon Bond

Value the coupon stream using the annuity formula:

$$P = \frac{c}{y}[1 - \frac{1}{(1+y/2)^{2T}}] + \frac{1}{(1+y/2)^{2T}}$$

- The closed-form expression simplifies computation.
- Note that if $c=y$, $P=1$ (the bond is priced at par).
- If $c>y$, $P>1$ (the bond is priced at a *premium* to par).
- If $c<y$, $P<1$ (the bond is priced at a *discount*).
- The yield on a zero is the zero rate: $c=0$; $y=r_T$

Class Problem: Suppose the 1.5-year 8.5%-coupon bond is priced to yield 9%. What is its price per $1 par?

Par Rates

- The *par rate* for a given maturity T is the coupon rate that makes a T-year coupon bond sell for par.
- Of course, the yield on the bond will also be the par rate.
- Since coupon bonds are usually issued at par, par rates are yields on newly issued bonds.

Par Rate in Terms of Zero Prices

- In practice, bond pricing data usually comes in the form of par rates – yields on newly issued bonds that are sold at par.
- In other cases, we might want to compute par rates from zero prices. This is one yield computation that is explicit:
- For each maturity T, the par rate c_T is the coupon rate that sets the bond price equal to par, i.e.,

$$(c_T/2)\times d_{0.5}+(c_T/2)\times d_1+(c_T/2)\times d_{1.5}+\ldots+(c_T/2)\times d_T+1\times d_T=1$$

so in terms of zero prices d_t, the T-year par c_T is

$$c_T=\frac{2(1-d_T)}{d_{0.5}+d_1+d_{1.5}+\ldots d_T}$$

Class Problem

Solve for the 2-year par rate in the term structure below:

Maturity	Zero Rate	Zero Price
0.5	5.54%	0.973047
1.0	5.45%	0.947649
1.5	5.47%	0.922242
2.0	5.50%	0.897166

Yield Curves for Zeroes and Par Bonds