CHAPTER TWO
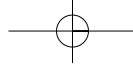# A REVIEW OF ELEMENTARY MATHEMATICS: FUNCTIONS AND OPERATIONS

2.1   INTRODUCTION

This chapter contains a review of some of the most basic mathematics functions and operations required for financial analysis. Anyone interested in pursuing study in finance should be very comfortable working through the material in this chapter (and chapter 3) prior to enrolling in an introductory finance course. Most of the concepts presented in the early part of this chapter will likely have been covered during the reader's high-school (or junior high-school) math sequence; however, many students will benefit from a review. Some readers will wish to merely skim this material before progressing to subsequent chapters. Readers with weaker mathematics preparation may find sections and applications followed by an asterisk (*) somewhat more difficult. Such readers may prefer to work through relevant exercises at the end of the chapter before spending too much time with this material. In some cases, the more difficult material may be skipped entirely.

Students in any introductory or intermediate finance course must be comfortable working with material in sections 2.2 through 2.8. Hence, students should be certain that they are comfortable with this material before attempting any study in finance. Other sections may be important as well, even in many introductory finance courses, depending on the reader's interests or exact course or program content. For example, material in section 2.9 will be needed by students planning study of material related to portfolio management, section 2.10 will be needed by students planning study in interest rate theory or fixed income management, and the remainder of the chapter will be needed by students interested in the pricing of options contracts and other derivatives.

2.2   VARIABLES, EQUATIONS, AND INEQUALITIES

Mathematics is the primary language of financial model-building, while variables, functions, equations, and inequalities are the building blocks of financial models. A

variable, usually represented as a letter or group of letters, denotes a quantity that can be assumed from a given set of numbers. A *function* is a rule that assigns to each element x of a set X a unique element y of a set Y. For example, the following reads "y is a function of x":

$$y = f(x), \tag{2.1}$$

where y is a dependent variable and x is an independent variable. The variable y may also be expressed as a function of a series of independent variables. Such a function might be described more explicitly by specifying a particular mathematical operation to define the rule relating the sets. For example, we could write one example of the above functional relationship as

$$y = 3x + 4.$$

In this equation, we used two arithmetic operations, multiplication and addition. An *equation* such as the one we just considered is a statement that two mathematical expressions are equal, while an inequality states that one expression is greater than ($>$) or less than ($<$) another.

2.3   EXPONENTS

After the four basic arithmetic operations, exponentiation is the most commonly used operation in finance. Exponents enable us to express a repetitive series of multiplications or divisions in a concise form. They are particularly useful in finance where we frequently need to evaluate a large number of payments extending over many years. Exponents are also useful for expressing various types of "roots" of numbers or variables. Exponents are sometimes referred to as "powers."

Consider the function $y = 3^2$, which reads "y equals three squared" or "y equals three to the power two." The value of y is $3 \cdot 3 = 9$. We refer to the "power" 2 as the exponent of 3. More generally, we may write functions of the following form:

$$y = x^n, \tag{2.2}$$

where n represents the exponent for variable x. If n is a positive integer (a whole number greater than zero), it is an exponent referring to the number of times minus one the variable x is to be multiplied by itself. Thus, for example, if n were to equal 4, y would equal $x \cdot x \cdot x \cdot x = x^4$; that is, x is multiplied by itself three times. Clearly, it is more efficient to write x^4 than $x \cdot x \cdot x \cdot x$. Where $y = x^n$ and x is 3 and n is 4, $y = 3^4 = 81$.

The exponent n can also be a fractional or negative number. A negative exponent associated with x means that y is the inverse of x^n; that is, $y = x^{-n}$, which is the inverse of x^n:

$$y = x^{-n} = \frac{1}{x^n}. \tag{2.3}$$

Thus, $x^{-1} = 1/x$ and $x^{-2} = 1/x^2$. If $y = 4^{-2}$, then $y = 1/4^2 = 1/16$. Note that one may multiply any term by its inverse to obtain 1. For example, $x^2 \cdot x^{-2} = x^2 \cdot 1/x^2 = 1$. If n is a fractional value, y requires finding the "nth root" of x:

$$y = x^{1/n} = \sqrt[n]{x}. \qquad (2.4)$$

Thus, $x^{1/2} = \sqrt{x}$ (the square root of x equals $x^{1/2}$) and $x^{1/3} = \sqrt[3]{x}$. For example, $4^{1/2} = 2$ and $64x^{1/3} = 4$. Furthermore, if $y = x^{m/n}$, then $y = \sqrt[n]{x^m}$. For example, if $y = 4^{3/2}$, $y = 8$. If $y = x^{-m/n}$, then $y = 1 \div \sqrt[n]{x^m}$.

When the exponent of a real number equals zero, the function has a value equal to one; that is, $x^0 = 1$ for any real x. The product of any base x with exponents n and m equals the base x with an exponent equal to the sum of the original exponents; that is, $x^n x^m = x^{n+m}$. For example, $y = 4^3 \cdot 4^2 = 4^{3+2} = 1{,}024$. Finally, $(x^n)^m$ equals x^{nm}.

APPLICATION 2.1: INTEREST AND FUTURE VALUE
(A more complete presentation of this material is provided in chapter 4)

Suppose we deposit $100 into a bank account paying interest at an annual rate of 5% compounded annually.[1] The total amount in our account after one year will be $105, or 1.05 times our original deposit amount of $100:

$$\$100 \cdot 1.05 = \$105.$$

If we left our $105 in the account for a second year, our account would have a future value of $105 \cdot 1.05 = \$110.25$. Thus, our account will have a future value of $\$100 \cdot 1.05^2$:

$$\$100 \cdot 1.05 \cdot 1.05 = \$100 \cdot 1.05^2 = \$110.25.$$

More generally, our account will have a future value of $\$100 \cdot 1.05^n$, where n is the number of years that the account will accumulate interest. Again, the concepts of interest and future value are dealt with more extensively in chapter 4.

We can use negative exponents to determine how much we would have to deposit now to obtain a given future value in the future. Suppose we wished to make a deposit now into a savings account paying 5% interest such that the amount in the account in two years were $100. Thus, how much would we have to deposit now to obtain a given future value in two years? We find the solution to this problem as follows: $\$100 \cdot 1.05^{-2} = \$90.70$. This problem can be written

$$\$100 \cdot 1.05^{-2} = \frac{\$100}{1.05^2} = \$90.70.$$

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[1] Compound interest means that interest accrues on accumulated interest on loans as well as principal.

2.4 THE ORDER OF ARITHMETIC OPERATIONS AND THE RULES OF ALGEBRA
(Background reading: section 2.3)

Many equations require combinations of different types and series of operations. Mathematicians use well-established rules for sequencing operations in more complicated equations, which may be summarized as follows:

A Operations within innermost sets of parentheses (or brackets) are performed first. After operations within innermost sets of parentheses are performed, operations may be performed within sets moving to outer levels, until operations within the outermost level are performed. For example, consider the expression: $10 \cdot \{[3 + (6 \cdot 4)] \cdot [5 \cdot (7 + 11)]\}$. This expression simplifies to $10 \cdot \{[3 + (24)] \cdot [5 \cdot (18)]\}$, and further simplifies to $10 \cdot \{[27] \cdot [90]\} = 10 \cdot \{2,430\} = 24,300$.

B Varying types of operations within a given level of parentheses are performed in the following order:
1 Exponents.
2 Multiplication and division (do these in any order).
3 Addition and subtraction (do these in any order).

Consider the equation $y = 12 + 4 \cdot 2^3$. This equation simplifies to $y = 12 + 4 \cdot 8 = 12 + 32 = 44$. The following equation may be simplified using a combination of rules A and B above:

$$y = 3 + \{4[3(2 \cdot 6 + 1) + 2(4 + 3 \cdot 4^{(1+2)}) + 7]^2\} + 3\{2 + [4 \cdot 2^3 \cdot (3 + 5^2)(2 + 4)]\},$$

$$y = 3 + \{4[3(12 + 1) + 2(4 + 3 \cdot 4^{(3)}) + 7]^2\} + 3\{2 + [4 \cdot 2^3 \cdot (3 + 25)(6)]\},$$

$$y = 3 + \{4[3(13) + 2(4 + 3 \cdot 64) + 7]^2\} + 3\{2 + [4 \cdot 8 \cdot (28)(6)]\},$$

$$y = 3 + \{4[39 + 2(4 + 192) + 7]^2\} + 3\{2 + [5,376]\},$$

$$y = 3 + \{4[39 + 2(196) + 7]^2\} + 3\{5,378\},$$

$$y = 3 + \{4[39 + 392 + 7]^2\} + 16,134,$$

$$y = 3 + \{4[438]^2\} + 16,134 = 3 + \{4 \cdot 191,844\} + 16,134$$
$$= 3 + 767,376 + 16,134,$$

$$y = 783,513.$$

Notice that in the earlier formulations of the equations, higher levels of parentheses are distinguished with brackets [ ] and braces { }.

Now, consider the following equation:

$$PVA = \$1,000 \cdot \left[\frac{1}{0.05} - \frac{1}{0.05 \cdot (1 + 0.05)^5} \right].$$

This equation, which is used for evaluating series of cash flows over time (see section 4.9 for more details on its application) can be simplified and solved as follows:

$$PVA = \$1{,}000 \cdot \left[\frac{1}{0.05} - \frac{1}{0.05 \cdot (1 + 0.05)^5}\right] = \$1{,}000 \cdot \left[\frac{1}{0.05} - \frac{1}{0.05 \cdot (1.05)^5}\right]$$

$$= \$1{,}000 \cdot \left[\frac{1}{0.05} - \frac{1}{0.05 \cdot 1.27628}\right] = \$1{,}000 \cdot \left[\frac{1}{0.05} - \frac{1}{0.063814}\right]$$

$$= \$1{,}000 \cdot \left[\frac{1}{0.05} - 15.670523\right] = \$1{,}000 \cdot [20 - 15.670523]$$

$$= \$1{,}000 \cdot 4.329476 = \$4{,}329.48.$$

APPLICATION 2.2: INITIAL DEPOSIT AMOUNTS
(A more complete presentation of this material is
provided in chapter 4)

Suppose that you wished to have \$1,000 in your bank account in five years. Your bank account pays interest at an annually compounded rate of 10% or 0.10. How much would you need to deposit in your account today for your balance to be \$1,000 in five years?

First, we note that the account balance $(1 + 0.10)$ times as great at any point in time as it was one year earlier because of the 10% interest rate. For example, the account will have a balance of \$1,000 after five years; in four years, the account will have a balance of \$909.91 (which is \$1,000 $\div$ $[1 + 0.1]$). Furthermore, the account balance will be $(1 + 0.1)^n$ times as large as n years earlier. We know that the future value of the account will be \$1000; what will be the initial deposit amount? Let X_0 represent the initial deposit amount of the account, solved for as follows:

$$X_0 = \frac{\$1{,}000}{(1 + 0.10)^5} = \frac{\$1{,}000}{(1.10)^5} = \frac{\$1{,}000}{1.61051} = 620.92.$$

Note the order of arithmetic operations in the above equation.

2.5   THE NUMBER E

The number e, also known as the base of the natural exponential function, has a non-repeating, nonterminating decimal value of approximately 2.7182818. The number e may be computed more precisely as follows:

$$e = \left(1 + \frac{1}{\infty}\right)^{\infty} \approx 2.7182818. \tag{2.5}$$

Table 2.1 Computing the number e

m	$y = \left(1 + \dfrac{1}{m}\right)^{m}$	y
1	$y = (1 + 1/2)^{2}$	2
2	$y = (1 + 1/3)^{3}$	2.25
3	$y = (1 + 1/4)^{4}$	2.37037037
4	$y = (1 + 1/1)^{1}$	2.44140625
5	$y = (1 + 1/5)^{5}$	2.48832
6	$y = (1 + 1/6)^{6}$	2.521626372
7	$y = (1 + 1/7)^{7}$	2.546499697
8	$y = (1 + 1/8)^{8}$	2.565784514
9	$y = (1 + 1/9)^{9}$	2.581174792
10	$y = (1 + 1/10)^{10}$	2.59374246
100	$y = (1 + 1/100)^{100}$	2.704813829
1,000	$y = (1 + 1/1,000)^{1,000}$	2.716923932
100,000	$y = (1 + 1/100,000)^{100,000}$	2.718268237
10,000,000	$y = (1 + 1/10,000,000)^{10,000,000}$	2.718281694

Obviously, one cannot use the arithmetic operations described thus far to divide by infinity or raise the sum to the power infinity. However, one can replace the value for infinity with the variable m and observe what happens to the value of the function y in the following as m is replaced by integers of increasing value:

$$y = \left(1 + \frac{1}{m}\right)^{m}.$$

As m approaches infinity, y approaches the value of the number e. Table 2.1 provides computations to demonstrate how y approaches e as m approaches infinity. The number e is defined again in chapter 8. This number e is used as the base for the natural log function (discussed in the next section) and is particularly useful for computations involving continuous growth or compounding (see chapter 4).

2.6 LOGARITHMS
(Background reading: sections 2.2 and 2.5)

A logarithm is a power to which another number (called the base) must be raised to obtain a given value. For example, suppose we wish to find the power n to which 10 must be raised in order to obtain 100; that is, we wish to solve $100 = 10^{n}$ for n. The number 2 is the appropriate power or exponent such that $100 = 10^{2}$. The number 10 is our base and 2 is the logarithm (or log) of 100. Thus we say that 2 is the base ten log of 100. The base ten logs are the second most commonly used in finance.
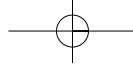
The most commonly used log in finance is the natural log (also called a Napierian log). The base of the natural log is the number e, which as we observed in section 2.5, has a nonrepeating, nonterminating decimal value of approximately 2.7182818. The natural log is particularly useful for certain types of computations involving continuous growth or compounding (see chapter 4). If $x = \ln(y)$, we say that x is the natural log of y and $y = e^x \approx 2.718^x$. For example, the natural log of 22 is approximately 3.091 (i.e., $\ln(22) \approx 3.091$), which means that $e^{3.091} \approx 22$ or $2.718^{3.091} \approx 22$.

The following rules of operation apply to operations involving logs of a given base:

$$\log(x \cdot y) = \log(x) + \log(y),$$

$$\log(x \div y) = \log(x) - \log(y),$$

$$\log(x^y) = y \cdot \log(x).$$

Logs are useful when solving for n in exponential functions such as $y = cx^n$. If $y = cx^n$, then from the rules of operation above, the following must be true: $\ln(y) = \ln(c) + n \cdot \ln(x)$. Suppose, for example, that we wished to solve for n in the following: $50 = 3 \cdot 7^n$. We may do so by finding the logs of both sides of the equation: $\ln(50) = \ln(3) + n \cdot \ln(7)$. Solving, we obtain $3.912 = 1.0986 + n \cdot 1.9459$; $n = 1.4458$.

APPLICATION 2.3: THE TIME NEEDED TO DOUBLE ONE'S MONEY (You may also wish to read related material in chapter 4)

How long would it take to double the balance of a bank account paying interest compounded at an annual rate of 5%? For example, how long would it take $1,000 to double to $2,000 in a 5% bank account? We determine this by solving the following for n:

$$\$2,000 = \$1,000(1 + 0.05)^n.$$

First, divide both sides by 1,000 and find the natural logs of both sides, to obtain

$$\ln\left(\frac{2,000}{1,000}\right) = \ln[(1 + 0.05)^n] = n \cdot \ln(1.05).$$

We rewrite this equation to obtain

$$\ln(2) = n \cdot \ln(1.05),$$

$$0.693147 = n \cdot 0.04879.$$

We solve for n, the number of years required to double our money in the bank account, as follows:

$$\frac{0.693147}{0.04879} = n = 14.2067.$$

$1,000 must be left in this 5% bank account for 14.207829 years to double. A simple rule of thumb based on this analysis is sometimes used by practitioners known as the *Rule of 72*. If one divides 72 (it is slightly higher than 100 times the natural log of 2) by the interest rate of the account expressed as a percentage (in this case, 5% which is slightly higher than 100 times the natural log of 1 plus the interest rate), one can *approximate* the length of time required for a bank account to double. In this case, our approximation would hold that the account would require 72/5 = 14.4 years to double.

2.7 SUBSCRIPTS

Consider the expression x_i, which might be read "x subscript i" or "x sub i." Subscripts are variables or numbers that serve to identify the variables with which they are associated. For example, in the expression $x_1 + x_2 + x_3$, which reads "x sub one plus x sub two plus x sub three," the subscripts distinguish the x variables from each other. Sometimes it is more practical to identify variables with subscripts than with different letters, especially when there are many variables or there is a well-defined order to them.

Frequently, the subscript t is used to identify time periods. For example, suppose that an investment will pay its owner $100 in one year, $200 in two years, $300 in three years, and $400 in four years. Let the variable CF_t refer to the cash flow (payment) generated by the investment for a given year t. The following table associates each cash flow with its appropriate identifying subscript:

t	CF_t
1	$CF_1 = \$100$
2	$CF_2 = \$200$
3	$CF_3 = \$300$
4	$CF_4 = \$400$

2.8 SUMMATIONS
(Background reading: sections 2.3 and 2.7)

The summation, or sigma, notation (Σ) is a "shorthand" means of expressing our intent to add a series of variables. Consider the following expression:

$$y = \sum_{i=1}^{n} x_i, \tag{2.6}$$

which reads "y equals the sum from i equals 1 to n of x sub i." This expression orders values of x from one (the first x) to n (the last x) and sums them. The subscript i serves as the counter (its value increases by one as we continue the operation) and the integer under the uppercase sigma (Σ) is the starting value for the order. The stopping point for
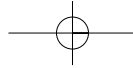
the order is the superscript n. The sigma means that we will add all of the x values within the stated range.

Suppose that we wish to sum the following ordered series of variables: x_1, x_2, x_3, and x_4. There are two ways to represent this operation:

$$\sum_{i=1}^{4} x_i = x_1 + x_2 + x_3 + x_4.$$

If the series is long or involves many variables, the summation notation is a more efficient means of representing the summation operation. For most functions involving the summation notation, it will be necessary to perform another series of operations before adding. For example, consider the following:

$$z = \sum_{i=1}^{3} x_i^2 = x_1^2 + x_2^2 + x_3^2,$$

which reads "z equals the sum from i equals one to three of x squared." Each of the x values must be squared before the summation takes place. Suppose, for example that $x_1 = 3$, $x_2 = 6$, $x_3 = 9$, and $x_4 = 12$. Then z would equal $3^2 + 6^2 + 9^2 + 12^2 = 9 + 36 + 81 + 144 = 270$.

APPLICATION 2.4: MEAN VALUES
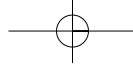### (You may also wish to read related material in section 5.1)

A stock generated the following profits over a five-year period:

Year	Profit ($)
1	15
2	5
3	2
4	24
5	4

Let the subscript t represent time (years 1 through 5) and let π_t represent profit for a given year. The stopping value n for our summation will equal 5. We can determine the average or mean ($\bar{\pi}$) profit over the five-year period as follows:

$$\bar{\pi} = \frac{\sum_{t=1}^{n} \pi_t}{n} = \frac{\sum_{t=1}^{5} \pi_t}{5} = \frac{15 + 5 + 2 + 24 + 4}{5} = 10.$$

The value that we have computed here is often referred to as the arithmetic mean. We simply add the observed values and divide by the number of observations.

2.9 DOUBLE SUMMATIONS
(Background reading: section 2.8)

Double summations are very useful when working with variables that are to be paired
and ordered for operations. The double summation notation is very useful for concisely
representing many of the types of long and repetitive series of operations that arise in
finance. The subscript notation can be generalized to denote a variety of character-
istics associated with a variable. For example, consider the following array or matrix
of numbers on the left, which can be identified by row and column numbers given in
the array on the right:

$$\begin{bmatrix} 7 & 4 & 9 \\ 6 & 4 & 12 \\ 3 & 2 & 17 \end{bmatrix} \quad \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

Each of the numbers in the first matrix is associated with a variable $a_{i,j}$, where the first
subscript i represents the row of element $a_{i,j}$ and j represents the column of element $a_{i,j}$.
Each of the three elements in a given row will have the same i subscript; each of the
elements in a given column will have the same j subscript. For example, since the num-
ber 12 is in the second row and third column, it is identified with variable $a_{2,3}$. Suppose
that we wished to add the elements in the three rows and columns. We could add ele-
ments across the top row, then across the second row and then across the bottom row
and finally add the three row totals. This operation is equivalent to

$$y = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{i,j} = [a_{1,1} + a_{1,2} + a_{1,3}] + [a_{2,1} + a_{2,2} + a_{2,3}] + [a_{3,1} + a_{3,2} + a_{3,3}],$$

$$y = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{i,j} = [7 + 4 + 9] + [6 + 4 + 12] + [3 + 2 + 17],$$

$$y = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{i,j} = [20] + [22] + [22] = 64.$$

We started with row 1 ($i = 1$) and summing the three variables in row 1. With $i = 1$,
we set $j = 1$, $j = 2$, and $j = 3$ in order. We summed the three variables $a_{1,1}$, $a_{1,2}$, and
$a_{1,3}$ to obtain 20. When we completed the summation in the first row, we moved to
the second row, setting i equal to 2. With $i = 2$, we set $j = 1$, $j = 2$, and $j = 3$ in order.
We summed the three variables $a_{2,1}$, $a_{2,2}$, and $a_{2,3}$ to obtain 23. We then repeated this
operation for the third row. Finally, we summed the three row totals $20 + 22 + 22$ to
obtain 64.
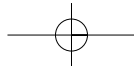
2.10 PRODUCTS
(Background reading: section 2.8)

The product, or $\prod$, notation provides a "shorthand" expression multiplying together a series of variables. Consider the following expression:

$$y = \prod_{i=1}^{n} x_i,$$

which reads "y equals the product from i equals 1 to n of terms labeled x sub i." This expression orders values of x from one to n and multiplies them. The subscript i serves as the counter (its value increases by one as we continue the operation) and the integer under the upper-case pi ($\prod$) is the starting value for the order. The stopping point for the order is the superscript n. The pi means that we will multiply together all of the x values.

Suppose that we wish to multiply the following ordered series of variables: x_1, x_2, x_3, and x_4. There are two ways to represent this operation:

$$\prod_{i=1}^{4} x_i = x_1 \cdot x_2 \cdot x_3 \cdot x_4.$$

If the series is long or involves many variables, the product notation is a more efficient means of representing the multiplication operations. For many functions involving the product notation, it will be necessary to perform another series of operations before multiplying. For example, consider the following:

$$z = \prod_{i=1}^{3} x_i^2 = x_1^2 \cdot x_2^2 \cdot x_3^2,$$

which reads "z equals the product from i equals one to three of x squared." Each of the x values must be squared before the multiplication takes place.

APPLICATION 2.5: GEOMETRIC MEANS
(Background reading: application 2.4)

Geometric means are particularly useful for situations involving compound interest, returns, or growth. The geometric mean of a series is computed as the nth root of the product of the series

$$\bar{x}_g = \sqrt[n]{\prod_{i=1}^{n} x_i}. \qquad (2.7)$$
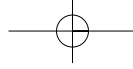
For example, suppose the value of a $1,000 security held by an investor grew by 10% in its first year and 20% in its second year, but lost 30% in its third year. Its final value would be determined as follows:

$$V_3 = 1{,}000 \cdot [(1 + 0.10) \cdot (1 + 0.20) \cdot (1 - 0.30)] = 924.$$

The arithmetic mean return on this investment is computed to be 0 ([0.10 + 0.20 − 0.30] ÷ 3 = 0), although we saw in the above equation that the investment actually lost value. This discrepancy is due to the fact that the arithmetic mean return does not account for the compounding of growth before the year it lost value. The geometric mean return, computed as follows, will account for the compounding of growth:

$$\bar{x}_g = \sqrt[3]{(1 + 0.10) \cdot (1 + 0.20) \cdot (1 - 0.30)} = \sqrt[3]{1.1 \cdot 1.2 \cdot 0.7} = \sqrt[3]{0.924} = 0.974.$$

We see that, on average, the investment will be 0.974 as large in each of the three years than in the previous year; thus, the geometric mean growth rate is computed as $1 - 0.974 = -0.026$.

APPLICATION 2.6: THE TERM STRUCTURE OF INTEREST RATES
(Background reading: application 2.5)

The term structure of interest rates is concerned with the change in interest rates on debt securities resulting from varying times to maturity on the debt. For example, it may be concerned with explaining why the interest rate on debt maturing in one year might be 4% versus 7% for debt maturing in 20 years. Generally, at a given point in time, we observe longer-term interest rates exceeding shorter-term rates, although this is not always the case (for example, the years 1980–3). There are several theories which attempt to explain relationship between long- and short-term interest rates. Among the best known is the Pure Expectations Theory, which states that long-term spot rates (interest rates on loans originating now) can be explained as a product of short-term spot rates and short-term forward rates (interest rates on loans committed to now but actually originating at later dates). For example, if we borrow now for three years at 5%, the three-year spot rate equals 5%. If we commit now with a bank to borrow in one year at 8% for one year, the one year forward rate for one year equals 8%. Thus, we may write $y_{0,3} = 0.05$ and $y_{1,2} = 0.08$, where the first subscript represents the year in which the loan is originated and the second subscript represents the year in which the loan is repaid. Where $y_{t,m}$ is the rate on a loan originated at time t to be repaid at time m, the Pure Expectations Theory defines the relationship between long- and short-term interest rates as follows:

$$(1 + y_{0,n})^n = \prod_{t=1}^{n} (1 + y_{t-1,t}). \tag{2.8}$$

Thus, the long-term spot rate $y_{0,t}$ is defined as the nth root of the product of the one-period spot rate $y_{0,1}$ and a series of one-period forward rates $y_{t-1,t}$ minus one. That is,
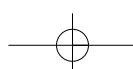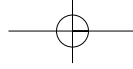
the long-term spot rate equals the geometric mean of the short-term (one-year) spot and forward rates minus one.

Consider an example where we can borrow money today for one year at 4%; $y_{0,1} = 0.04$. Suppose that we are able to obtain a commitment to obtain a one-year loan one year from now at an interest rate of 8%. Thus, the one-year forward rate on a loan originated in one year equals 8%. According to the Pure Expectations Theory, we compute the two-year spot rate as follows:

$$(1 + y_{0,2})^2 = \prod_{t=1}^{2} (1 + y_{t-1,t}) = (1 + 0.05)(1 + 0.08) = 1.134,$$

$$y_{0,2} = (1 + 0.05)(1 + 0.08)^{1/2} - 1 = \sqrt{1.134} - 1 = 0.0648944.$$

2.11 FACTORIAL PRODUCTS
(Background reading: section 2.10)

Factorial products are very useful for determining numbers of possible combinations of objects and for working through different types of probability distributions. A factorial product ($n!$) is defined as follows:

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3) \cdot \ldots \cdot 2 \cdot 1 = \prod_{i=0}^{n-1} (n - i). \qquad (2.9)$$

For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

Suppose that we wished to purchase five given stocks one at a time. How many possible orderings of transactions are there? There are five different stocks that we could purchase first. Given our first purchase, there are four different stocks that we could purchase second. Thus, there are $5 \cdot 4 = 20$ possible orderings for the first two purchases. Then, there are three possible orderings for the third purchase, given our first two purchases. This process continues until we notice that there are $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ possible orderings for the five stock purchases. Thus, $n!$ represents the number of potential orderings or permutations of a series of n events. We will make extensive use of factorial products later when we discuss probability and option pricing. We also note that $0! = 1$ and that $69! \approx 1.7112 \cdot 10^{98}$.

APPLICATION 2.7: DERIVING THE NUMBER E
(Background reading: section 2.5)

The number e was defined in section 2.5 and approximated at 2.718. A second definition which is often quite useful in finance and probability computations is given as follows:

$$e = \sum_{i=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \ldots \qquad (2.10)$$
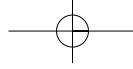
2.12 PERMUTATIONS AND COMBINATIONS
(Background reading: section 2.11)

A permutation of a set of objects is an ordered arrangement of some or all of the objects in the set. In section 2.11, we demonstrated that a set of n objects can be ordered $n!$ ways. More generally, a set of r objects taken from a set of n objects can be ordered $_nP_r$ ways where $_nP_r$ is defined as follows:

$$_nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1). \tag{2.11}$$

For example, suppose that an investor wishes to purchase 5 stocks from 12 in a particular industry. Suppose that she will select 5 stocks from the 12 and then enter purchases in sequence one at a time. How many possible orderings (permutations) of 5 purchases from an industry of 12 must she select from? This number is determined as follows:

$$_nP_r = {}_{12}P_5 = \frac{12!}{(12-5)!} = \frac{479{,}001{,}600}{5{,}040} = 95{,}040.$$

There are 12 possible selections for the first stock, 11 for the second, 10 for the third, and so on:

$$_nP_r = {}_{12}P_5 = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 95{,}040.$$

We have determined the possible number of orderings of stock purchases; now we will determine the number of possible combinations of stocks. We will now assume that the purchase order for the five stocks combines all five stocks, rather than assume five separate sequential purchase orders. Where $_nP_r$ represented the number of orderings of sequential purchases, $_nC_r$ (which reads "n choose r") will represent the number of possible combinations irrespective of order:

$$_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{_nP_r}{r!}. \tag{2.12}$$

In our example, there will be 792 possible combinations of 12 stocks from which the investor can select 5:

$$_{12}C_5 = \binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(5\cdot 4\cdot 3\cdot 2\cdot 1)\cdot(7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)}$$

$$_{12}C_5 = \binom{12}{5} = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{95{,}040}{120} = 792.$$
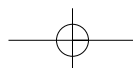
EXERCISES

2.1. Solve each of the following for y:

(a) $y = 5^2$;   (e) $y = 5^{-2}$;   (i) $y = 5^{1/2}$;   (m) $y = 5^{3/2}$;
(b) $y = 5^3$;   (f) $y = 5^{-3}$;   (j) $y = 5^{1/3}$;   (n) $y = 5^{-1/3}$;
(c) $y = 5^4$;   (g) $y = 5^{-4}$;   (k) $y = 5^{1/4}$;   (o) $y = 5^{3/4}$;
(d) $y = 25^2$;   (h) $y = 25^{-2}$;   (l) $y = 25^{1/2}$;   (p) $y = 25^{3/4}$.

2.2. Solve each of the following for y:

(a) $y = 5^2 \cdot 5^{-2}$;   (e) $y = 5^{1/2} \cdot 5^{3/2}$;
(b) $y = 5^3 \cdot 5^{-3}$;   (f) $y = 5^{1/3} \cdot 5^{-3}$;
(c) $y = 5^4 \cdot 5^{-4}$;   (g) $y = 5^{1/4} \cdot 5^{3/4}$;
(d) $y = 25^2 \cdot 25^{-2}$;   (h) $y = 25^{1/4} \cdot 25^{3/4}$.

2.3. Solve the following for *FV*: $FV = 50(1 + 0.3)^4$

2.4. Solve the following for *PV*: $PV = 180(1 + 0.2)^{-0.5}$

2.5. Simplify the following:

(a) $y = 4x^3 \cdot 6(5x + 7x)$;
(b) $y = 7(2x^2 + 3x - 5x^2) + 4x$;
(c) $y = 25 + \{2x[3x + 4(x + 5)^2 \cdot 6] - 5\} - 7x^2$.

2.6. Solve each of the following for y:

(a) $y = 9(4(7 + 3^2)^2)^{0.5} + 6^3 \cdot 5(2 + 7)$;
(b) $y = ((7 + 2^4)(3^3 \cdot 4^2 + 2)^3)^{-0.5}$.

2.7. Solve the following for *PVA*:

$$PVA = \$250 \cdot \left[\frac{1}{0.08} - \frac{1}{0.08 \cdot (1 + 0.08)^7} \right]$$

2.8. Find the following natural logs:

(a) $\ln(0.000001)$;   (d) $\ln(1.1)$;   (g) $\ln(3)$;   (j) $\ln(1,000)$.
(b) $\ln(0.5)$;   (e) $\ln(2.7)$;   (h) $\ln(10)$;
(c) $\ln(1)$;   (f) $\ln(2.718)$;   (i) $\ln(100)$;

2.9. Determine how long it takes to double the money in a bank paying interest compounded annually at each of the following rates:

(a) 0.03;   (c) 0.07;   (e) 0.12;
(b) 0.05;   (d) 0.10;   (f) 0.15.

2.10.   Work through each of the parts of problem 2.8 using the Rule of 72.

2.11.   The future value of an account whose interest is compounded can be approximated with the following future-value formula:

$$FV_n = PV \cdot e^m.$$

A simple rule used by practitioners holds that the length of time that it takes for an account to double when interest is compounded equals 0.72 divided by the interest rate (expressed as a decimal). Demonstrate this rule to be approximately correct using the future-value equation expressed above.

2.12.   Demonstrate that the number e is the future value of an account with an initial deposit of $1 and an interest rate of 0.10 compounded continuously.

2.13.   Consider the following series:

t	X
1	15
2	25
3	35
4	45

(a) Identify each of the variables X with a subscript t.
(b) What is the value of $\sum_{t=1}^{4} X_t$?
(c) What is the value of $\sum_{t=1}^{4} 3X_t$?
(d) What is the value of $\sum_{t=1}^{4} 3X_t^2$?
(e) What is the value of $\prod_{t=1}^{4} 3X_t^2$?
(f) What is the arithmetic mean of X_t?
(g) What is the geometric mean of X_t?

2.14.   Let $w_1 = 0.4$, $w_2 = 0.5$, and $w_3 = 0.1$. Furthermore, the following table (or "matrix") provides values for $x_{i,j}$:

$$\begin{bmatrix} 7 & 4 & 9 \\ 6 & 4 & 12 \\ 3 & 2 & 17 \end{bmatrix}.$$

(a) Compute $\sum_{i=1}^{3} w_i$.
(b) Compute $\sum_{i=1}^{3} \sum_{j=1}^{3} w_i w_j x_{i,j}$.
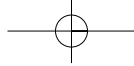
2.15. Suppose that the one-year spot rate $y_{0,1}$ of interest is 5%. Investors are expecting that the one-year spot rate one year from now will increase to 6%; thus, the one-year forward rate $y_{1,2}$ on a loan originated in one year is 6%. Furthermore, assume that investors are expecting that the one-year spot rate two years from now will increase to 7%; thus, the one-year forward rate $y_{2,3}$ on a loan originated in two years is 7%. Based on the pure expectations hypothesis, what is the three-year spot rate?

2.16. Solve the following for *FV*: $FV = \sum_{t=1}^{4} 25(1 + 0.10)^t$.

2.17. Solve each of the following for *x*:

(a) $x = 8!$;
(b) $x = 9!$;
(c) $x = 0!$.

2.18. Solve each of the following:

(a) $_6P_4$;   (d) $_6C_4$;
(b) $_4P_4$;   (e) $_4C_4$;
(c) $_8P_2$;   (f) $_8C_2$.

APPENDIX 2.A   AN INTRODUCTION TO THE EXCEL™ SPREADSHEET

This appendix is intended to introduce the reader to a few of the bare essentials for working with spreadsheets such as Excel™, Lotus™, and Quattro®Pro. We will emphasize Excel here, because it is currently the best-selling spreadsheet program, but most of these basics will also apply to other spreadsheet programs as well. Excel is an electronic spreadsheet program designed to perform mathematics, statistics, and financial computations as intended by the user. The spreadsheet is a rectangular array comprising cells that are organized into rows identified by numbers and columns identified by letters. Normally, when one loads a new Excel file, the cursor is initiated in cell A1, the uppermost and leftmost cell on the spreadsheet. Arrow keys on the keyboard can be used to move the cursor about the screen.

The user creates a spreadsheet file by moving the cursor to appropriate cells and making entries with the keyboard or mouse. Three of the most important types of entries are values, labels, and formulas. Suppose, for example, that I wished to compose a simple income statement for a firm. The firm had $1,000 in sales and $600 in costs. I could enter labels in column A to identify the numbers and formula that I will enter in column B. As we see in table 2.A.1, I enter labels in cells A1, A2, and A3 for Sales, Costs and Profits. We enter numbers in cells B1 and B2 for Sales and Costs. In cell B3, I enter in the formula =b2-b3 to calculate profits.

The user should be aware of a few spreadsheet conventions. Label entries either begin with a letter or an apostrophe. Number entries begin with a number, decimal sign, plus

Table 2.A.1   A simple income statement example

	A	B
1	Sales	1000
2	Costs	600
3	Profits	=b2-b1

What you enter

	A	B
1	Sales	1000
2	Costs	600
3	Profits	400

What appears on your screen.

Table 2.A.2   A simple formula example

	A	B
1	Coefficient	3
2	Base	4
3	Exponent	2
4	Result	=b1*b2^b3

What you enter

	A	B
1	Coefficient	3
2	Base	4
3	Exponent	2
4	Result	48

What appears on your screen

sign, or minus sign. Formula entries can begin with an equals sign or a number. The sign for the multiplication operation is * and the exponent operation is ^. Thus, 3 times 4 squared would be entered in the spreadsheet as 3*4^2. If the user wished to allow the coefficient, base and exponent to vary, he could create a spreadsheet like that in table 2.A.2. I enter numerical values for my coefficient, base, and exponent in cells B1, B2, and B3. The result to my equation is obtained by typing the formula =b1*b2^b3 into cell B4. Now, the user may change values in cells B1, B2, and B3 to obtain the result for any equation of the form $y = cx^n$.

To edit a formula or other entry, the user may move the cursor to the appropriate cell and left click, then move the cursor to the Formula Bar near the top of the screen (the entry in the cell should appear there). One can edit the cell in the Formula Bar. To save a worksheet file, the user may either click the diskette icon near the top of the screen or click File in the top left-hand corner, then Save As and follow instructions. To print a file, the user may either click the printer icon near the top of the screen or click File in the top left-hand corner, then Print and follow instructions. One exits by clicking the X button in the top right-hand corner of the screen. The Escape key (Esc) in the upper left-hand corner of the keyboard may sometimes be used to block an action that has been initiated.

Among the most useful functions for finance applications in Excel is the Paste Function (f_x) on the toolbar near the top of the screen. This function will return a menu of categories of formulas including financial, statistical, math, and trig functions. Work through the menu and follow instructions to use a particular function. Function definitions and help can be obtained by clicking Help or a question mark button on a page or pallette.

Perhaps the best way to learn how to use a spreadsheet is simply to experiment. Modern electronic spreadsheets are menu-driven. Menus and toolbars near the top of the screen will provide many options and categories of commands and functions. One of the most important things to remember while working is to *save frequently* and to back up your spreadsheet files, because accidents frequently occur that can cause you to lose work.