CHAPTER THREE

A Review of Elementary Mathematics: Algebra and Solving Equations

3.1 Algebraic Manipulations

(Background reading: section 2.4)

Algebraic manipulations are series or combinations of arithmetic operations on equations. The usual purpose for manipulating an equation is to solve for a given variable. Two types of manipulations will be used throughout this text:

1 *Addition or subtraction*. Here we either add or subtract a constant, variable or another equation to the equation we wish to manipulate.
2 *Multiplication*. Here we multiply (or divide) the equation by a constant (other than zero) or a variable.

These operations may usually be applied in any sequence or combination. In order to maintain the equality stated by an equation, we must perform the same series of arithmetic operations on both sides of the equality.

Suppose that we wished to solve the following for x: $24 = 15 + 2x$. We may subtract a constant (15) from both sides of the equation to obtain: $24 - 15 = 2x$, or $9 = 2x$. We may also divide both sides of our new equation by a constant (2) to obtain: $4.5 = x$. We now have successfully solved our original equation for x.

Consider the following example:

$$900 = \frac{1{,}000}{1 + r}.$$

We may solve for r by first multiplying both sides of the equation by $(1 + r)$ to obtain $900(1 + r) = 1{,}000$. Next, divide both sides of the equation by 900 to obtain $(1 + r) = 1{,}000/900 = 1.1111$. Finally, subtract 1 from both sides of the equation to obtain $r = 0.1111$.

Next, consider a slightly more complex equation:

$$900 = 500 + \frac{500}{(1+r)^2}.$$

To solve this equation for x, subtract 500 from both sides:

$$900 - 500 = \frac{500}{(1+r)^2}.$$

Simplify the left side and find the positive square root of both sides:[1]

$$\sqrt{400} = \frac{\sqrt{500}}{1+r}.$$

Continue to rewrite as follows:

$$\sqrt{400}(1+r) = \sqrt{500},$$

$$r = \frac{\sqrt{500}}{\sqrt{400}} - 1 = \frac{22.36068}{20} - 1 = 0.118034.$$

Geometric expansions

If your algebra is particularly "rusty," you may wish to work through a few of the exercises following this chapter and applications 3.1 through 3.3 before proceeding with this example, applying a technique known as a geometric expansion. In any case, suppose that we wished to solve the following equation for x: $1 = 0.5 + 0.5x + 0.5x^2 + 0.5x^3 + 0.5x^4 + 0.5x^5$. It will not be possible in this example to isolate x by itself on one side of the equation with everything else on the other side; that is, this equation has no closed form solution for x. We will need to find another solution algorithm or substitute values for x until we find one that works. However, this substitution process may be substantially simplified by reducing the length of the original equation. One method for reducing the length of this equation is to first multiply both sides by our variable (x) to obtain: $x = 0.5x + 0.5x^2 + 0.5x^3 + 0.5x^4 + 0.5x^5 + 0.5x^6$. If we subtract our original equation from our new equation, we are able to cancel a number of terms, thus simplifying the equation:

$$\begin{array}{rl} x = & \;\; 0.5x + 0.5x^2 + 0.5x^3 + 0.5x^4 + 0.5x^5 + 0.5x^6 \\ -1 = & -0.5 - 0.5x - 0.5x^2 - 0.5x^3 - 0.5x^4 - 0.5x^5 \\ \hline (x-1) = & -0.5 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad + 0.5x^6 \end{array}$$

[1] A positive number will have two square roots, one positive and one negative. We should take care to use the appropriate value to solve our equation. In finance examples, positive square roots will more frequently (although not always) provide the more meaningful solution than the negative square root.

Notice again how subtracting the equations canceled a number of terms. Our equation is much less cumbersome now. We have performed this procedure, known as a geometric expansion, to simplify our original equation. To simplify it further, we add 1 to both sides, to obtain $x = 0.5 + 0.5x^6$, or $0.5x^6 - x = -0.5$. This is probably the simplest form of our original equation (although the process of simplifying it did change the solution set for the equation, as we will see later). We may now substitute values for x until we find one (or more) that will work. As $0.5x^6 - x$ approaches -0.5 with our substituted values for x, our substituted values will approach the true value for x. In this example, we will find that our solution for x is approximately 0.5086603917. Verify this by substituting 0.5086603917 for x in our original equation. Although our original equation has only one solution (0.5086603917), our simplified equation created with the geometric expansion does have a second solution (check the value $x = 1$). We should be careful to test our solutions when we perform complicated algebraic manipulations, since our final simplified equations do not always retain the same values as our original equations. We will provide a more complete description of geometric expansions in section 3.4, discuss finance applications of the geometric expansion in chapter 4, and consider applications of substitution methods in chapter 5.

APPLICATION 3.1: PURCHASE POWER PARITY

An important concept in international finance is that a given commodity must sell for the same price (after adjusting for currency prices) in two countries. Of course, this law must be adjusted for differences in costs of providing the commodity, taxes, and so on. One well known (though somewhat tongue-in-cheek) test of the purchase power parity is the "Big Mac Standard," popularized by *The Economist*. McDonald's Corporation's Big Mac hamburgers are generally regarded to be more or less identical all over the world. If purchase power parity holds, then the Big Mac should sell for the same price in each country. For example, in the April 9, 1994 issue, *The Economist* reported that the Big Mac cost \$2.30 in a U.S. restaurant. The Big Mac cost £1.81 in the U.K. At the then prevailing exchange rate of \$1.46/£, the dollar equivalent cost was \$2.64; the British pound appeared overvalued by approximately 15%. In Thailand, the Big Mac cost Baht 48. At an exchange rate of Baht 25.3/\$, this represented a dollar cost of \$1.90 per Big Mac. The Baht appeared undervalued by 17%. Thus, the Law of One Price did not hold with respect to Big Macs. However, one must note that Big Macs are not easily exported out of countries where they are underpriced; thus, it is difficult for markets to adjust to purchase power parity violations. As we suggested earlier, this relationship among prices account for differences in taxes, subsidies, labor, and other production costs.

Suppose that we expected that purchase power parity should hold between the United States and Canada. Assume that the exchange rate between U.S. dollars and Canadian dollars is $CanD1/USD0.64$. That is, one dollar Canadian will purchase 0.64 dollars U.S. If $USD2.30$ purchases one Big Mac in the U.S., how much should a Big Mac cost in Canada? This problem is formulated as follows:

$$CanD\ Big\ Mac\ Cost = USD2.30 \cdot \frac{CanD1.00}{USD0.64} = 2.30 \cdot 1.5625 = CanD3.59375.$$

Now, consider an example involving three currencies concerning cross rates of exchange. Assume that 2.5 Swiss francs are required to purchase one U.S. dollar. If 0.64 U.S. dollars are required to purchase one Canadian dollar, how many Canadian dollars are required to purchase ten Swiss francs? If we assume that purchase power parity holds, how many Swiss francs are required to purchase one Big Mac? First, we will determine the Canadian dollar/Swiss franc exchange rate as follows:

$$SFr\ per\ CanD = 2.5SFr\ per\ USD \div \frac{CanD1.00}{USD0.64} = 2.5 \div 1.5625 = SFr1.6.$$

Thus, 10/1.6 = 6.25 Canadian dollars are required to purchase ten Swiss francs. In Switzerland, one Big Mac should cost 2.5 *SFr per USD* times 2.30 *USD* per Big Mac, or *SFr*5.75, when purchase power parity holds.

APPLICATION 3.2: FINDING BREAK-EVEN PRODUCTION LEVELS

Suppose that a firm produces a product that can be sold for a price of $P = \$10$ per unit such that its total annual revenues equals $TR = PQ = 10Q$. The variable Q represents the number of units of the product produced and sold by the firm. Further suppose that the firm incurs two types of costs in its production process, fixed and variable. Fixed costs FC include overhead expenditures totaling \$100,000 per year. Variable costs VC incurred by the firm include raw materials and direct labor. These costs are \$6 per unit produced. Total variable costs are simply the product of variable costs per unit VC and total production output Q. The firm's profit function is defined as follows as total revenues minus the sum of fixed and variable costs:

$$\pi = P \cdot Q - (FC + VC \cdot Q) = 10Q - (100{,}000 + 6 \cdot Q).$$

We can observe from the above equation that if the firm were to maintain an output level $Q = 0$, its profit would equal −100,000 due to its fixed costs. As the firm's output (and sales level) Q increases, profits π will increase. The firm's break-even production level is determined by solving for Q when profits π equal zero:

$$0 = 10Q - (100{,}000 + 6 \cdot Q).$$

First, we will add 100,000 to both sides:

$$100{,}000 = 10Q - 6Q.$$

Next, note that $10Q - 6Q = 4Q$; so that we will divide both sides of the above equation by 4, to obtain

$$100{,}000 \div 4 = Q^* = 25{,}000,$$

where Q^* is the break-even production level (the asterisk does not represent a product symbol here). Thus, the firm must produce 25,000 units to recover its fixed costs in order to break even. More generally, the following expression can be used to determine the break-even production level when the firm has linear revenue and cost functions:

$$Q^* = \frac{FC}{P - VC}.$$

The above expression is derived algebraically as follows:

$$0 = PQ^* - (FC + VC \cdot Q^*),$$
$$FC = PQ^* - VC \cdot Q^* = Q^*(P - VC).$$

Finally, divide both sides of the above equation by $P - VC$ to solve for Q^*.

APPLICATION 3.3: SOLVING FOR SPOT AND FORWARD INTEREST RATES (Background reading: sections 1.3 and 2.10, and application 2.6)

Application 2.6 described the long-term interest rate as a geometric mean of the short-term spot rate and a series of short-term forward rates of interest. If current long- and short-term spot rates of interest are known, we may be able to solve algebraically for forward rates of interest. Suppose, for example, that the one-year spot rate of interest equals 8% ($y_{0,1} = 0.08$) and the two-year spot rate equals 12% ($y_{0,2} = 0.12$). What would these spot rates imply about the one-year spot rate anticipated one year hence? That is, what is the one-year forward rate on a loan originated one year from now? Given the pure expectations structure discussed in application 2.6, we wish to solve for $y_{1,2}$ in the following:

$$(1 + y_{0,2})^2 = (1 + y_{0,1})(1 + y_{1,2}) = (1 + 0.12)^2 = (1 + 0.08)(1 + y_{1,2}).$$

We can solve algebraically for the one-year forward rate $y_{1,2}$ by dividing both sides of the above equation by $1 + 0.08$, then subtracting 1 as follows:

$$\frac{(1+0.12)^2}{1+0.08} = 1 + y_{1,2},$$
$$\frac{(1+0.12)^2}{1+0.08} - 1 = y_{1,2} = 0.16148.$$

3.2 THE QUADRATIC FORMULA (Background reading: section 3.1)

A *quadratic* equation is an equation of order two; that is, the highest exponent in the equation is two. The usual form of the quadratic equation written in polynomial form (in descending order of exponents) is $0 = ax^2 + bx + c$, where a, b, and c are coefficients. Note that the x^2 appears first, the x term second, and the constant third on the right side of the equation. Consider the following example: $2 = 5x^2 + 10x$. We subtract 2 from both sides and express this equation in polynomial form (set equal to zero and write it in descending order of exponents) as follows: $0 = 5x^2 + 10x - 2$. Now, we will generalize this equation by changing its numerical coefficients to variables. If we let $a = 5$, $b = 10$, and $c = 2$, we write this equation as $0 = ax^2 + bx + c$. This is a more general form of a quadratic equation. A useful formula known as the quadratic formula for solving a quadratic equation for the variable x is given as follows:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We can use this formula to solve for x when $b^2 \geq 4ac$. When $b^2 > 4ac$, we will obtain two real solutions for x, and when $b^2 = 4ac$, we will obtain one real solution. When $b^2 < 4ac$, there will be no real solution for x. Substituting in values from our original example for a, b, and c, we obtain the following two solutions for x:

$$x = \frac{-10 \pm \sqrt{10^2 - 4 \cdot 5 \cdot (-2)}}{2 \cdot 5} = \frac{-10 \pm \sqrt{100 - (-40)}}{10} = \frac{-10 \pm \sqrt{140}}{10}$$

$$= \frac{-10 \pm 11.83216}{10} = \frac{1.183216}{10} \text{ and } \frac{-21.83216}{10} = 0.1183216 \text{ and } -2.183216,$$

where ± signifies two operations: add terms following the ±, and then, as a separate operation, subtract terms following the ±.

APPLICATION 3.4: FINDING BREAK-EVEN PRODUCTION LEVELS
(Background reading: application 3.2)

Suppose that a firm's price and per-unit variable costs are functions of sales and output levels. For example, as the firm sells additional units, its market may become saturated and the price it receives for each additional unit of production may drop. Consider a firm whose price function is related to sales: $TR = 10 - 0.00001Q$. Total revenues equal the product of price and output $PQ = 10Q - 0.00001Q^2$. In addition, assume that this firm's per-unit variable costs increase with production: $4 + 0.00002Q$. Further, assume that this firm's fixed costs equal \$100,000, such that total costs equal $100{,}000 + 4Q + 0.00002Q^2$. Thus, the firm's profit function, total revenues minus total costs, is given by

$$\pi = 10Q - 0.00001Q^2 - (100{,}000 + 4Q + 0.00002Q^2),$$

which simplifies to

$$\pi = -0.00003Q^2 + 6Q - 100{,}000.$$

Note that the terms are arranged in descending order of exponents for Q. Our coefficients for this quadratic equation are $a = -0.00003$, $b = 6$, and $c = -100{,}000$. We can solve for the break-even production level by setting π equal to zero using the quadratic formula, as follows:

$$Q = \frac{-6 \pm \sqrt{6^2 - 4 \cdot (-0.00003) \cdot (-100{,}000)}}{2 \cdot (-0.00003)} = \frac{-6 \pm \sqrt{36 - (12)}}{-0.00006} = \frac{-6 \pm \sqrt{24}}{0.00006}$$

$$= \frac{-6 \pm 4.89898}{-0.00006} = \frac{-10.89898}{-0.00006} \text{ and } \frac{-1.10102}{-0.00006} = 181{,}633.33 \text{ and } 18{,}350.33 \text{ units.}$$

Either of the above production levels will enable the firm to break even with a profit level equal to zero (the arithmetic may round to a slightly different figure).

APPLICATION 3.5: FINDING THE PERFECTLY HEDGED PORTFOLIO

Certain parts of this example are fairly technical and not covered until chapter 5. However, the algebra application is still useful and, if you wish to avoid the more technical part of the discussion, you may simply skip to the equation beginning with zero. If you choose to read on, think of a portfolio as a combination of two or more securities with given proportions. The assets may provide for an opportunity to hedge risk through diversification. In certain extreme cases, two risky assets can be combined to hedge out all of the risk in the portfolio. Our problem here is to determine what proportions of total investment should be put into each of the assets.

Suppose that an investor wishes to combine two risky stocks into a portfolio that is perfectly safe. This perfectly safe combination of two risky stocks is actually possible if the two stocks' returns or profits are perfectly inversely correlated. This means that two risky stocks can be combined into a riskless portfolio if one of the securities always does well when the second security performs poorly and if the second security performs well whenever the first security performs poorly. There is a formula that can be used to measure the risk of a portfolio (also called portfolio variance, or σ_p^2) whose two securities' returns are perfectly inversely correlated:

$$\sigma_p^2 = (\sigma_2^2 + \sigma_2^2 + 2 \cdot \sigma_1 \cdot \sigma_2)w_1^2 - (2\sigma_2^2 + 2 \cdot \sigma_1 \cdot \sigma_2)w_1 + \sigma_2^2,$$

where σ_1^2 is the risk or variance of the first stock, σ_2^2 is the risk or variance of the second stock, and w_1 is the proportion of the portfolio invested in the first stock. The terms σ_1 and σ_2 are simply the square roots of the variances of the two stocks' returns (also known as standard deviations).

Now, suppose that the stocks' variances or risk levels are $\sigma_1^2 = 0.01$ and $\sigma_2^2 = 0.0324$. What must be the proportion of the investor's money invested in stock 1 (w_1) for the portfolio to be riskless? We shall assume that the remainder of the portfolio will be invested in stock 2. We want to find that w_1 value that will set portfolio variance equal to zero. We rewrite the above equation as follows:

$$0 = (0.01 + 0.0324 + 0.036)w_1^2 - (0.0648 + 0.036)w_1 + 0.0324,$$

$$0 = 0.0784w_1^2 - 0.1008w_1 + 0.0324.$$

We need to solve this equation for w_1 using the quadratic formula. This is accomplished as follows:

$$w_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad \text{where } a = 0.0784,\ b = -0.1008, \text{ and } c = 0.0324.$$

We fill in our coefficients' values to determine the proportion of the investor's money to be invested in stock 1:

$$w_1 = \frac{0.1008 \pm \sqrt{-0.1008^2 - 4 \cdot 0.0784 \cdot 0.0324}}{2 \cdot 0.0784} = \frac{0.1008 \pm \sqrt{0}}{0.1568} = 0.64286.$$

The value under the square root sign (radical) will be zero. Hence, there will be only one value for $w_1 = 0.64286$. Therefore, we find that the portfolio is riskless when $w_1 = 0.64286$. Thus, 64.286% of the riskless portfolio should be invested in stock 1 and 35.714% of the portfolio should be invested in stock 2. There will be more discussion of risk and portfolio mathematics in chapters 5 and 6.

3.3 Solving Systems of Equations that Contain Multiple Variables (Background reading: section 3.1)

Systems of linear equations with multiple variables are often solved algebraically using the addition method. This method uses addition and multiplication principles. In order to use this method to solve the system completely, we normally must have the same number of equations as variables. That is, for two variables, we need two equations, for three variables, we need three equations, and so on.

For a two-variable system, we simply add or multiply the equations to cancel out or eliminate one variable. We then solve for the second variable, and plug it into either equation to solve for the first variable which we originally canceled. Consider the following two-equation system:

$$0.05 = 0.05x + 0.12y, \tag{A}$$

$$0.08 = 0.10x + 0.30y. \tag{B}$$

This system is easily solved by eliminating one of the unknown variables, x or y. One easy way to accomplish this is to multiply the first equation by -2, then add the two equations. This will enable us to eliminate the x variable. First, multiply both sides of (A) by -2,

$$-2 \cdot 0.05 = -2 \cdot 0.05x + -2 \cdot 0.12y,$$

which gives us the following equation:

$$-0.10 = -0.10x + -0.24y.$$

We then add this equation to our original equation (B):

$$\begin{array}{rcrcr} -0.10 = & -0.10x & + & -0.24y \\ +0.08 = & 0.10x & + & 0.30y \\ \hline -0.02 = & 0 & + & 0.06y \end{array}$$

Now we can easily solve for y:

$$0.06y = -0.02,$$
$$y = -0.02/0.06,$$
$$y = -0.333.$$

Now that we have found y, we can easily solve for x. This can be done by substituting -0.333 for y into either of our original equations:

$$0.05 = 0.05x + -0.333 \cdot 0.12,$$
$$0.05 = 0.05x + -0.04,$$
$$0.05x = 0.09,$$
$$x = 0.09/0.05,$$
$$x = 1.8.$$

To check our solution, we can also substitute -0.333 for y into our original second equation (B) to ensure that we obtain a value of 1.8 for x:

$$0.08 = 0.10x + 0.30 \cdot (-0.333),$$
$$0.10x = 0.18,$$
$$x = 1.8.$$

Solving systems of equations for three unknown variables is quite similar to solving for two variables. Solving a system of three variables requires three equations. We will attempt to substitute an equation for one of the variables and reduce the three equations to a two-equation system. When the system is reduced to two equations with two variables, we will solve just as we did in the previous example. Consider the following three-equation, three-variable example:

$$0.05 = 0.04x + 0.09y + 0.15z, \qquad \text{(A)}$$

$$0.15 = 0.08x + 0.12y + 0.10z, \tag{B}$$

$$0.30 = 0.12x + 0.06y + 0.25z. \tag{C}$$

To begin, we solve one equation for x. We could have started with any other variable, but x in this example is eliminated slightly more easily. Solving the first equation for x results in the following:

$$0.04x = 0.05 - 0.09y - 0.15z,$$
$$x = 0.05/0.04 - 0.09y/0.04 - 0.15z/0.04,$$
$$x = 1.25 - 2.25y - 3.75z.$$

Now we substitute for x (our revised version of equation (A)) into the other two equations:

$$0.15 = 0.08(1.25 - 2.25y - 3.75z) + 0.12y + 0.10z, \tag{B1}$$

$$0.30 = 0.12(1.25 - 2.25y - 3.75z) + 0.06y + 0.25z. \tag{C1}$$

Simplifying these two equations results in the following:

$$0.05 = -0.06y - 0.20z, \tag{B2}$$

$$0.15 = -0.21y - 0.20z. \tag{C2}$$

Now we have two equations with two variables. This is solved exactly as our first example. Multiply the first equation by −1 and then add the two equations:

$$-0.05 = 0.06y + 0.20z \tag{B3}$$
$$+ \quad 0.15 = -0.21y - 0.20z \tag{C3}$$
$$0.10 = -0.15y$$
$$y = -0.66667$$

Now substitute for y in either equation (B3) or (C3) and solve for z:

$$0.05 = -0.06 \cdot (-0.66667) - 0.20z,$$
$$z = -0.05.$$

Finally, we have solved for y and z, and we can substitute these values into any of our three original equations to solve for x:

$$0.15 = 0.08x + 0.12 \cdot (-0.66667) + 0.10 \cdot (-0.05),$$
$$x = 2.9375.$$

These values for x, y, and z can be substituted into any of the other two original equations to verify our results.

APPLICATION 3.6: PRICING FACTORS

An investor has a theory that oil company stock prices are a function of the Dow Jones Industrial Average and the price of a gallon of gasoline. The investor believes that no other factors affect prices and that if the *DJIA* and gas prices were zero, stock prices would be zero. For example, Greaser Company stock is currently selling for \$32.20 per share and Slick Oil Company stock is currently selling for \$63.30 per share. The investor has determined that a one-point change in the Dow Jones Industrial Average (*DJIA*) changes the stock price of the Greaser Company by 0.01 and changes the price of the Slick Oil Company by 0.02. Thus, the sensitivity (often called beta) of Greaser Company stock to the *DJIA* equals 0.01 and the sensitivity or beta of the Slick Oil Company to the *DJIA* equals 0.02. A one cent change in the price of gasoline changes the stock price of the Greaser Company by \$2 and changes the price of the Slick Oil Company by \$3. Thus, the sensitivity or beta of Greaser Company stock to the gas price equals 2 and the sensitivity or beta of the Slick Oil Company to the gas price equals 3. In fact, the investor uses the following equations to price the Greaser and Slick Oil Companies:

$$P_G = \$32.20 = 0.01 \cdot DJIA + 2 \cdot (Gas\ Price), \qquad \text{(A1)}$$

$$P_S = \$63.30 = 0.02 \cdot DJIA + 3 \cdot (Gas\ Price). \qquad \text{(B1)}$$

By solving these two equations simultaneously, we are able to value the two factors that affect oil company stock prices, *assuming that the investor's theory is correct*. We will first multiply equation (A1) by 2, to obtain:

$$2 \cdot P_G = \$64.40 = 0.02 \cdot DJIA + 4 \cdot (Gas\ Price). \qquad \text{(A2)}$$

Now, subtract equation (B1) from equation (A2):

$$P_G = \$1.10 = 1 \cdot (Gas\ Price). \qquad \text{(C1)}$$

Thus, the price of a gallon of gasoline must be \$1.10. We find the level of the Dow Jones Industrials Average by plugging \$1.10 for (*Gas Price*) into either equation (A1), (B1), or (A2). We find that *DJIA* = 3,000.

APPLICATION 3.7: EXTERNAL FINANCING NEEDS
(Background reading: section 3.3)

Consider the Albert Company, whose financial statements are given in tables 3.1 and 3.2. The firm's Earnings Before Interest and Tax (*EBIT*) level is projected to be \$300,000 next year. This *EBIT* level (sometimes referred to as Net Operating Income) represents the sum of funds available to pay interest to bondholders, taxes, dividends to shareholders, and earnings to retain. The firm has previously borrowed \$600,000 by issuing bonds which will require \$50,000 in interest payments. Management expects the firm to remain in the 40% corporate income tax bracket and pay out one

Table 3.1 Albert Company financial statements

Income statement, this year		*Pro-forma income statement, next year*	
Sales (*TR*)	$500,000	Sales (*TR*)	$700,000
Cost of Goods Sold	200,000	Cost of Goods Sold	300,000
Gross Margin	300,000	Gross Margin	400,000
Fixed Costs	100,000	Fixed Costs	100,000
EBIT	200,000	*EBIT*	300,000
Interest Payments	50,000	Interest Payments	
Earnings Before Taxes	150,000	Earnings Before Taxes	
Taxes (@ 40%)	60,000	Taxes (@ 40%)	
Net Income After Tax	90,000	Net Income After Taxes	
Dividends (@ 33%)	30,000	Dividends (@ 33%)	
Retained Earnings	60,000	Retained Earnings	

Balance sheet, December 31, this year

ASSETS		LIABILITIES AND EQUITY	
Cash	$100,000	Accounts Payable	$100,000
Accounts Receivable	100,000	Accrued Wages	50,000
Inventory	100,000	Current Liabilities	150,000
Current Assets	300,000	Bonds Payable	600,000
Plant and Equipment	700,000	Equity	250,000
Total Assets	1,000,000	Total Capital	1,000,000

Pro-forma balance sheet, December 31, next year

ASSETS		LIABILITIES AND EQUITY	
Cash	$140,000	Accounts Payable	$140,000
Accounts Receivable	140,000	Accrued Wages	70,000
Inventory	140,000	Current Liabilities	210,000
Current Assets	420,000	Bonds Payable	
Plant and Equipment	980,000	Equity	
Total Assets	1,400,000	Total Capital	1,400,000

third of its after-tax earnings in dividends. However, since the firm's production level is expected to increase next year, management has determined that each asset account must also increase by 40%. Assets currently total $1,000,000; thus, total assets must increase by $400,000. Current liabilities will also increase from its present level of $150,000 by 40% to $210,000. The firm pays no interest on its current liabilities. Managers have already decided to sell bonds at an interest rate of 10% to provide any external capital necessary to finance the asset level increase. Management's problem is to determine how much additional capital to raise through this 10% bond issue. Based on this information, we may determine the Albert Company's external financing needs (*EFN*) for next year.

Table 3.2 Albert Company pro-forma income statement

Pro-forma income statement, next year	
Sales (*TR*)	$700,000
Cost of Goods Sold	300,000
Gross Margin	400,000
Fixed Costs	100,000
EBIT	300,000
Interest Payments	50,000 + (0.10 · EFN)
Earnings Before Taxes	250,000 – (0.10 · EFN)
Taxes (@ 40%)	100,000 – (0.04 · EFN)
Net Income After Taxes	150,000 – (0.06 · EFN)
Dividends (@ 33%)	50,000 – (0.02 · EFN)
Retained Earnings	100,000 – (0.04 · EFN)

Since management has determined that it must increase its asset total by $400,000, it must determine how these assets will be financed. That is, management must determine the total sum of capital required to support the change in the total asset level. Some of this necessary capital can be derived from internal sources such as retained earnings (*RE*) or current liabilities (*CL*). These sources are likely to change simultaneously with the firm's production level and provide capital directly from the increase in the firm's level of operation. For example, an increase in the firm's sales level may result directly in an increase in the firm's level of retained earnings, since revenues, variable costs, and profits can be expected to increase. Furthermore, as the firm's sales level increases, it may be reasonable to anticipate an increase in the firm's number of employees, further resulting in an increase in the firm's accrued wages level. Other current liability levels are likely to increase in a similar manner. The remaining funds must be obtained through some external source, such as the sale of long-term bonds or equity. In summary, the amount of money the firm must raise from external sources is determined by the following equation:

$$EFN = \Delta Assets - \Delta CL - RE,$$

where *EFN* is the firm's external financing need; ΔAssets is the anticipated change in the firm's asset level from the prior year to the year of the increased operating level; ΔCL is the anticipated change in the firm's current liability debt level, assuming that current liabilities change spontaneously with the firm's sales level; and *RE* is the firm's anticipated retained earnings level for the year of the increased level of operation.

Our first problem is to compute Net Income After Taxes (*NIAT*) and Retained Earnings (*RE*) for Albert next year. However, we don't know what the company's interest expenditure (*INT*) next year will be until we know how much money it will borrow (*EFN*). At the same time, we cannot determine how much money the firm needs to borrow until we know its interest expenditure (so we can solve for retained earnings). Therefore, we must solve simultaneously for *EFN* and interest expenditure (or solve simultaneously for *EFN* and *RE*).

EFN can be found with the following:

$$EFN = \Delta Assets - \Delta CL - RE,$$
$$EFN = \$400{,}000 - \$60{,}000 - RE,$$
$$EFN = \$340{,}000 - RE.$$

Retained Earnings (*RE*) is determined by first subtracting total interest payments (\$50,000 on existing debt plus $0.10 \cdot EFN$ for newly issued debt) from Earnings Before Interest and Taxes (*EBIT*). This difference, $\$300{,}000 - \$50{,}000 - 0.10 \cdot EFN$ results in Earnings Before Taxes (*EBT*), or taxable income. Since the corporate tax rate equals 40%, the firm realizes 60% in Net Income after Taxes (*NIAT*), or $0.6(\$300{,}000 - \$50{,}000 - 0.10 \cdot EFN) = \$150{,}000 - 0.06 \cdot EFN$. Since the firm pays one third of its Net Income After Taxes (*NIAT*) to shareholders in dividends, its retained earnings are 0.667 times its *NIAT*, or $RE = 0.6667(\$150{,}000 - 0.06 \cdot EFN) = \$100{,}000 - 0.04 \cdot EFN$. We have two equations to work with here, one for *EFN* and a second for *RE*:

$$EFN = \$340{,}000 - RE,$$
$$RE = \$100{,}000 - 0.04 \cdot EFN.$$

We have two equations with two unknown variables, *EFN* and *RE*. We may use the addition method discussed in section 3.3 to solve this system. Therefore, the External Financing Needs expression may be written as follows:

$$EFN = \$340{,}000 - (\$100{,}000 - (0.04 \cdot EFN)).$$

Thus, we have written an expression for *RE*, $(\$100{,}000 - (0.04 \cdot EFN))$, and inserted it into our expression for *EFN*. We now proceed to solve for *EFN* as follows:

$$EFN = \$240{,}000 + 0.04 \cdot EFN,$$
$$0.96 \cdot EFN = \$240{,}000,$$
$$EFN = \$250{,}000.$$

Our *EFN* problem is complete. We now know that the firm must borrow \$250,000. Thus, the firm's total interest payments for next year must be \$50,000 plus 10% of \$250,000, or \$75,000. Retained Earnings will be 90,000, computed by substituting 250,000 for *EFN* into the following expression for *RE*: $RE = \$100{,}000 - 0.04 \cdot EFN$.

3.4 Geometric Expansions
(Background reading: section 3.1)

At the end of section 3.1, we introduced the concept of the geometric expansion as a technique to simplify a polynomial consisting of a repetitive series of terms. These terms, arranged in a series of terms with a single variable and exponents arranged in descending order of exponents is called a geometric series. A geometric expansion is an algebraic procedure used to simplify a geometric series. This procedure is most useful when the number of terms is large. Suppose that one intended to solve the following finite geometric series for S:

$$S = c + cx + cx^2 + cx^3 + \ldots + cx^n. \tag{A}$$

In this series, c is a constant, or parameter, and x is a quotient, or variable. If n is large, direct calculations on this series may be time-consuming and repetitive. Simplifying the series to reduce the number of terms may save a significant amount of time performing routine calculations. The geometric expansion is a two-stage procedure:

1 First, multiply both sides of the equation by the quotient:

$$Sx = cx + cx^2 + cx^3 + cx^4 + \ldots + cx^{n+1}. \tag{B}$$

This first step is intended to obtain a very similar type of expression with repetitive terms that will be eliminated in the second step.

2 Second, to eliminate these repetitive terms, subtract the above product (B) from the original equation (A) and then simplify the result:

$$\begin{aligned} Sx - S = {} & cx + cx^2 + cx^3 + cx^4 + \ldots + cx^{n+1} \\ & - c - cx - cx^2 - cx^3 - \ldots - cx^n. \end{aligned} \tag{C}$$

The following simplification completes the geometric expansion. Notice the set of terms that should cancel when we simplify:

$$Sx - S = -c + cx^{n+1}, \tag{D}$$

$$S(x - 1) = c(x^{n+1} - 1). \tag{E}$$

Continue the process of simplification by dividing both sides by $(x - 1)$:

$$S = c\left(\frac{x^{n+1} - 1}{x - 1}\right) \qquad \text{for } x \neq 1. \tag{F}$$

Consider the following example, where we set x to equal $(1 + i)$. Equations (G) and (H) will be identical:

$$S = c + c(1 + i) + c(1 + i)^2 + c(1 + i)^3 + \ldots + c(1 + i)^n, \tag{G}$$

$$S = c\left(\frac{1 - (1 + i)^{n+1}}{1 - (1 + i)}\right) = c\frac{(1 + i)^{n+1} - 1}{i}. \tag{H}$$

Thus, any geometric series where $x \neq 1$ can be simplified with the following right-hand side formula:

$$S = c + cx + cx^2 + cx^3 + \ldots + cx^{n-1} = c\frac{x^n - 1}{x - 1}. \tag{I}$$

Geometric expansions are most helpful in time value mathematics with many periods, and in situations involving series of potential outcomes with associated probabilities. Such situations occur very frequently in finance. The geometric expansion procedure can save substantial amounts of computation time for problems involving these situations.

APPLICATION 3.8: MONEY MULTIPLIERS

Commercial banks play a most important role in the world economy. Among their important functions is their role in creating money by extending credit or loans. It may be reasonable to assume that the central bank of a country (the Federal Reserve system in the U.S.) issues a fixed amount of currency (paper money) K to the public and allows commercial banks to loan funds entrusted to them by depositors (in checking accounts, also known as demand deposits) of amount DD. Thus, businesses and consumers receive the currency from the central bank and deposit it into the commercial banking system in the form of demand deposits. Typically, the central bank requires that commercial banks hold on reserve a proportion r of their demand deposits. This reserve cannot be loaned to the general public. More specifically, commercial banks leave on deposit (or reserve) with the central bank nonloanable reserves totaling $r \cdot DD$. The bank loans the remainder. After one round of deposits and loans, money supply in the economy is determined:

$$M_1 = K + (1 - r)K \qquad \text{with a single round of deposits and loans.} \tag{A}$$

Thus, the economy's money supply thus far consists of paper currency K issued by the government which is deposited into the bank plus proportion $(1 - r)$, which is loaned by the bank to its customers. The loaned funds are then spent and then redeposited by their recipients, allowing the process to repeat itself:

$$M_1 = K + (1 - r)K + (1 - r)^2K \qquad \text{with two rounds of deposits and loans.} \tag{B}$$

This process can continue perpetually. Whenever funds are loaned by a commercial bank, they are spent by the borrower. The borrower purchases goods from a seller; the seller then deposits its receipts into the commercial banking system, creating more funds available to loan. However, each deposit requires that the commercial bank increase its reserve left with the central bank by the proportion r. The level of money supplied, M_1, in such a system is determined as follows:

$$M_1 = K + (1 - r)K + (1 - r)^2K + \ldots + (1 - r)^{\infty}K. \tag{C}$$

K is the currency originally issued by the central bank to the public and deposited in the commercial banking system. The amount rK fulfills the initial reserve requirement and the remainder $(1 - r)K$ is loaned to the public. The public redeposits this sum back into the commercial banking system. Of the $(1 - r)K$ redeposited into the banking system, $(1 - r)(1 - r)K = (1 - r)^2K$ is available to loan after the reserve requirement is fulfilled on the second deposit. This process continues forever; that is, it continues through $(1 - r)^\infty K$. Where K is the level of currency originally issued by the central bank and r is its reserve requirement, what is the total money supply for this economy? Obviously, since series (A) above is extended through an infinite number of repetitions, its exact computation is impossible without simplification. We can determine total money supply through the following geometric expansion, where we first multiply by $1 - r$:

$$(1 - r)M_1 = (1 - r)K + (1 - r)^2K + (1 - r)^3K + \ldots + (1 - r)^{\infty+1}K. \qquad \text{(D)}$$

The coefficient $1 - r$ is analogous to c in the geometric expansions in section 3.3. We subtract equation (C) from equation (D), to obtain

$$(1 - r)M_1 - M_1 = (1 - r)^1K + (1 - r)^2K + \ldots + (1 - r)^{\infty+1}K - (K + (1 - r)^1K + (1 - r)^2K + \ldots + (1 - r)^\infty K), \qquad \text{where } (1 - r)^\infty K = 0, \qquad \text{(E)}$$

which simplifies to

$$(1 - r)M_1 - M_1 = -K + (1 - r)^{\infty+1}K, \qquad \text{where } (1 - r)^{\infty+1}K = 0. \qquad \text{(F)}$$

The process of simplification continues as follows:

$$rM_1 = K, \qquad \text{(G)}$$

$$M_1 = K/r, \qquad \text{(H)}$$

where we assume that K is positive and $0 < r \leq 1$. Thus, the money multiplier here equals $1/r$. Money supply M_1 is simply K/r. Thus, for example, a central bank issuing \$100 in currency with a reserve requirement equal to 10% will have a total money supply equal to \$1,000:

$$M_1 = \frac{100}{0.10} = 1{,}000. \qquad \text{(I)}$$

3.5 Functions and Graphs
(Background reading: section 2.3)

A *function* is a rule that assigns to each number in a set a unique second number. Functions may be represented by equations, graphs, or tables. An example of a "generic" functional relationship in equation form is given by $y = f(x)$, which reads

"y is a function of x." If y increases as x increases, then y is said to be a direct or increasing function of x. In the following examples, y is an increasing function of x:

(a) $y = 5x$, (b) $y = 12x + 3$,
(c) $y = 0.9x$, (d) $y = 3e^x$,
(e) $y = 10x^2 + 20x + 10$ (when $x > -1$), (f) $y = 12x^3 + 2x^2$.

Functions (a), (b), and (c) are said to be linear functions. This means that graphs depicting the relationships between x and y would plot lines. In other words, the graphs or plots would be linear. Equation (d) represents an exponential function because of its use of the number e. Equation (e) represents a quadratic function because it has a single independent variable x and its exponents are integers ranging from zero to two. Equation (f) is a cubic function because its exponents are integers ranging from zero to three. If y decreases as x increases, it is said that y is a decreasing or inverse function of x. In the following examples, y is a decreasing function of x:

$$y = \frac{6}{x} \quad (\text{where } x > 0), \qquad y = -9x + 5,$$

$$y = -3x^2 - 6x \quad (\text{where } x > -1), \qquad y = 6e^{-x},$$

$$y = \frac{1}{7x} \quad (\text{where } x > 0), \qquad y = -(x^2).$$

Consider the functions depicted in figures 3.1 and 3.2. To plot functions on a graph, simply solve for y in terms of x and plot corresponding coordinates on x/y axes as in the graphs depicted in figures 3.1 and 3.2. Coordinates for two points are sufficient to plot out linear functions; many other functions may require more points or other information in order to estimate placement of the appropriate curve.

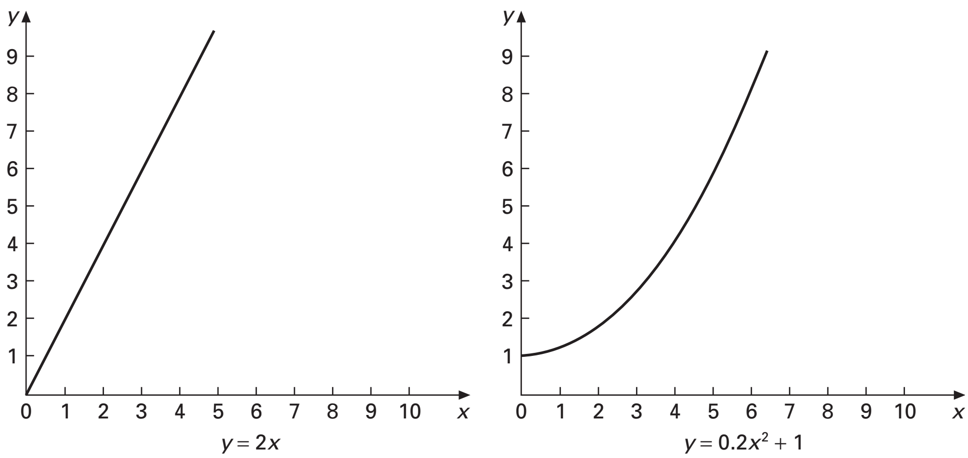

Figure 3.1 Increasing or direct functions.

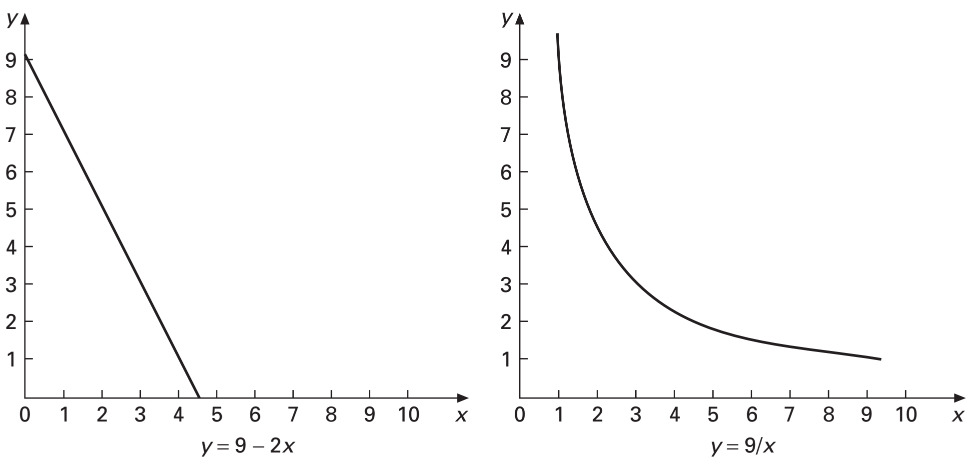

Figure 3.2 Decreasing or inverse functions.

APPLICATION 3.9: UTILITY OF WEALTH

In finance, we typically assume that individuals obtain some sort of satisfaction from acquiring wealth. In this application, we shall assume that an investor can associate some measurable level of personal satisfaction (or, in economic terms, utility) with any given wealth level. Furthermore, it is reasonable to assume that this level of utility or satisfaction increases as the investor's level of wealth increases; that is, an investor becomes more satisfied as his level of wealth increases. Next, we shall assume in this application that we can mathematically define the relationship between an investor's wealth and utility levels. Thus, utility will be characterized as a function (which will be specified precisely shortly) of the investor's level of wealth:

$$U = f(W).$$

Finally, we shall make the additional and perhaps less realistic assumption that utility is measurable and that we can specify its exact functional relationship with wealth. An example of such a utility function might be

$$U = 0.5 \cdot \sqrt{W}.$$

An investor with this utility function and whose wealth is given by W (W and $U \geq 0$) will have a utility level equal to one-half times the square root of his wealth level. Figure 3.3 represents a utility of wealth curve for this function. The investor with this particular utility of wealth function whose wealth level is \$2,000,000 would have a utility level of 1,414.21. If the investor's wealth level were to increase to \$3,000,000, his utility level would increase to 1,732.05. Clearly, this investor's utility level

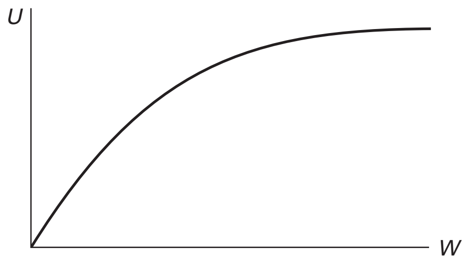

Figure 3.3 Utility of wealth. Utility-of-wealth function for risk-averse individual: $f'(W) > 0$; $f'(W) < 0$. For example, $U_W = 0.5W^{0.5}$.

increases as he becomes wealthier. There is a positive or relationship between the investor's wealth and utility levels. Of course, all of this is based on the rather unrealistic assumption that utility is observable and measurable. Even though the assumptions of observability and measurability may be unrealistic, utility models can be most useful in describing how investors behave and react to changing conditions. These models can also be used to demonstrate how investors react to risk, how they might value insurance policies, and how they mix bonds and stocks in portfolios.

EXERCISES

3.1. Solve each of the following for x:

(a) $100 = 5x + 10$;
(b) $500 = 3{,}000 - 2x$;
(c) $100 = 15 + 2x + 6x$;
(d) $50 = 300/5 + 5x$;
(e) $20 = 10 - 15/x + 25$;
(f) $25 - 3x = 100 - 2x + 5x$.

3.2. Solve each of the following for x:

(a) $100 = 90(1 + x)^2$;
(b) $100 = 90(1 + x)^3$;
(c) $100 = 90/(1 + x)^2$;
(d) $90 = 100/(1 + x)^2$.

3.3. Assume that \$1 will purchase £0.60 and ¥108; that is, one U.S. dollar will purchase 0.6 U.K. pounds and 108 Japanese yen. Assume that goods in the three countries are identically priced after adjusting for currency exchange rates.

(a) What is the value of £1 in ¥?
(b) What is the value of ¥1 in £?
(c) If one ounce of gold costs \$300 in the United States, what is its cost in the U.K. and in Japan?

3.4. Smith Company produces a product that can be sold for a price of $P = \$80$ per unit such that its total annual revenues equals $TR = PQ = 80Q$. The variable Q represents the number of units of the product produced and sold by the firm. The company incurs two types of costs in its production process, fixed and variable. Fixed costs total \$500,000 per year. Variable costs VC are \$50 per unit produced. The firm's profit function is defined as follows as total revenues minus the sum of fixed and variable costs:

$$\pi = P \cdot Q - (FC + VC \cdot Q) = 80Q - (500{,}000 + 50 \cdot Q).$$

What is the firm's break-even production level?

3.5. Suppose that the one-year spot rate $y_{0,1}$ of interest is 5%. Investors are expecting that the one-year spot rate one year from now will increase to 6%; thus, the one-year forward rate $y_{1,2}$ on a loan originated in one year is 6%. Furthermore, assume that investors are expecting that the one-year spot rate two years from now will increase to 7%; thus, the one-year forward rate $y_{2,3}$ on a loan originated in two years is 7%. Based on the pure expectations hypothesis, what is the three-year spot rate?

3.6. Suppose that the one-year spot rate $y_{0,1}$ of interest is 5%. Investors are expecting that the one-year spot rate one year from now will increase to 7%; thus, the one-year forward rate $y_{1,2}$ on a loan originated in one year is 7%. Furthermore, assume that the three-year spot rate equals 7% as well. What is the anticipated one-year forward rate $y_{2,3}$ on a loan originated in two years based on the pure expectations hypothesis?

3.7. Consider a firm whose price function is related to sales: $TR = 50 - 0.00002Q$. Total revenues equal the product of price and output $PQ = 50Q - 0.00002Q^2$. In addition, assume that this firm's per-unit variable costs increase with production, $20 + 0.00001Q$, and that this firm's fixed costs equal \$500,000, such that total costs equal $500{,}000 + 20Q + 0.00001Q^2$.

(a) What is the firm's profit function?
(b) What is the firm's break-even production level?

3.8. Suppose that the risk of a portfolio is given by the following equation:

$$\sigma_p^2 = 0.25w_1^2 - 0.3w_1 + 0.09.$$

What proportion of the portfolio should be invested in asset one (w_1) so that the portfolio risk σ_p^2 equals zero?

3.9. Solve the following for x_1 and x_2:

$$0.04x_1 + 0.04x_2 = 0.01,$$

$$0.04x_1 + 0.16x_2 = 0.11.$$

3.10. Solve the following for x and λ:

$$20x - 5\lambda = -3$$
$$-5x - 0\lambda = -100$$

3.11. Solve the following for x_1, x_2, and x_3:

$$962 = 100x_1 + 100x_2 + 1{,}100x_3,$$
$$1{,}010.4 = 120x_1 + 120x_2 + 1{,}120x_3,$$
$$970 = 100x_1 + 1{,}100x_2.$$

3.12. Stock A, which currently sells for \$16.80, has a sensitivity or beta with respect to GNP equal to 4 and a beta with respect to interest rates i equal to 80. Stock B, which currently sells for \$16.80, has a beta with respect to GNP equal to 4 and a beta with respect to interest rates i equal to 80. Thus, two factors drive stock prices in this economy. No other factors affect prices.

(a) Write one equation for each stock depicting the relationship between stock prices and the two explanatory variables GNP and i.
(b) Solve these two equations simultaneously to determine current GNP and interest rate levels.

3.13. The Victoria Company's financial statements are given below. Management is forecasting an increase in the company's sales level by 40% to \$1,050,000. Managers predict that this 40% sales increase will increase the firm's Cost of Goods Sold level by 50% to \$450,000. Fixed costs will remain constant at \$150,000. The firm will continue to make the \$50,000 interest payments necessary to sustain its \$600,000 in bonds outstanding. Management expects the firm to remain in the 40% corporate income tax bracket and pay out one third of its earnings in dividends. In order to sustain this 40% increase in sales, management has determined that each asset account must also increase by 40%; that is, the total must increase by \$400,000. Current Liabilities will also increase by 40%. The firm pays no interest on its current liabilities. Managers have already decided to sell bonds at an interest rate of 10% to provide any external capital necessary to finance the asset level increase. Management's problem is to determine how much additional capital to raise through this 10% bond issue. Based on this information and the company's financial statements given below, determine the Victoria Company's 2001 External Funding Needs (*EFN*).

VICTORIA COMPANY FINANCIAL STATEMENTS

Income statement, 2000		*Pro-forma income statement, 2001*	
Sales (*TR*)	$750,000	Sales (*TR*)	$1,050,000
Cost of Goods Sold	300,000	Cost of Goods Sold	450,000
Gross Margin	450,000	Gross Margin	600,000
Fixed Costs	150,000	Fixed Costs	150,000
EBIT	300,000	*EBIT*	450,000
Interest Payments	50,000	Interest Payments	
Earnings Before Taxes	250,000	Earnings Before Taxes	
Taxes (@ 40%)	100,000	Taxes (@ 40%)	
Net Income After Tax	150,000	Net Income After Taxes	
Dividends (@ 33%)	50,000	Dividends (@ 33%)	
Retained Earnings	100,000	Retained Earnings	

Balance sheet, December 31, 2000

ASSETS		LIABILITIES AND EQUITY	
Cash	$100,000	Accounts Payable	$100,000
Accounts Receivable	100,000	Accrued Wages	50,000
Inventory	100,000	Current Liabilities	150,000
Current Assets	300,000	Bonds Payable	600,000
Plant and Equipment	700,000	Equity	250,000
Total Assets	1,000,000	Total Capital	1,000,000

Pro-forma balance sheet, December 31, 2001

ASSETS		LIABILITIES AND EQUITY	
Cash	$140,000	Accounts Payable	$140,000
Accounts Receivable	140,000	Accrued Wages	70,000
Inventory	140,000	Current Liabilities	210,000
Current Assets	420,000	Bonds Payable	
Plant and Equipment	980,000	Equity	
Total Assets	1,400,000	Total Capital	1,400,000

3.14. Perform a geometric expansion to simplify the following series:

$$1{,}200 = 100(1 + x + x^2 + x^3 + \ldots + x^9).$$

What is the value of x?

3.15. The Keynesian Macroeconomic Model postulates a relationship between autonomous consumer consumption (consumption expenditures independent of income) and total income. Suppose that the following depicts the relationships among income Y, autonomous consumption $\overline{C}$, and income-dependent consumption cY:

$$Y = \overline{C} + cY. \tag{A}$$

If autonomous consumption were to increase by a given amount, this would increase income, resulting in an increase in income-dependent consumption. This would further increase income and consumption, and the process would replicate itself perpetually:

$$\Delta Y = \Delta\overline{C} + Y(c + c^2 + c^3 + \ldots + c^{\infty}). \tag{B}$$

Perform a geometric expansion on (B) to derive an income multiplier to determine the full amount of the change in income resulting from a change in autonomous consumption.

3.16. For each of the following, $y = f(x)$. Plot each of these four functions:

(a) $y = 10 - 0.5x$; (c) $y = 9 + 25x$;
(b) $y = 20/3x$; (d) $y = 8 + 0.3x^2$.

APPENDIX 3.A SOLVING SYSTEMS OF EQUATIONS ON A SPREADSHEET

(Background reading: section 3.3 and appendix 2.A)

Solving larger systems of equations containing larger numbers of variables by hand or with a calculator can be an extremely time-consuming and frustrating process. However, spreadsheets can be used quite effectively to solve systems of linear equations simultaneously. Suppose that we wished to solve the following system of two equations simultaneously for x and y using an Excel™ spreadsheet:

$$10 = 8x + 4y,$$

$$20 = 2x + 6y.$$

We have two equations with two variables, x and y, with unknown values. Notice that the numbers for each of the two equations are arranged in a particular order: solutions (10 & 20) on the left, coefficients for x (8 & 2) then coefficients for y (4 & 6). Allowing one equation per row and noting that the x coefficients line up in the first column and the y coefficients line up in the second column, insert the coefficients in the spreadsheet as follows:

	A	B	C	D	E
1	8	4			
2	2	6			
3					
4					
5					

To solve the system, we will perform a matrix invert procedure which we will discuss in chapter 7. However, the mechanics are quite simple. First, use the mouse to highlight cells A4 to B5. With most desktop computers, this means to move the cursor to cell A4, hold down the left click, and while holding the left click, move the cursor to cell B5. After releasing the left click with four cells highlighted, left click on the toolbar at the top of the screen the Paste Function button (f_x). A Paste Function menu should appear on the screen, from which you should select the MATH & TRIG submenu. In the MATH & TRIG submenu, scroll down to select MINVERSE. The MINVERSE function will prompt you for an array; you should enter A1:B2. To fill all four cells A4 to B5, simultaneously hit the Ctrl, Shift, and Enter keys. Your spreadsheet should then appear as follows:

	A	B	C	D	E
1	8	4			
2	2	6			
3					
4	0.15	–0.1			
5	–0.05	0.2			

Now, enter into cells C4 and C5 the equation solutions 10 and 20. Then highlight cells D4 and D5, left click the Paste Function key in the Toolbar, select the MATH & TRIG menu and scroll down to and select the MMULT function. Now, you will be prompted for two arrays. The first will be A4:B5 – hit the Tab key; the second array will be C4:C5. Then hit the Ctrl, Shift, and Enter keys simultaneously to fill cells D4 and D5. The result will be –0.5 and 3.5. Thus, $x = -0.5$ and $y = 3.5$. The final spreadsheet will appear as follows:

	A	B	C	D	E
1	8	4			
2	2	6			
3					
4	0.15	–0.1	10	–0.5	
5	–0.05	0.2	20	3.5	

The process of expanding this solution procedure to three or more equations and variables is quite simple. First, be certain that each equation in the system is linear (no exponents other than 0 or 1 on the variables) and that there are exactly the same number of variables as equations. Arrange the terms in the equations so that the columns contain coefficients for the same variables. In some instances, the systems cannot be solved. Reasons for this will be discussed in chapter 7. Consider the following four-equation, four-variable system:

$$10 = 8x + 4y + 2z + 10q,$$

$$20 = 2x + 4y + 1z + 12q,$$

$$30 = 0x + 4y + 2z + 16q,$$

$$40 = 5x + 6y + 8z + 20q.$$

Now, examine the following spreadsheet, which solves the system:

	A	B	C	D	E	F	G
1	8	4	2	10			
2	2	4	1	12			
3	0	4	2	16			
4	5	6	8	20			
5							
6	0.235294	−0.29412	0.147059	−0.05882	10	−1.470588	
7	−0.52941	1.578431	−1.12255	0.215686	20	1.2254902	
8	−0.11765	−0.01961	−0.15686	0.196078	30	1.5686275	
9	0.147059	−0.39216	0.362745	−0.07843	40	1.372549	

Thus, $x = -1.470588$, $y = 1.2254902$, $z = 1.5686275$, and $q = 1.372549$. This issue of solving systems of linear equations simultaneously will be revisited in chapter 7.