CHAPTER 9
Stock Selection and Portfolio Fit

A: Portfolio Return and Risk: A Review

A portfolio is simply a collection of investments. The entire set of an investor's holdings is considered to be his portfolio. It may be reasonable for an investor to be concerned with the performance of individual securities only to the extent that their performance affects the performance of his overall portfolio of investments. Thus, the performance of the portfolio is of primary importance to the investor. The return of an investor's portfolio is simply a weighted average of the returns of the individual securities that comprise his portfolio. The expected return of a portfolio may be calculated either as a function of potential portfolio returns and their associated probabilities (a weighted average of potential returns) or as a simple weighted average of the expected individual security returns. However, the risk of the portfolio is somewhat more complicated to determine. Generally, the portfolio variance or standard deviation of returns will be less than a weighted average of the individual security variances or standard deviations. This reduction on portfolio risk will be intensified as the portfolio becomes more diversified; that is, portfolio risk is reduced when the selected securities are more dissimilar and when the number of securities in the portfolio increases.

Portfolio Return

The expected return of a portfolio can be calculated using Equation (1) where the subscript (p) designates the portfolio and the subscript (j) designates a particular outcome out of (m) potential outcomes:

$$E\left[R_p\right] = \sum_{j=1}^{m} R_{pj} \cdot P_j \tag{1}$$

Thus, the expected return of a portfolio is simply a weighted average of the potential portfolio returns where the outcome probabilities serve as the weights.

For many portfolio management applications, it is useful to express portfolio return as a function of the returns of the individual securities that comprise the portfolio. This is often because we want to know how a particular security will affect the return and risk of our overall holdings or portfolio. For example, consider a portfolio made up of two securities, one and two. The expected return of security one is 10% and the expected return of security two is 20%. If forty percent of the dollar value of the portfolio is invested in security one (that is, [w_1] = .40), and the remainder is invested in security two ([w_2] = .60), the expected return of the portfolio may be determined by Equation (2):

$$E\left[R_p\right] = \sum_{i=1}^{n} w_i \cdot E\left[R_i\right] \tag{2}$$

$$E\left[R_p\right] = (.4 \cdot .10) + (.6 \cdot .20) = .16$$

The subscript (i) designates a particular security, and weights [w_i] are the portfolio proportions. That is, a security weight (w_i) specifies how much money is invested in Security (i) relative to the total amount invested in the entire portfolio. For example, [w_1] is:

$$w_1 = \frac{\text{\$ invested in security 1}}{\text{Total \$ invested in the portfolio}}$$

Thus, portfolio return is simply a weighted average of individual security returns.

<u>Portfolio Risk</u>

Because risky securities often behave quite differently from one another, the variance of portfolio returns is not simply a weighted average of individual security variances. In fact, in this section, we will demonstrate that combining securities into portfolios may actually result in risk levels lower than those of any of the securities comprising the portfolio. That is, in some instances, we can combine a series of highly risky assets into a relatively safe portfolio. The risk of this portfolio in terms of variance of returns can be determined by solving the following double summation:

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_i \cdot \sigma_j \cdot \rho_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{ij} \tag{3}$$

Consider the portfolio constructed in the previous section. If the standard deviation of returns on securities one and two were .20 and .30, respectively, and the correlation coefficient ($\mathbf{D}_{ij}$) between returns on the two securities were .5, the resulting standard deviation of the portfolio would be .23, the square root of its .0504 variance level:

$$(4)\ \sigma_p^2 = (.4 \cdot .4\ \ .2 \cdot .2 \cdot 1) + (.4 \cdot .6 \cdot .2 \cdot .3 \cdot .5)\ + (.6 \cdot .4 \cdot .3 \cdot .2 \cdot .5)\ +\ (.6 \cdot .6 \cdot .3 \cdot .3 \cdot 1) = .0532\,; \sigma_p = .23$$

More generally, when n=2:

$$\begin{aligned}(5)\qquad \sigma_p^2 &= (w_1 \cdot w_1 \cdot \sigma_1 \cdot \sigma_1 \cdot \rho_{11})\ +\ (w_1 \cdot w_2 \cdot \sigma_1 \cdot \sigma_2 \cdot \rho_{12}) + (w_2 \cdot w_1 \cdot \sigma_2 \cdot \sigma_1 \cdot \rho_{21}) + \\ &(w_2 \cdot w_2 \cdot \sigma_2 \cdot \sigma_2 \cdot \rho_{22})\end{aligned}$$

Notice that both counters i and j are set equal to one to begin the double summation. Thus, in the first set of parentheses of Equations 4 and 5, since both i and j equal one, both portfolio weights w_i and w_j equal .4; σ_i and σ_j equal .2. The coefficient of correlation between any variable and itself must be one; therefore, ($\mathbf{D}_{11}$) equals one. After variables are substituted into Equation (3) for (i) equals 1 and (j) equals 1, the counter of the inside summation is increased to two. Thus, in the second set of parentheses, (i) equals 1 and (j) equals 2. Hence, (w_i) equals .4, (w_j) equals .6, (σ_i) equals .2, and ($\mathbf{F}_j$) equals .3. Since the number of securities comprising the portfolio (n) is two, the inside summation is completed. We now increase the counter of the outside summation (i) to two and begin the inside summation over again (by setting [j] equal to 1). Thus, in the third set of parentheses, (i) equals two and (j) equals one. The correlation coefficient ($\rho_{2,1}$) must equal .5 because it must be identical to ($\rho_{1,2}$). We now increase the counter of the inside summation to two; in the fourth set of parentheses both (i) and (j) equal two. Since both counters now equal (n), (3) can be simplified and solved. It is important to realize that (i) and (j) are merely counters; they do not necessarily refer to any specific security consistently throughout the summation process. By simplifying the expressions in the first and fourth sets of parentheses, and combining the terms in the second and third sets, one can simplify Equation (5):

(6) $$\sigma_p^2 = (.42 \cdot .22) + (.62 \cdot .32) + 2(.4 \cdot .6 \cdot .2 \cdot .3 \cdot .5) = .0532$$

Therefore, when a portfolio is comprised of two securities, its variance can be determined by Equation (7):

(7) $$\sigma_p^2 = (\mathrm{w}_1^2\ \sigma_1^2) + (w_2^2\ \sigma_2^2) + 2(w_1\ w_2 \sigma_1 \sigma_2 \rho_{1,2})$$

Equation (7) allows us to determine portfolio variance without having to work through the double summation only when (n) equals two. Larger portfolios require the use of some form of Equation (3). However, the number of sets of parentheses to work through and then add is equal to the number of securities in the portfolio squared (n^2). Equation (3) can be simplified to an equation with a form similar to that of (7). For example, if the portfolio were to be comprised of three securities, Equation (7) would change to:

(7.a)
$$\sigma_p^2 = (\mathrm{w}_1^2\ \sigma_1^2) + (w_2^2\ \sigma_2^2) + (w_3^3 \sigma_3^3) + 2(w_1 w_2\ \sigma_1 \sigma_2 \rho_{12}) + 2(w_1 w_3 \sigma_1 \sigma_3 \rho_{13}) + 2(w_2 w_3 \sigma_2 \sigma_3 \rho_{23}).$$

You may find it useful to derive Equation (7.a) from Equation (3). In any case, notice the similarity in the patterns of variables between equations (7) and (7.a).

If fifty securities were to be included in the investor's portfolio, 2500 expressions must be solved and then added for Equation (3). This portfolio would require solutions to 1275 expressions for solving the more simple equation (7.a). Obviously, as the number of securities in the portfolio becomes large, computers become quite useful in working through the repetitive calculations. The equations are not difficult to solve, they are merely repetitive and time-consuming.

In our first example, the weighted average of the standard deviation of returns of the two securities one and two is 26%, yet the standard deviation of returns of the portfolio they combine to make is only 23%. Clearly, some risk has been diversified away by combining the two securities into the portfolio. In fact, the risk of a portfolio will almost always be lower than the weighted average of the standard deviations of the securities that comprise that portfolio.

For a more extreme example of the benefits of diversification, consider two securities, three and four, whose potential return outcomes are perfectly inversely related. Data relevant to these securities is listed in Table (1). If outcome one occurs, security three will realize a return of 30%, and security four will realize a 10% return level. If outcome two is realized, both securities will attain returns of 20%. If outcome three is realized, securities three and four will attain return levels of 10% and 30%, respectively. If each outcome is equally likely to occur ([P_i] is .333 for all outcomes), the expected return level of each security is 20%; the standard deviation of returns for each security is .08165. The expected return of a portfolio combining the two securities is 20% if each security has equal portfolio weight ([w_3] = [w_4] = .5), yet the standard deviation of portfolio returns is zero. Thus, two relatively risky securities have been combined into a portfolio that is virtually risk-free.

TABLE 1: **Portfolio return with perfectly inversely correlated securities. $w_3 = w_4 = 0.5$**

i	R_{3i}	R_{4i}	R_{pi}	P_i
1	.30	.10	.20	.333
2	.20	.20	.20	.333
3	.10	.30	.20	.333

Notice in the previous paragraph that we first combined securities three and four into a portfolio and then found that portfolio's return given each outcome. The portfolio's return is 20% regardless of the outcome; thus, it is risk free. The same result could have been obtained by finding the variances of securities three and four, the correlation coefficient between their returns, then solving for portfolio variance with Equation (6) as in Table (2).

TABLE 2: Portfolio return with perfectly inversely correlated securities.

Given:

$\overline{R}_3 = 0.20$ $\qquad$ $\overline{R}_4 = 0.20$

σ_3=0.08165 $\quad$ σ_4=0.08165 $\quad$ w_3=0.50 $\quad$ w_4=0.50 $\quad$ $\rho_{3,4} = -1$

Then: $\overline{R}_p = w_3\overline{R}_3 + w_4\overline{R}_4 = (0.5\times 0.20) + (0.5\times 0.20) = 0.20$

$$\sigma_p = \sqrt{w_3^2\sigma_3^2 + w_4^2\sigma_4^2 + 2w_3w_4\sigma_3\sigma_4\rho_{3,4}}$$

$$\sigma_p = \sqrt{0.5^2 \times .0066667 + 0.5^2 \times 0.0066667 \ + 2\times 0.5\times 0.5\times 0.08165\times 0.08165\times(-1)}$$

$$\sigma_p = \sqrt{0.0016667 + 0.0016667 - 0.003333} = \sqrt{0} = 0$$

The implication of the two examples provided in this chapter is that security risk can be diversified away by combining the individual securities into portfolios. Thus, the old stock market adage "Don't put all your eggs in one basket" really can be validated mathematically. Spreading investments across a variety of securities does result in portfolio risk that is lower than the weighted average risks of the individual securities. This diversification is most effective when the returns of the individual securities are at least somewhat unrelated; or better still, inversely related as were securities three and four in the previous example. For example, returns on a retail food company stock and on a furniture company stock are not likely to be perfectly positively correlated; therefore, including both of them in a portfolio may result in a reduction of portfolio risk. From a mathematical perspective, the reduction of portfolio risk is dependent on the correlation coefficient of returns ($\rho_{i,j}$) between securities included in the portfolio. Thus, the lower the correlation coefficients between these securities, the lower will be the resultant portfolio risk. In fact, as long as (ρ_{ij}) is less than one, which, realistically is always the case, some reduction in risk can be realized from diversification.

Consider Figure (1). The correlation coefficient between returns of securities C and D is one. The standard deviation of returns of any portfolio combining these two securities is a weighted average of the returns of the two securities' standard deviations. Diversification here yields no benefits. In Figure (2), the correlation coefficient between returns on Securities A and B is .5. Portfolios combining these two securities will have standard deviations less than the weighted average of the standard deviations of the two securities. Given this lower correlation coefficient, which is more representative of "real world" correlations, there are clear benefits to diversification. In fact, we can see in Figures (3) and (4) that decreases in correlation coefficients result in increased diversification benefits. Lower correlation coefficients result in lower risk levels at all levels of expected return. Thus, an investor may benefit by constructing his portfolio of securities with low correlation coefficients.

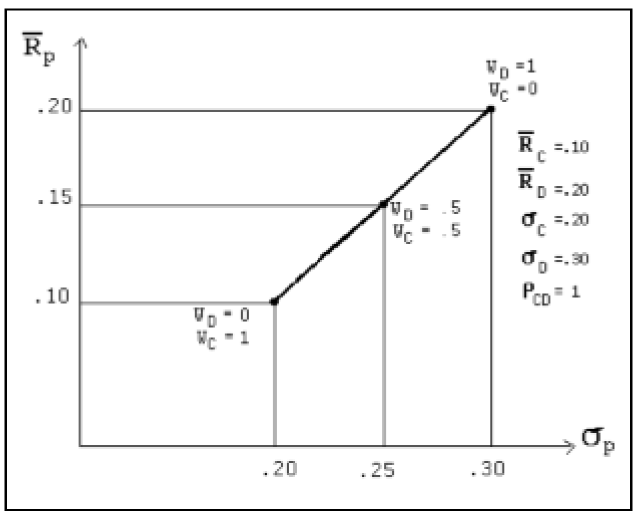

Figure 1: Relationship between portfolio return and risk when ρ_{CD}=1

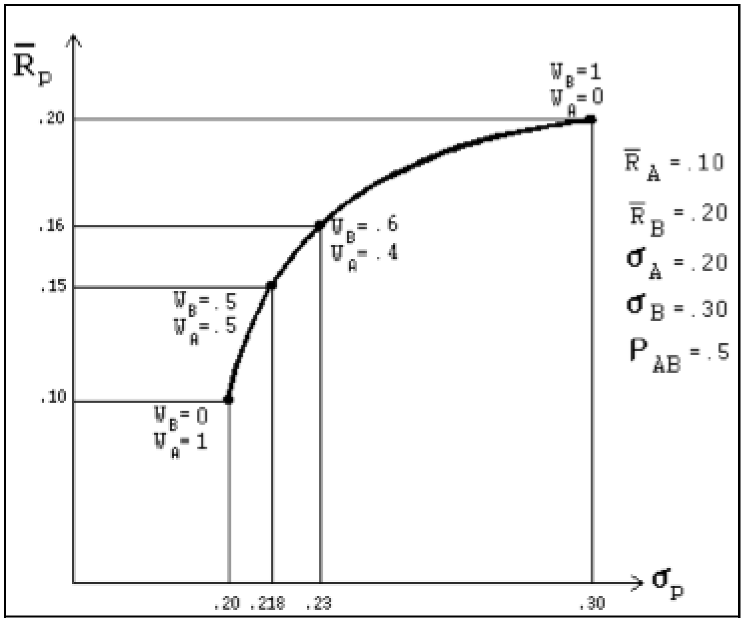

Figure 2: Relationship between portfolio return and risk when ρ_{AB}=.5

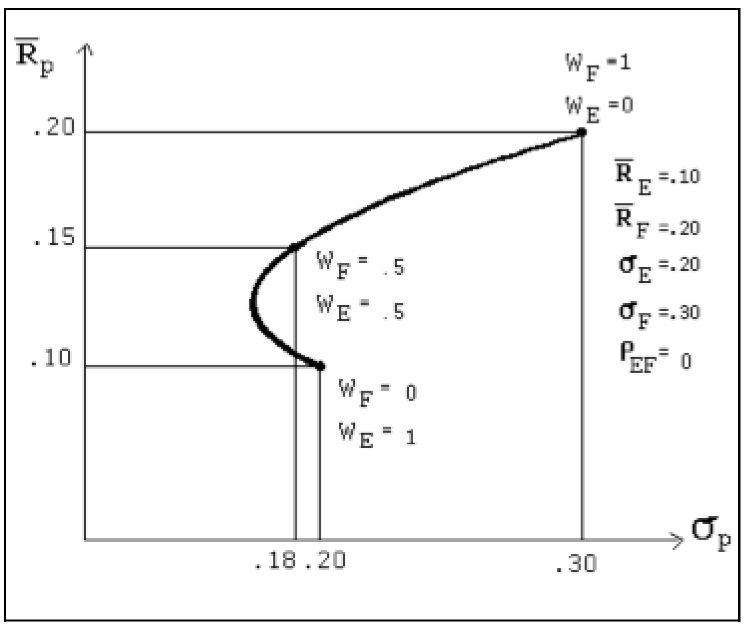

Figure 3: The relationship between portfolio return and risk when ρ_{EF}=0

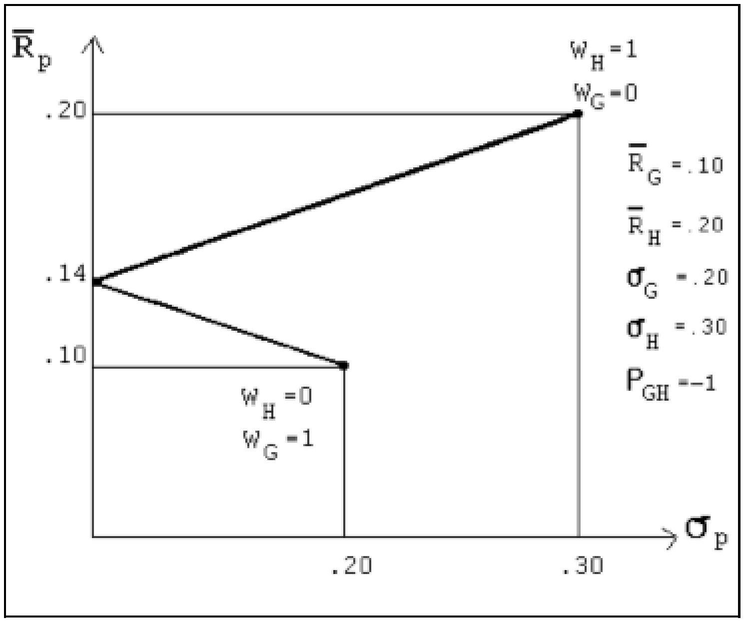

Figure 4: The relationship between portfolio return and risk when ρ_{GH}=-1

Shortly, we will see how adding additional securities beyond 2 further decreases portfolio risk. This relationship is demonstrated in Derivation Box 2. The two keys for diversifying portfolio risk are to select securities with low correlation coefficients with respect to one another and to select many securities. Hence, in the next section, we will discuss portfolio efficiency and how it improves as additional securities are added to a portfolio.

Derivation Box 1
Deriving Portfolio Variance

The variance of security returns can be computed based on either potential or actual historical returns. Portfolio variance may also be found as a function of potential or historical portfolio returns. However, it is often useful to express portfolio variance as a function of individual security characteristics. For example we may have estimates of security variance and covariance levels (based on historical estimates) but have no information regarding probabilities to associate with outcomes. Furthermore, it is useful to know exactly how changing portfolio weights will affect portfolio variances.

To derive the variance of portfolio (p) as a function of security variances, covariances and weights, we begin with our standard variance expression as a function of (m) potential portfolio return outcomes (j) and associated probabilities.

(1) $\sigma_p^2 = \sum_{j=1}^{m} (R_{pj} - E[R_p])^2 P_j$

Derivation Box 1, Continued

For sake of simplicity, let the number of securities (n) in our portfolio equal two. From our portfolio return expression, we may compute portfolio variance as follows:

(A) $\sigma_p^2 = \sum_{j=1}^{m}(w_1 R_{1j} + w_2 R_{2j} - w_1 E\left[R_j\right] - w_2 E\left[R_2\right])^2 P_j$

Next, we complete the square for Equation (A) and combine terms multiplied by the two weights to obtain:

(B) $\sigma_p^2 = \sum_{j=1}^{m}[(w_1^2 (R_{1j} + E\left[R_j\right])^2 + w_2^2 (R_{2j} - E\left[R_2\right])^2 + 2w_1 w_2 (R_{1j} - E\left[R_1\right])(R_{2j} - E\left[R_2\right])]P_j$

Next, we bring the summation term inside the brackets:

(C)

$\sigma_p^2 = w_1^2 \sum_{j=1}^{m}(R_{1j} + E\left[R_j\right])^2 P_j + w_2^2 \sum_{j=1}^{m}(R_{2j} - E\left[R_2\right])^2 P_j + 2w_1 w_2 \sum_{j=1}^{m}(R_{1j} - E\left[R_1\right])(R_{2j} - E\left[R_2\right])P_j]$

We complete our derivation by noting our definitions from Chapter Four for variances and covariances as follows:

(7) $\sigma_p^2 = (\mathrm{w}_1^2\ \sigma_1^2) + (w_2^2\ \sigma_2^2) + 2(w_1\ w_2 \sigma_1 \sigma_2 \rho_{1,2})$

The process for deriving variances for larger portfolios would be analogous.

B. <u>Stock Fit and Optimal Portfolio Weights</u>

Because investors prefer as much return and as little risk as possible, the most efficient portfolios are those with the following characteristics:

1. Less risk than all portfolios with identical or larger returns and
2. Greater return than all portfolios with identical or less risk.

One portfolio dominates a second when one of the following three conditions is met:

a. the first portfolio has both higher return and smaller risk levels than does the second,

b. both portfolios have identical variance but the first portfolio has a higher return level than does the second, or

c. both portfolios have identical returns but the first portfolio has a smaller variance than does the second.

A portfolio is considered dominant if it is not dominated by any other portfolio. Thus, the most efficient portfolios are all dominant.

Consider an investor who has the opportunity to invest in a combination of a risk-free asset and one of several risky portfolios (A) through (E) depicted in Figure 9. Which of these five portfolios is the best to combine with the risk-free asset? Notice that the portfolios with risk-return combinations on the line connecting the risk-free asset and portfolio (C) dominate all other portfolios available to the investor. Thus, any portfolio whose risk-return combination falls on lines extending through portfolios (A), (B), (D), and (E) will be dominated by some portfolio whose risk-return combination is depicted on the line extending through portfolio(C). This line has a steeper slope than all other lines between the risk-free asset and risky portfolios. The investor's objective is to choose that portfolio of risky assets enabling him to maximize the slope of this line; that is, the investor should pick that portfolio with the largest possible ($\mathbf{1}_p$), where ($\mathbf{1}_p$) is defined by Equation (8):

$$\frac{E\lfloor R_p \rfloor - r_f}{\sigma_p} = \Theta_p \tag{8}$$

Therefore, the investor should invest in some combination of portfolio (C) and the risk-free asset. If the curve connecting portfolios (A) through (E) were the Efficient Frontier, then portfolio (C) would be referred to as the market portfolio. This is because every risk averse investor in the market should select this portfolio of risky assets to combine with the riskless asset. Notice that the line extending through portfolio (C) is tangent to the curve at point (C).

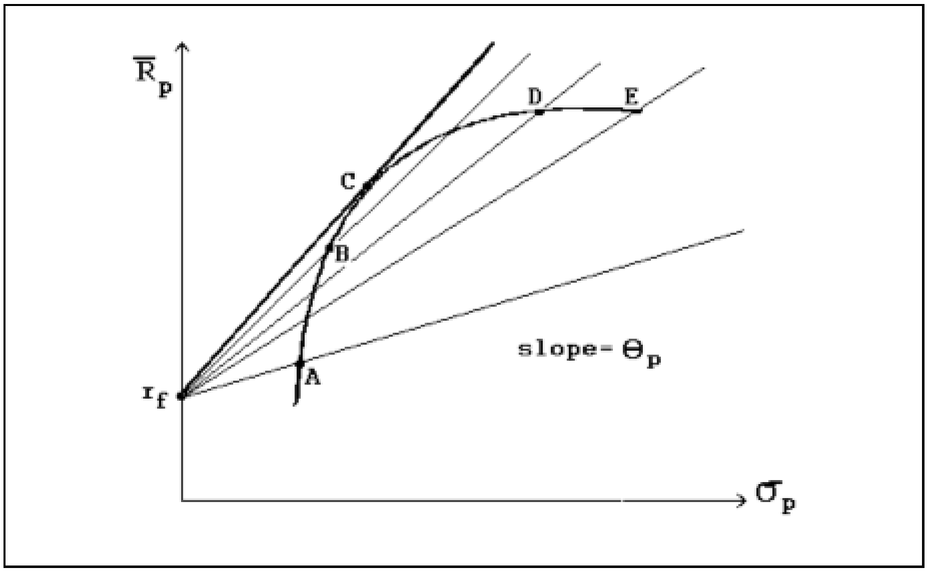

Figure 9: Combination of risk-free asset with one of five portfolios of risky assets

The Capital Market Line

The best portfolio of risky assets to combine with the risk-free security lies on the Efficient Frontier, tangent to the line extending from the risk-free security. This line is referred to as the Capital Market Line (CML). Notice that portfolios on the Capital Market Line dominate all portfolios on the Efficient Frontier. If a risk-free security exists, the Capital Market Line represents risk-return combinations of the best portfolios of securities available to investors. Thus, an investor's risk-return combinations are constrained by the Capital Market Line.

The most efficient portfolio on the Efficient Frontier to combine with the riskless asset is referred to as the Market Portfolio (depicted by [M] in Figure 10). Thus, the Market Portfolio lies at a point of tangency between the Efficient Frontier and the Capital Market Line. All investors should hold portfolios of risky assets whose weights are identical to those of the Market Portfolio. The Capital Market Line combines the Market Portfolio with the riskless asset. This line can be divided into two parts: the lending portion and the borrowing portion. If an investor invests at point (M) on the Capital Market Line, all of his money is invested in the Market Portfolio. If he invests to the left of (M), his portfolio is a lending portfolio. That is, he has purchased treasury bills, in effect, lending the government money, and invested the remainder of his funds in the Market Portfolio. If he invests to the right of point (M), he has a borrowing portfolio. In this case, he has invested all of his funds in the Market Portfolio and borrowed additional money at the risk-free rate to invest in the Market Portfolio. All investors will invest at some risk-return combination on the Capital Market Line. Exactly which risk-return combination an investor will choose will depend on the investor's level of risk aversion.

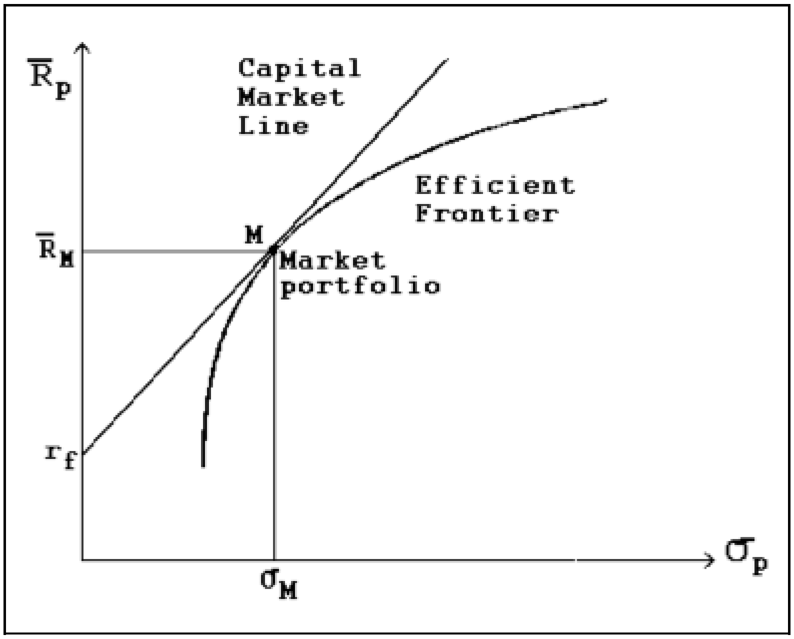

Figure 10: The Capital Market Line

Consider a simple example where there exist two risky securities 1 and 2 in the stock market of Noplacia. A particular investor in this market has projected the following characteristics for these stocks along with a riskless treasury bill:

$E[R_1] = .12$ $\qquad\qquad$ $\sigma_1 = .20; \sigma_2 = .40; \sigma_{1,2} = -.01$

$E[R_2] = .18$

There also exists a riskless treasury instrument (bill) available for investors of Noplacia. The expected return or implied interest rate on this bill is 8%. Given this interest rate and the above stock projections, determine:

1. the stock weightings for the optimal portfolio of risky securities for this investor
2. the expected return of his portfolio of stocks,
3. the risk of his stock portfolio, as measured by standard deviation,
4. the characteristics of the Capital Market Line faced by this investor; that is, what is the equation for the Capital Market Line?

Since our objective is to select a portfolio of the two risky assets such that the slope of the Capital Market Line is maximized, we will select stock portfolio weights such that $\mathbf{1}_p$ is maximized. To accomplish this, we will find partial derivatives of $\mathbf{1}_p$ with respect to weights of each of the two stocks, set the partial derivatives equal to zero and solve for the weight values w_1 and w_2. We accomplish this in Derivation Box 3.

The system of equations we solve to obtain the Capital Market Line in our 2-security economy is:

(9) $$E[R_1] - r_f = z_1\sigma_1^2 + z_2\sigma_{1,2}$$

$$E[R_2]-r_f = z_1\sigma_{2,1} + z_2\sigma_2^2$$

Substituting in appropriate values from our example, we find that:

$$.12-.08=.2^2 z_1 - .01z_2$$
$$.18-.08=-.01z_1 + .4^2 z_2$$

which yields:

$$.04=.04z_1 - .01z_2$$
$$.10=-01z_1 + .16z_2$$

Solving the above simultaneously yields z_1 = 1.174603 and z_2 = .698412. Since $E[R_p]$, r_f and σ_p are the same for both z_1 and z_2, portfolio weights w_1 and w_2 will be linearly related to their z values. Thus, the portfolio weights are determined as follows:

$$w_1 = z_1 \div (z_1 + z_2) = .627$$
$$w_2 = z_2 \div (z_1 + z_2) = .373$$

The return and risk levels of the portfolio (m) of risky stocks are simply:

$$E[R_m] = .627 \cdot .12 + .373 \cdot .18 = .142$$
$$\sigma_m = \left[.627^2 \cdot .04 + .373^2 \cdot .16 + 2 \cdot .627 \cdot .373 \cdot (-.01)\right]^{.5} = .1735$$

Thus the answers to the four problems proposed earlier are as follows:

1. w_1 = .627, w_2 = .373
2. $E[R_m]$ = .142
3. σ_m = .1735
4. The equation for the Capital Market Line is given as follows:

$$E[R_p] = r_f + \frac{E[R_m - r_f]\cdot \sigma_p}{\sigma_m} = .08 + \frac{(.142-.08)\sigma_p}{.1735} = .08 + .3594\sigma_p$$

This process is easily expanded to include as many securities as mayq exist in the market. Matrix mathematics may simplify computations. The Efficient Frontier can be plotted by varying the riskless rate; an additional "market portfolio" is obtained each time a new riskless return is used in the computations.

Derivation Box 3
Deriving the Capital Market Line

Since our objective is to select a portfolio of the two risky assets such that the slope of the Capital Market Line is maximized, we will select stock portfolio weights such that $\mathbf{1}_p$ is maximized. To accomplish this, we will find partial derivatives of $\mathbf{1}_p$ with respect to weights of each of the two stocks, set the partial derivatives equal to zero and solve for the weight w_1 and w_2. First, we write $\mathbf{1}_p$ for the simple two stock portfolio as follows:

$$\text{(A)}\quad \Theta_p = \frac{E\left[R_p\right]-r_f}{\sigma_p} = \frac{w_1(E\left[R_1\right]-r_f)+w_2(E\left[R_2\right]-r_f)}{(w_1^2\sigma_1^2+w_2^2\sigma_2^2+2w_1w_2\sigma_{12})^{1/2}}$$

Next, we use the quotient rule to find the derivative of $\mathbf{1}_p$ with respect to w_1 and w_2:

$$\text{(B)}\quad \frac{\partial\Theta_p}{\partial w_1} = \frac{\frac{\partial(E\left[R_p\right]-r_f)}{\partial w_1}\sigma_p - \frac{\partial\sigma_p}{\partial w_1}(E\left[R_p\right]-r_f)}{\sigma_p^2} = 0$$

$$\frac{\partial\Theta_p}{\partial w_2} = \frac{\frac{\partial(E\left[R_p\right]-r_f)}{\partial w_2}\sigma_p - \frac{\partial\sigma_p}{\partial w_2}(E\left[R_p\right]-r_f)}{\sigma_p^2} = 0$$

However, we need to use the chain rule to find the derivative of the denominator with respect to w_1 and w_2:

$$\text{(C)}\quad \frac{\partial\sigma_p}{\partial w_1} = \frac{1}{2}(\sigma_p^2)^{-1/2}(2w_1\sigma_1^2+2w_2\sigma_{12}) = \frac{w_1\sigma_1^2+w_2\sigma_{12}}{\sigma_p}$$

$$\frac{\partial\sigma_p}{\partial w_2} = \frac{1}{2}(\sigma_p^2)^{-1/2}(2w_2\sigma_1^2+2w_1\sigma_{12}) = \frac{w_2\sigma_2^2+w_1\sigma_{12}}{\sigma_p}$$

Notice from equation (A) that the derivative of ($E[R_p]$-r_f) with respect to w_1 equals ($E[R_1]$-r_f). Next, we substitute our results of equation set (C) into equation set (B) making use of the far right hand side of equation (A):

$$\text{(D)}\quad \frac{\partial\Theta_p}{\partial w_1} = \frac{(E\left[R_1\right]-r_f)\sigma_p - E\left[R_p\right]-r_f)(w_1\sigma_1^2+w_2\sigma_{12})/\sigma_p}{\sigma_p^2} = 0$$

$$\frac{\partial\Theta_p}{\partial w_2} = \frac{(E\left[R_2\right]-r_f)\sigma_p - E\left[R_p\right]-r_f)(w_2\sigma_2^2+w_2\sigma_{12})/\sigma_p}{\sigma_p^2} = 0$$

Because the derivatives from Equation Set D are both set equal to zero, we may multiply the numerator by $\mathbf{F}_p$ and maintain the equality. Next, we re-write Equation Set D as follows:

$$\text{(E)}\quad E\left[R_1\right]-r_f = \frac{(E\left[R_p\right]-r_f)(w_1\sigma_1^2+w_2\sigma_{12})}{\sigma_p^2}$$

$$E\left[R_2\right]-r_f = \frac{(E\left[R_p\right]-r_f)(w_2\sigma_2^2+w_1\sigma_{12})}{\sigma_p^2}$$

To continue the process of simplification, define the variable z_i to be $w_i(E[R_p]-r_f)/\sigma$ and re-write equation set (D) as follows:

$$E\left[R_1\right]-r_f = z_1\sigma_1^2+z_2\sigma_{1,2}$$

$$E\left[R_2\right]-r_f = z_1\sigma_{1,2}+z_2\sigma_2^2$$

This is the general format for deriving the set of equations needed to solve for characteristics and composition of the Capital Market Line. Extending this system to accommodate more risky securities is straightforward; there will be one z-value and one equation for each risky security.

C. Internationalization of Equity Portfolios

In their text on portfolio analysis, Elton and Gruber report that the average correlation coefficient between returns on two randomly selected stocks of U.S. corporations is approximately .40.[1] This correlation is substantially higher than the dollar return correlation between randomly selected U.S. stocks and randomly selected stocks from other countries, which is likely to range from about .1 to .35. Since the U.S. stock market comprises somewhere between thirty and forty percent of world stock markets, ample opportunity exists for American investors to diversify their portfolio risk without sacrificing portfolio return.

The American investor faces several additional risks investing outside of American markets:

1. Country Risk: Many other countries do not have the political and economic stability that exists in the U.S. Thus, stock return variances are frequently higher in other countries than in the U.S. However, country risk between many countries will often be quite low.
2. Currency Exchange Risk: Currency exchange is simply the trading or swapping of currencies. The currency exchange rate is simply the number of units of one currency that must be exchanged for another; the exchange rate represents the costs of currencies. The exchange rate between dollars and a foreign currency will certainly affect the dollar denominated return on an investment made in that country. For example, if an American firm invests in the United Kingdom, all of the British profits will be generated in pounds. These pounds must be exchanged for dollars before they can be spent in the U.S. Since the dollar exchange rate (the value of the dollar) varies over time, one cannot be certain exactly how many dollars can be purchased with profits denominated in pounds. This is clearly a source of risk to American investors. However, currency exchange risk between many countries will often be quite low.

Thus, in the internationally diversified portfolio, the two sources of risk are country risk (σ_c) and foreign exchange risk (σ_{fx}). Because these risks are not perfectly correlated ($\rho_{c,fx} < 1$), overall portfolio risk (σ) as measured by Equation 16 is likely to be less than the risk of a domestic portfolio:

$$(16) \quad \sigma_P^2 = \sigma_c^2 + \sigma_{fx}^2 + 2\sigma_c \sigma_{fx} \rho_{cfx}$$

Although foreign investments are likely to have higher risk (variance) levels for the American investor than the typical domestic investment, they still represent an opportunity for Americans to reduce portfolio risk without sacrificing return. This is due to the particularly low correlation coefficients between American and foreign securities. In fact the following is offered in support of globalizing investment portfolios:

1. Portfolio risk at any return level will be lower for a globally diversified portfolio

[1] Elton, Edwin J. and Martin J. Gruber. Modern Portfolio Theory and Investment Analysis, fourth edition. New York: John Wiley & Sons, Inc.: p.252.

than for a domestic portfolio.

2. Portfolio return at any risk level will be higher for a globally diversified portfolio than for a domestic portfolio.
3. Fewer securities from global markets will be required to attain a given portfolio diversification and risk level than would be required from only a domestic market. This is significant because larger portfolios typically require larger brokerage fees to acquire and are more costly and time consuming to manage.

Elton and Gruber [1992] report that the average correlation coefficient between returns on U.S. securities is approximately .40. The correlation coefficient between two randomly selected 100 security portfolios, one drawn from NYSE stocks and the other selected from AMEX stocks exceeds .90. However, dollar correlations between stock indices of different international markets is significantly smaller than these values, as indicated by the following table:

DOLLAR CORRELATION COEFFICIENTS BETWEEN MARKET INDICES[2]

Correlation of Stock Markets, January 1992–January 2002
Monthly Returns in U.S. Dollars (bottom left) and Currency Hedged (top right)

	USA	Canada	U.K.	France	Germany	Italy	Switz.	Japan	Hong Kong	Europe	EAFE	World	S&P/ IFCG
USA	1.00	0.69	0.63	0.58	0.59	0.34	0.60	0.39	0.51	0.69	0.63	0.86	0.56
Canada	0.70	1.00	0.58	0.63	0.59	0.42	0.53	0.39	0.58	0.67	0.66	0.76	0.63
U.K.	0.66	0.54	1.00	0.74	0.62	0.49	0.69	0.30	0.53	0.74	0.63	0.71	0.56
France	0.58	0.59	0.72	1.00	0.82	0.64	0.68	0.36	0.40	0.74	0.63	0.67	0.54
Germany	0.61	0.59	0.61	0.77	1.00	0.61	0.63	0.34	0.44	0.70	0.56	0.63	0.56
Italy	0.32	0.41	0.34	0.50	0.48	1.00	0.45	0.31	0.23	0.48	0.44	0.44	0.40
Switzerland	0.51	0.42	0.61	0.58	0.50	0.25	1.00	0.30	0.43	0.65	0.55	0.63	0.52
Japan	0.40	0.42	0.44	0.39	0.31	0.27	0.36	1.00	0.17	0.40	0.69	0.62	0.35
Hong Kong	0.51	0.60	0.48	0.40	0.44	0.22	0.37	0.24	1.00	0.49	0.50	0.56	0.70
Europe	0.69	0.65	0.88	0.90	0.84	0.56	0.72	0.45	0.49	1.00	0.87	0.87	0.55
EAFE	0.63	0.65	0.78	0.77	0.70	0.53	0.65	0.79	0.50	0.87	1.00	0.93	0.56
World	0.86	0.76	0.80	0.76	0.72	0.48	0.64	0.69	0.56	0.87	0.93	1.00	0.62
S&P/IFCG	0.56	0.66	0.46	0.48	0.49	0.35	0.36	0.32	0.69	0.55	0.56	0.62	1.00

These correlation coefficients are all based on amounts converted into U.S. dollars. However, cross-country correlation coefficients are not particularly stable over time and have risen somewhat in recent years due to increased globalization and integration of capital markets. Furthermore, cross-country correlation coefficients are highest when markets are more volatile. In addition, some countries still impose significant barriers to outside investment.

Lessard (in Elton and Gruber *International Capital Markets*, 1975) demonstrate the extra return that investors from various countries might obtain from investing in internationalized portfolios with risk levels comparable to their own countries' indices:

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[2] Taken from Solnik and MacLeavey [2003], *International Investments*, 5th edition. Addison Wesley, p. 462.

Country	%Loss in Return from Holding Domestic Portfolio
Australia	4.06
Austria	3.88
Belgium	2.08
Canada	0.85
Denmark	3.96
France	4.47
Germany	3.87
Italy	6.14
Japan	5.01
Netherlands	1.83
Norway	5.19
Spain	4.50
Sweden	3.27
Switzerland	3.22
U.K.	3.30
U.S.A.	0.31

The benefits to be gained from international diversification are quite clear. However, to what extent do investors diversify their portfolios internationally? A study by Cooper and Kaplanis [1994] suggests that 98% of U.S. investors' portfolios are comprised of U.S. securities, even though U.S. equity markets value represents less than one half the world total. The same study reports that 86.7% of Japanese assets are Japanese securities, 75.4% of German portfolios are German securities and 78.5% of U.K. portfolios are invested in U.K. securities. These figures indicate a clear home-country bias in investing. Such a bias would seem not to reflect the advantages of portfolio internationalization discussed above. Two possible (though, probably not fully satisfactory) explanations have been offered in the literature for this home-country bias:[3]

1. Costs of internationalization, including information, management and transactions expenses offset the benefits of diversification.
2. Domestic equity securities provide a better hedge against inflation.

Emerging Markets Equity Investment

The International Finance Corporation (IFC) states that as of the end of 1995, over 19,000 companies were exchange listed in developing companies, representing almost 10% of total world capitalization.[4] This figure reflects a 100% increase over ten years earlier. An emerging market might be defined as a market that is small relative to its country's GNP with limited depth and breadth of traded instruments. The IFC defines an emerging stock market as one that meets the following standards:

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[3]See also Tesar and Werner [1995]

[4]*The Economist*, July 27, 1996, p.66.

1. The stock market has demonstrated growth in terms of volume and sophistication and is expected to continue this growth and
2. The market exists within a low to middle income economy.

Several Asian, Latin American and Eastern European markets meet these criteria. Equity investment in emerging markets may provide for substantial internationally diversified growth opportunities. Consider the following table regarding seven selected emerging markets adapted from Solnick [1996]:

Return and Risk of Emerging Stock Markets in $U.S.
January 1985-December 1993

Country	Annual Return %	%Standard Deviation	Corre. with World
U.S.A.	15.8	15.6	0.71
World	16.9	15.4	1.00
Argentina	40.5	106.2	-0.06
Brazil	13.3	69.7	0.12
India	19.4	34.8	-0.15
Malaysia	18.5	26.5	0.42

Most importantly, notice on this table the relatively low correlation coefficients with the world market. These low correlation coefficients have obvious implications for portfolio diversification.

Although returns associated with emerging markets equity investment can be quite high, many of these markets will have poorly developed regulatory systems. This can frequently mean that the potential for fraud and other market abuses is quite high. For example, a major equity market scandal occurred in Russia in 1994 when it was revealed that Russia's largest investment company, MMM was little more than a Ponzi scheme where investor "profits" were merely proceeds from new investors buying into the fund. The fund's president, Sergei Mavrodi was arrested for tax evasion, then elected to parliament.

The Japanese stock market might have been characterized as an emerging market in the early 1950s. Its total market capitalization to GDP ratio has increased from approximately .1 in 1951 to over 1 by 1990. This growth was fueled by overall strong performance in the Japanese economy, government policy that succeeded in promoting growth and stability that, in turn, increased investor confidence in Japanese equity markets. Depth and breadth in equity markets also increased considerably, such that, the Japanese markets might be characterized as being mature.

D. Equity Portfolio Performance Evaluation

Portfolio evaluation is likely to be based on return calculations. Furthermore, a proper return measure will be crucial to the evaluation of portfolio risk. The measurement of an average annual return for a portfolio held for n years is fairly straightforward when there are no funds added to or withdrawn from the portfolio:

(1) $$R_p = \frac{V_n - V_0}{nV_0}$$

However, matters become significantly more complicated when the investment base of the portfolio is changing. When a total of m cash flows moving in or out of the portfolio, one might measure return with a variation of the internal rate of return (r), solving the following for r:

(2) $$V_n = V_0(1+r)^n - \sum_{t=1}^{n} CF_t(1+r)^{n-t}$$

For example, if a given cash flow CF_t were positive, the portfolio received a deposit from an investor. This increases the investment base of the portfolio, lowering the portfolio's return if no compensating increase in the terminal value (V_n) is realized. A return computed in this manner is called a *dollar weighted return*. A second means for computing a return, the *time weighted average*, requires periodic return computations each time a change to the investment base occurs:

(3) $$ROI_{g,p} = \sqrt[n]{\prod_{t=1}^{n}(1+r_t) - 1}$$

Generally, the time weighted return will be a more meaningful measure because the returns on funds initially deposited into the portfolio will not be affected by future deposits or withdraws. We note that any errors in computing returns will bias measured betas downwards and will "slop" over into unsystematic variances. Even seemingly minor problems can significantly bias beta measures. However, there do exist reasonably good correction procedures for betas measured with error.

Proper benchmarks for comparison must be established when evaluating portfolio returns. Ideally, a benchmark should make allowances for portfolio risk. The following are three well known risk adjusted measures for portfolio performance:

$$S_p = \frac{R_p - r_f}{\sigma_p} \qquad T_p = \frac{R_p - r_f}{\beta_p}$$

Sharpe Measure *Treynor Measure*

Jensen Measure: $\alpha_p = [R_p - r_f]-[\beta_p(r_m-r_f)]$; $[R_p - r_f] = \alpha_p + \beta_p(r_m-r_f)$

However, each of these measures have associated with them difficulties in addition to the proper computation of returns. For example, what exactly is the riskless rate of return for a given period? The Jensen and Treynor measures require computation of market portfolio returns. What is the appropriate index for the market? Roll shows that different indexes (even if highly correlated) will lead to very different performance rankings. Note that the Sharpe measure does not require a market proxy. Once returns have been computed, exactly how are portfolio betas computed? This may be a significant problem, given that the composition of the portfolio is likely to constantly be in transition. Should portfolio beta be computed as a "time-weighted" average of individual security betas, or based on historical portfolio returns? Again, beta mis-measurement here will bias betas downwards, causing apparent performance to be understated.

The following represent additional issues regarding the difficulties of using the above risk adjusted portfolio performance measures:

1. Given that portfolio managers change jobs rather frequently, is it really reasonable to measure fund performance rather than manager performance?
2. How frequently are we able to obtain enough data to obtain statistically significant measures of performance?
3. The Capital Asset Pricing Model is only a one time period model. Multiple time periods and multiple cash flows cause problems in its application.
4. Investors holding funds representing only market segments may find that any measure based on the Capital Asset Pricing Model is inappropriate.
5. The Sharpe Index will understate portfolio performance of undiversified portfolios in a setting where investors, in sum, hold numerous undiversified portfolios.

Portfolio Performance Measurement Example:

Proper benchmarks for comparison must be established when evaluating portfolio returns. Ideally, a benchmark should make allowances for portfolio risk. Here, we will illustrate the Jensen alpha measure to evaluate the performance of a portfolio:

$$[R_p - r_f] = \alpha_p + \beta_p(r_m - r_f)$$

Presumably, a positive alpha (α) that is statistically significant will indicate that the portfolio outperforms the market on a risk-adjusted basis. Consider Table 1, which records returns over a twenty-year period for a portfolio (p) and the market along with riskless return rates. To compute the Jensen alpha measure, we first convert returns to risk premiums by subtracting riskless rates from returns on the portfolio and the market. We then run a regression of portfolio risk premiums against market risk premiums to obtain the following:[1]

$$\begin{array}{ccccc} [R_{p,t} - r_{f,t}] & = & -.076253 & + & .966203(r_{m,t} - r_{f,t}) \\ & & (2.27267) & & (12.5324) \end{array}$$

$$\text{r-square} = .0897 \quad d.f. = 18$$

Thus, based on a t-test, we can conclude that the portfolio beta (.966203) is statistically significant at the 1% level since 12.5324 exceeds its critical value of 2.552. But more importantly, we can conclude that Jensen's alpha is statistically significant at the 5% level since 2.27267 exceeds its critical value of 1.734.

[1] See Appendix A to Chapter 5 for an introduction to simple OLS regressions and their interpretation.

Year	R_p	R_m	r_f	R_p-r_f	R_m-r_f
1985	0.14	0.05	0.03	0.11	0.02
1986	0.11	0.03	0.02	0.09	0.01
1987	0.04	-0.01	0.02	0.02	-0.03
1988	0.16	0.11	0.03	0.13	0.08
1989	0.03	-0.12	0.02	0.01	-0.14
1990	0.14	0.09	0.03	0.11	0.06
1991	0.26	0.13	0.04	0.22	0.09
1992	0.26	0.18	0.05	0.21	0.13
1993	0.13	0.04	0.05	0.08	-0.01
1994	-0.08	-0.11	0.04	-0.12	-0.15
1995	0.11	0.07	0.05	0.06	0.02
1996	0.22	0.17	0.06	0.16	0.11
1997	0.22	0.16	0.07	0.15	0.09
1998	-0.01	-0.05	0.06	-0.07	-0.11
1999	0.04	-0.08	0.05	-0.01	-0.13
2000	0.28	0.21	0.06	0.22	0.15
2001	0.22	0.11	0.07	0.15	0.04
2002	0.21	0.11	0.08	0.13	0.03
2003	-0.04	-0.11	0.07	-0.11	-0.18
2004	0.18	0.12	0.05	0.13	0.07

Table 1: Jensen Alpha Measure Illustration

Investment Companies

A variety of institutional investors manage huge sums of money for millions of clients. Among these institutions are pension funds, life insurance companies, trust departments of banks and investment companies. An *investment company* is an institution which accepts funds from investors for the purpose of investing on their behalf. One might expect (at least in an ideal world) that the investment company would provide professional and competent management services and enjoy cost advantages resulting from economies of scale. These cost advantages should reduce transactions costs, provide for more efficient record-keeping and enable improved diversification. As suggested above, there are a number of different types of investment companies.

A *Unit Investment Trust* invests its funds in a set of securities when it is established and does not normally re-balance its portfolio. The portfolio does not require much management after its establishment, therefore the annual fees tend to be quite low. This is the primary motivation for creation of these trusts. A unit investment trust may also be created to permit investors the opportunity to purchase shares that otherwise would be either unavailable or unattractive to them. For example, by January 1996, a single share of Berkshire Hathaway stock reached a price of approximately $33,000. These shares were simply unaffordable to many individual investors who wished to benefit from Berkshire Hathaway's CEO Warren Buffet's expertise. An investment company created a unit investment trust to pool investors' purchases of Berkshire Hathaway stock.[2] As in the case of this trust, secondary markets often exist for unit investment trusts. In fact, one of their advantages is that, unlike the case for many mutual funds, intra-day trading markets exist for unit investment trusts.

Other unit investment trusts have been created to replicate indices. For example, *Diamonds* are shares of a trust created to replicate the Dow Jones Industrial Average. These shares are traded on the American Stock Exchange. Spiders (Standard & Poors Depositary Receipts) track the S&P 500. Midcap Spiders, which track the S&P Midcap 400 Index, and Webs, which track the Morgan Stanley world equity benchmark indices also trade on the ASE. Combined, these four products accounted for approximately 15% of ASE volume. The "Dogs of the Dow" track the ten DJIA stocks with the highest dividend yields. Although these trusts are designed to track a particular index, they are often more volatile. One explanation for this volatility is that there is substantial trading volume when the market swings. ASE specialist intervention during these swings to maintain liquidity often increases the magnitude of their swings.

Managed investment companies maintain a board of directors and typically retain a management company to manage the fund's assets. Shareholders of management companies incur management fees, miscellaneous administrative expenses and perhaps other costs as well. *Closed-end investment companies* are corporations which issue a specified number of shares that may be traded in secondary markets. One may purchase shares through a broker. There exists substantial evidence that closed end fund IPO's sell at a premium to net asset value and later fall in market price to below net asset value. At present, there is significant controversy as to why this phenomenon persists.[3] *Open-end investment companies* (or, *Mutual funds*) frequently accept additional funds and are willing to repurchase outstanding shares.

Mutual funds that accept funds from investors without a sales charge are called *no-load* funds; investors buy and sell funds at net asset value. Institutions which charge a sales fee are called *load* funds. An investor should be aware that fund performance computations are frequently overstated because returns are usually calculated only on the investment net of fees. In addition, load charge percentages are generally understated in that the loads are determined as a fraction of the amount invested in the fund in addition to any loads charged. In addition, after accounting for fees, one might generally expect that no-load funds will out-perform load funds. Furthermore, these sales fees usually are not paid to the analysts and money managers, so that they may have very little incentive value. Some funds will impose a *back-end load* or redemption fee when redeeming shares. A number of funds adopt a 12b-1 plan which enables them to use fund assets to market their shares. Such 12b-1 expenses are often included with the sum of administrative expenses when computing annual expense ratios.

Mutual funds may be categorized according to their objectives or types of securities they

[2]In early 1996, Buffet created a second class of shares with a substantially lower price to compete with this trust.

[3]See, for example, Charles M.C. Lee, Andrei Schleifer and Richard Thaler "Investor Sentiment and the Closed-end Fund Puzzle,"*Journal of Finance*, 46, March 1991, pp.75-109.

may invest in. Among the categories of mutual funds are aggressive growth funds, growth funds, income funds, balanced funds, gold funds, money market funds, fixed income funds, global funds, high yield bond funds and municipal bond funds. Some larger fund management companies like Dreyfus and T. Rowe Price manage entire families of funds with a variety of investment objectives. In addition, index funds such as those in the Vanguard family have increased in popularity in recent years. These funds employ a passive management technique where managers simply attempt to maintain a fund composition matching a particular market index rather than attempt to "out-guess" the market. This passive management technique is also intended to keep fund management expenses low. Shares of mutual funds are valued, typically ad the end of each day, with net asset value computed as follows:

$$NAV = \frac{\text{Market values of assets minus liabilities}}{\text{\# Shares Outstanding}}$$

Exchange-traded funds (ETFs) are funds whose shares trade on exchanges. The first was the *S&P Depository Receipt* (*SPDR* or "Spider") sponsored by State Street Bank and Merrill Lynch. This fund is intended to mimic performance of the S&P 500 Index by maintaining the same portfolio of securities as is comprised by the index. Unlike most mutual funds, investors can trade this fund throughout the day at prices that vary as the market index varies. Hence, investors can more easily trade the market portfolio throughout the day rather than closing NAVs (net asset values) like mutual funds. Additionally, because this fund is not actively managed, investors can benefit from low management expenses. On the other hand, investors will have to pay brokerage expenses to trade the fund and will be faced with a bid-offer spread. Other well-known ETFs include *DIA* "Diamonds" that mimic the Dow Jones Industrial average and a number of narrow funds like the *i-shares energy sector* that mimics the portfolio of Dow Jones Energy companies.

Hedge funds are private funds that allow investors to pool their investment assets. To avoid SEC registration and regulations, hedge funds usually only accept funds from small numbers (often less than 100) of *qualified* investors, normally high net worth individuals and institutions. Because most hedge funds normally have only a small number of managers, they typically focus their investment strategies on the expertise of a few key managers. Many hedge funds seek investment opportunities or niches where larger institutions are prohibited or constrained due to regulatory restrictions. For example, because many banks, pension funds and other institutions cannot focus activities in the securities of distressed corporations facing reorganization or bankruptcy, some hedge funds will specialize in such investments. Other funds may specialize in short sales and derivatives to hedge against market downturns and others will simply focus on searches for arbitrage opportunities. Hedge fund managers typically take a proportion of assets invested (2% is a norm) and another proportion of profits (20% is typical) as compensation. While hedge funds are frequently able to report results that beat the market, investors should realize that performance results do not usually include the last several months of a hedge fund's existence, which is when a fund is most likely to fail.

A *real estate investment trust (REIT)* is a fund that invests in real estate and/or real estate mortgages, providing investors opportunities to diversify into real estate with relatively small investment sums. REITs also normally borrow large sums of money. REIT shares are frequently traded by the general public on exchanges. To qualify for exemption from corporate income taxation, the must satisfy the following:

Be structured as a corporation, business trust, or similar organization
Be managed by a board of directors or trustees
Have fully transferable shares and a minimum of 100 shareholders
Must have at least 75% of total investment assets in real estate
Must obtain at least 75% of its gross income from rents and mortgage interest
Must pay at least 90% of net income in dividends
Cannot have more than 50% of its shares held by five or fewer individuals during the last half of the year

Commingled funds are partnerships whose investors have pooled their resources for investment purposes. Partners in these funds tend to invest substantially more than is required by mutual funds and there is often there is some special need characteristic shared by these partners. In some instances, banks will pool client funds where it might not be appropriate to manage them separately. In some other instances these partners desire to pool their assets without actually selling them. Some of these partnerships are referred to as exchange or swap funds. In this case, investors can improve diversification without actually selling them and incurring capital gains obligations.

Morningstar, publisher of the *Mutual Fund Sourcebook*, is one of the best sources of information and analysis on mutual funds. For each of the large number of funds that it covers, Morningstar provides analyses of returns, risk, management quality and style and a large mount of other statistical data.

Mutual Fund Performance

Mutual funds have been among the fastest growing financial intermediaries during the past twenty years, with deposits by investors growing at an annual compound rate of 22% (See Gruber [1996]. Over $2.1 trillion was invested in mutual funds by year end 1994. Equity funds accounted for 40.1% of this total, holding 12.2% of all outstanding corporate equity. Their performance, when compared to relevant indices, has been mixed. Index funds' performance have generally tracked relevant indices (less management expenses of between .22% and 1.24% per year). Actively managed funds as a group have trailed the performance of relevant indices. Based on a study by Gruber [1996], mutual fund performance has underperformed the market by 1.94% during the period 1985-94, and after adjusting for beta, by 1.56%. Fund performance seemed inversely related to expense ratios (even before these are deducted) and directly related to past performance. Gruber concludes that poorly performing funds continue to exist because of unsophisticated investors, inability to short-sell funds and investors who face substantial capital gains liabilities when withdrawing money from funds.

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EXERCISES

1. An investor is considering combining Douglas Company and Tilden Company common stock into a portfolio. Fifty percent of the dollar value of the portfolio will be invested in Douglas Company stock; the remainder will be invested in Tilden Company stock. Douglas Company stock has an expected return of six percent and an expected standard deviation of returns of nine percent. Tilden Company stock has an expected return of twenty percent and an expected standard deviation of thirty percent. The coefficient of correlation between returns of the two securities has been shown to be .4. Compute the following for the investor's portfolio:
   a. expected return
   b. expected variance
   c. expected standard deviation

2. Work through each of your calculations in Problem 1 again assuming the following weights rather than those given originally:
   a. 100% Douglas Company stock; 0% Tilden Company stock
   b. 75% Douglas Company stock; 25% Tilden Company stock
   c. 25% Douglas Company stock; 75% Tilden Company stock
   d. 0% Douglas Company stock;100% Tilden Company stock

3. How do expected return and risk levels change as the portfolio proportions invested in Tilden Company stock increase? Why? Prepare a graph with expected portfolio return on the vertical axis and portfolio standard deviation on the horizontal axis. Plot the expected returns and standard deviations for each of the portfolios whose weights are defined in Problems 1 and 2. Describe the slope of the curve connecting the points on your graph.

4. The common stocks of the Landon Company and the Burr Company are to be combined into a portfolio. The expected return and standard deviation levels associated with the Landon Company stock are five and twelve percent, respectively. The expected return and standard deviation levels for Burr Company stock are ten and twenty percent. The portfolio weights will each be 50%. Find the expected return and standard deviation levels of this portfolio if the coefficient of correlation between returns of the two stocks is:
   a. 1
   b. .5
   c. 0
   d. -.5
   e. -1

5. Describe how the coefficient of correlation between returns of securities in a portfolio affects the return and risk levels of that portfolio.

6. An investor is considering combining Securities A and B into an equally weighted portfolio. This investor has determined that there is a twenty percent chance that the economy will perform very well, resulting in a thirty percent return for security A and a twenty percent for security B. The investor estimates that there is a fifty percent chance that the economy will perform only adequately, resulting in twelve percent and ten percent returns for Securities A and B. The investor estimates a thirty percent probability that the economy will perform poorly, resulting in

a negative nine percent return for Security A and a zero percent return for Security B. These estimates are summarized as follows:

outcome	probability	R_{ai}	R_{bi}	R_{pi}
1	.20	.30	.20	
2	.50	.12	.10	
3	.30	-.09	0	

a. What is the portfolio return for each of the potential outcomes?
b. Based on each of the outcome probabilities and potential portfolio returns, what is the expected portfolio return?
c. Based on each of the outcome probabilities and potential portfolio returns, what is the standard deviation associated with portfolio returns?
d. What are the expected returns of each of the two securities?
e. What are the standard deviation levels associated with returns on each of the two securities?
f. What is the covariance between returns of the two securities?
g. Based on your answers to part d in this problem, find the expected portfolio return. How does this answer compare to your answer in part b?
h. Based on your answers to parts e and f, what is the expected standard deviation of portfolio returns? How does this answer compare to your answer in part c?

7. An investor has combined securities X, Y and Z into a portfolio. He has invested \$1000 in Security X, \$2000 into Security Y and \$3000 into Security Z. Security X has an expected return of 10%; Security Y has an expected return of 15% and security Z has an expected return of 20%. The standard deviations associated with Securities X, Y and Z are 12%, 18% and 24%, respectively. The coefficient of correlation between returns on Securities X and Y is .8; the correlation coefficient between X and Z returns is .7; the correlation coefficient between Y and Z returns is .6. Find the expected return and standard deviation of the resultant portfolio.

8. An investor wishes to combine Stevenson Company stock and Smith Company stock into a riskless portfolio. The standard deviations associated with returns on these stocks are 10% and 18% respectively. The coefficient of correlation between returns on these two stocks is -1. What must be each of the portfolio weights for the portfolio to be riskless?

9. Assume that the coefficient of correlation between returns on all securities equals zero in a given market. There are an infinite number of securities in this market, all of which have the same standard deviation of returns (assume that it is .5). What would be the portfolio return standard deviation if it included all of these infinite numbers of securities in equal investment amounts? Why? (Demonstrate your solution mathematically.)

10. Investors have the opportunity to invest in any combination of the securities given in the table below:

(i)	$E[R_i]$	σ_i	$\sigma_{1,i}$	$\sigma_{2,i}$	$\sigma_{3,i}$
1	.25	.40	.16	.05	0

2	.15	.20	.05	.04	0
3	.05	0	0	0	0

Find the slope of the Capital Market Line.

11. Investors have the opportunity to invest in varying combinations of riskless treasury bills and the market portfolio. Investors' investment portfolios will have expected returns equal to [R_p] and standard deviations of returns equal to σ_p. Let w_m be the proportion of a particular investor's wealth invested in the market portfolio. Obviously, the investor's proportional investment in the riskless asset is $w_f = (1-w_m)$. Prove (or derive) the following:

a. $\sigma_P = w_m \sigma_m$

b. $E[R_p] = r_f + \frac{\sigma_p}{\sigma_m}(E[R_m] - r_f)$

Note: If you successfully complete parts a and b, you have derived the equation for the Capital Market Line (where the market portfolio characteristics are known). Now, complete part c:

c. What happens to the slope of the Capital Market Line as each investor's level of risk aversion increases?

12. How would you expect transactions costs to affect borrowing rates of interest? How would lending rates of interest be affected? How would the Capital Market Line be affected by transactions costs?

13. A securities analyst has recommended the purchase of two stocks, A and B to include in the portfolio for one of your clients. The analyst has forecasted returns and risk levels as measured by standard deviation of returns and covariances as follows:

Security	Expected Return	Standard Deviation	
A	.08	.30	COV(A,B) = 0
B	.12	.60	

Your client can borrow money at a rate of 6%, lend money at 4% and has $30,000 to invest. Your client, while not particularly risk averse, wishes to minimize the risk of his portfolio given that his expected return is at least 18%. How much money should he borrow or lend? How much should he invest in each of the two stocks?

14. Are extremely risk averse investors likely to be borrowers or lenders? How does risk aversion affect borrowing levels? Why?

15. An investor with $100,000 has the opportunity to invest in any combination of the securities given in the table below: He wishes to select the most efficient portfolio of these four assets such that his portfolio standard deviation equals .10.

SECURITY(i)	E[R]	COV(i,1)	COV(i,2)	COV(i,3)	COV(i,4)
1	.05	0	0	0	0

2	.10	0	.10	.01	.01
3	.15	0	.01	.20	0
4	.20	0	.01	0	.30

a. How much should the investor invest in each security?
b. *Assuming that the four securities above are the only ones available in the market, what is the Beta of Security 3? (See Chapter 7)

16. Given that correlation coefficients between domestic securities exceed correlation coefficients between domestic and foreign securities, how would expanding the feasible region to include foreign securities affect the Efficient Frontier? How would this expansion affect the Capital Market Line?

SOLUTIONS

1.a. $\overline{R}_P = (w_T \cdot \overline{R}_T) + (w_D + \overline{R}_D) = (.5 \cdot .20) + (.5 \cdot .06) = .13$

b. $\sigma_P^2 = w_D^2 \cdot \sigma_D^2 + w_T^2 + \sigma_T^2 + 2 \cdot W_D \cdot W_T \cdot \sigma_D \cdot \sigma_T \cdot \rho_{D,T}$

$\sigma_P^2 = .5^2 \cdot 09^2 + .5^2 + .30^2 + 2 \cdot .5 \cdot .5 \cdot .09 \cdot .30 \cdot .4$

= .002025 + .0225 + .0054 = .029925

c. $\sigma_P = \sqrt{.029925} = .1729884$, since standard deviation is the square root of variance.

2.a. $\overline{R}_P = .06, \sigma_P^2 = .0081, \sigma_P = .09$

b. $\overline{R}_P = .095, \sigma_P^2 = .0142312, \sigma_P = .1192948$

c. $\overline{R}_P = .165, \sigma_P^2 = .0551812, \sigma_P = .2349067$

d. $\overline{R}_P = .20, \sigma_P^2 = .09, \sigma_P = .3$

3. As proportions of funds invested in the Tilden Company increase, both expected portfolio return and portfolio variance (risk) levels will increase. Portfolio expected return increases because Tilden Company stock has a higher expected return. Portfolio variance increases because the correlation coefficient of .4 is not low enough to offset the high variance of returns on the Tilden Company stock. The slope of the curve should be positive, although, it should be more steep at the bottom.

4.a. $\overline{R}_P = .075, \sigma_P = .16$

b. $\overline{R}_P = .075, \sigma_P = .14$

c. $\overline{R}_P = .075, \sigma_P = .116619$

d. $\overline{R}_P = .075, \sigma_P = .0871770$

e. $\overline{R}_P = .075, \sigma_P = .04$

5. Correlation coefficients have no effect on the expected return of the portfolio. However, a decrease in the correlation coefficients between security returns will decrease the variance or risk of that portfolio.

6.a. R_{p1} = .25 , R_{p2} = .11 , R_{p3} = -.045 ; Since the portfolio weights are equal, each weight is .5.

b. $\overline{R}_P = (.20 \cdot .25) + (.50 \cdot .11) + (.30 \cdot -.045) = .0915$

c. $\sigma_P^2 = (.25 - .0915)^2 \cdot .20 + (.11 - .0915)^2 \cdot .50 + (-.045 - .0915)^2 \cdot .30$

$\sigma_P^2 = .0050244 + .0001711 + .0055896 = .0107851; \sigma_P = .1038517$

d. $\overline{R}_A = .093; \overline{R}_B = .09$

e.

$\sigma_A^2 = (.30 - .093)^2 \cdot .20 + (.12 - .093)^2 \cdot .50 + (-.09 - .093)^2 \cdot .30$

$\sigma_A^2 = .018981; \sigma_A = .1377715$

$\sigma_B^2 = (.20 - .09)^2 \cdot .20 + (.10 - .09)^2 \cdot .50 + (.0 - .09)^2 \cdot .30$

$\sigma_B^2 = .0049; \sigma_B = ..07$

f.

$\sigma_{AB} = (.30 - .093) \cdot (.20 - .09) \cdot .20 + (.12 - .093) \cdot (.1 - .09) \cdot .50 + (.09 - .093) \cdot (0 - .09) \cdot .30 = .00963$

$\overline{R}_P = (.5 \cdot .093) + (.5 \cdot .09) = .0915;$

It is the same, though found using portfolio weights and expected security returns rather than portfolio return outcomes and associated probabilities.

h. $\sigma_P^2 = .5^2 \cdot .1377715^2 + .5^2 \cdot .07^2 + 2 \cdot .5 \cdot .5 \cdot .1377715 \cdot .07 \cdot .998548$

where .998548 $\rho_{A,B} = .00963/(.1377 \times .07)$

$\sigma_P^2 = .0107851; \sigma_P = .1038517;$ the same as part c

7. Security weights are: $w_X = .167$, $w_Y = .333$, $w_Z = .5$

$\overline{R}_P = (.167 \cdot .10) + (.333 \cdot .15) + (.05 \cdot .20) = .167$

$\sigma_P^2 = (.167 \cdot .167 \cdot .12 \cdot .12 \cdot 1) + (.167 \cdot .333 \cdot .12 \cdot .18 \cdot .8)$
$+ (.167 \cdot .5 \cdot .12 \cdot .24 \cdot .7) + (.333 \cdot .167 \cdot .18 \cdot .12 \cdot .8)$
$+ (.333 \cdot .333 \cdot .18 \cdot .18 \cdot 1) + (.333 \cdot .5 \cdot .18 \cdot .24 \cdot .6)$
$+ (.5 \cdot .167 \cdot .24 \cdot .12 \cdot .7) + (.5 \cdot .333 \cdot .24 \cdot .18 \cdot .6)$
$+ (.5 \cdot .5 \cdot .24 \cdot .24 \cdot 1) = .0323144; \sigma_P = .179762$

8. Here, we want to find that w_A value that will set portfolio variance equal to zero. Remember that portfolio weights must sum to one. Thus, w_B is simply 1 - w_A. First, take what we know and substitute into the 2-security portfolio variance equation:

$\sigma_p^2 = w_A^2 \cdot .10^2 + w_B^2 \cdot .18^2 + 2 \cdot w_A \cdot w_B \cdot .10 \cdot .18 \cdot -1 = 0$

Since w_B is simply 1 - w_A, we substitute and simplify as follows:

$0 = .01 w_A^2 + .0324 \cdot (1 - w_A)^2 - .036 \cdot w_A \cdot (1 - w_A)$

Now, we separate out the (1 - w_A) terms:

$0 = .01 w_A^2 + .0324 + .324 w_A{}^2 - .0648 w_A - .036 w_A + .036 w_A^2$

Next, we combine similar terms:

$0 = .0784 w_A^2 - .1008 w_A + .0324$

Note that this expression is set up in descending order of exponents. Now let a=.0784, b=-.1008 and c=.0324. Solve for w_A using the quadratic formula:

$w_A = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, where a=.0784, b=-.1008 and c=.0324.

$w_A = \dfrac{.1008 + \sqrt{.1008^2 - 4 \times .0784 \times .0324}}{2 \times .0784} = \dfrac{.1008 + 0}{.1568} = .64286,$

Plugging in for a, b and c, we find that the portfolio is riskless when w_A = .64286. Thus, w_B = .35714.

Riskless portfolios can be constructed from risky securities only when their returns are perfectly inversely correlated. Even in this case, only one combination of weights results in a riskless portfolio.

9. This would be a perfectly diversified portfolio; its standard deviation will be zero. Portfolio variance is determined as follows:

$$\sigma_P^2 = 2[\sum_{i=1}^{\infty}\sum_{j=1, j\neq i}^{\infty}(1/n \cdot 1/(n-1) \cdot 0] + \sum_{j=1}^{\infty}(1/n)^2 \cdot \sigma_i^2$$

$$\sigma_P^2 = 2[\sum_{i=1}^{\infty}\sum_{j=1, j\neq i}^{\infty}(1/\infty \cdot 1/(\infty-1) \cdot 0] + \sum_{j=1}^{\infty}(1/\infty)^2 \cdot \sigma_i^2 = 0 + 1/\infty \cdot \sigma_i^2 = 0 + 0 = 0$$

10. Solve the following for z(1) and z(2):

.16z(1) + .05z(2) = (.25-.05)

.05z(1) + .04z(2) = (.15-.05)

z(1) = .7692308 ; z(2) = 1.5384616

w(1) = 7692308 /(1.5384616+7692308) ; w(2) = 1.5384616/(1.5384616+7692308) = .666666667

Thus, w(1) = .333333333 and w(2) = .666666667

E[R(m)] = .1833 ; σ^2 = .0577 ; σ_m = .2403 ; Θ = .5547

11.a. $\sigma_P = \sqrt{w_F^2\sigma_F^2 + w_M^2\sigma_M^2 + 2w_F w_M \sigma_{F,M}}$

$$\sigma_P = \sqrt{w_F^2 \cdot 0 + w_M^2\sigma_M^2 + 2w_F w_M \cdot 0} = w_M \sigma_M$$

b. $E[R_P] = w_f E[r_f] + w_m E[R_m]; w_m = \sigma_p / \sigma_m ; w_f = 1 - w_m$

$$E[R_P] = (1 - \frac{\sigma_p}{\sigma_m} \cdot r_f + \frac{\sigma_p}{\sigma_m} E[R_m]); r_f + \frac{\sigma_p}{\sigma_m}(E[R_m - r_f])$$

c. If the derived CML was for a single investor, an increase in risk aversion will lead to an increased required risk-premium on the market (R_m), increasing the slope of the CML. If the derived CML was for a market of many investors, no single investor will be able to affect its slope. In this case, the slope will remain unchanged; an investor will simply vary his holdings of the riskless asset.

12. Increase the cost of borrowing - if the borrower pays this cost, borrowing rates increase; decrease the return from lending - lending rates decrease if the lender pays this cost; In summary, transactions costs increase interest rates; get two separate lines - one for borrowing and one for lending

13. Use borrowing rate of .06 since he is likely to be a borrower - his 18% required return exceeds the return of either security.

.02 = .09z_1 + 0z_2 ; z_1= .2222

.06 = 0 z_1 + .36z_2 ; z_2= .1667 ; w_1=.571 ; w_2=.429

E[R_m] = .097 ; w_f=(1-w_m); E[R_p] = .18 = (1-w_m).06 + w_m.097

.18 = .06 + w_m.037 ; w_m = 3.243 ; Borrow \$67,297

Invest \$97,297 in the market - \$55,556 in security 1 and \$41,741 in Security 2

14. More Risk Averse: Lenders: Increasing risk aversion decreases borrowing; decreasing borrowing decreases risk.

15. This problem may be regarded as being a bit more difficult than the others. For part a, first, note that Security 1 is the riskless asset. We will first find the weights of the optimal portfolio of stocks and then find the optimal combination of stocks and bonds. We begin by solving the following for z(2), z(3) and z(4):

.10z(2) + .01z(3) + .01z(4) = (.10 -.05)
.01z(2) + .20z(3) + 0 z(4) = (.15 -.05)
.01z(2) + 0 z(3) + .30z(4) = (.20 - .05)

We find that z(2) = 0.403361, z(3) = 0.479832 and z(4) = 0.486555. This implies that for the optimal portfolio of stocks (the market portfolio), w(2) = 0.294479, w(3) = 0.350307 and w(4) = 0.355215. Now, the investor needs to determine the optimal mix of stocks and bonds. First, compute the standard deviation of the three-stock portfolio:

$$\sigma_m = \left[w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_{1,2} + 2w_1w_3\sigma_{1,3} + 2w_2w_3\sigma_{2,3}\right]^{.5} = .274268$$

Any portfolio on the Capital Market Line must be a linear combination of riskless bonds and the market portfolio of stocks. Our target portfolio with a standard deviation equal to .10 must be a linear combination of the standard deviations of the riskless asset and of the market portfolio of stocks:

$$\sigma_P = .10 = \left[w_{rf}^2\sigma_{rf}^2 + w_m^2\sigma_m^2 + 2w_{rf}w_m\sigma_{rf,m}\right]^{.5} = \left[0 + w_m^2 \cdot .274268^2 + 0\right]^{.5}$$

$$w_m = .10/.274268 = .364606$$

Thus, 36.4606% of the investor's money should be invested in the stock portfolio and the remaining \$63,539.4 in riskless bonds. Thus, of the \$36,460.6 in stocks, \$10,736.9 should be invested in Stock 2, \$12,772.4 in Stock 3 and \$12,951.4 in Stock 4. For part b (which is a bit more difficult still), we know from the derivation of the Capital Asset Pricing Model that:

$$\sigma_{3,m} = (w_2\sigma_{2,3} + w_3\sigma_3^2 + w_4\sigma_{3,4})$$

This implies that: $\sigma_{3,m}$ = .294479*.01 + .350307*.2 + 355215*0 = .0730187

Since the variance of the market is its standard deviation squared = .075223 and the beta of an asset equals its covariance with the market divided by the market variance, the beta of Security 3 is determined to be:

$$\beta_3 = \sigma_{3,m} \div \sigma_m^2 = .0730187/.075223 = .97069$$

16. Globalizing portfolios will shift the feasible region, efficient frontier and the Capital Market Line upwards and to the left.