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DIFFUSION APPROXIMATIONS FOR THE MAXIMUM OF A PERTURBED RANDOM WALK

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Abstract

Consider a random walk $S = (S_n : n \geq 0)$ that is "perturbed" by a stationary sequence $(\xi_n : n \geq 0)$ to produce the process $S = (S_n + \xi_n : n \geq 0)$. This paper is concerned with developing limit theorems and approximations for the distribution of $M_n = \max\{S_k + \xi_k : 0 \leq k \leq n\}$ when the random walk has a drift close to zero. Such maxima are of interest in several modeling contexts, including operations management and insurance risk theory. The associated limits combine features of both conventional diffusion approximations for random walk and extreme value limit theory.

1. Introduction

Let $S = (S_n : n \geq 0)$ be a random walk sequence, so that $S_0 = 0$ and $S_n = X_1 + ... + X_n$ where the X_i's are independent and identically distributed (iid). Given a sequence $(\xi_n : n \geq 0)$ of "perturbations", we call the process $(S_n + \xi_n : n \geq 0)$ a "perturbed random walk". This paper is concerned with developing limit theorems and related approximations for the

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maximum

$$M_n = \max_{0 \le k \le n} (S_k + \xi_k).$$

"Perturbed random walks" have been previously studied in the insurance risk theory literature; see, for example, Gerber (1970), Schmidli (1995), and Schlegel (1998). In contrast with the perturbations studied in those papers, which have generally been themselves of random walk type, we shall assume that $(\xi_n : n \ge 0)$ is a stationary sequence. Our stationary assumption also means that our perturbed random walks cannot be viewed through the prism of nonlinear renewal theory (see for instance Woodroofe (1982)), because our perturbations typically violate the "uniform continuity in probability" hypothesis that is common to that literature. The class of stationary perturbations discussed here is both natural from a modeling standpoint, and reflects a middle ground between the perturbations of random walk type (in which the perturbations are of the same magnitude as the unperturbed random walk) and those arising in nonlinear renewal theory (in which the perturbations are, in a sense, asymptotically constant; see Glasserman and Liu (1997) for a particular case of such perturbations appearing in an inventory production context).

For our class of perturbed random walks, the large-time behavior of $S_n + \xi_n$ is largely explained by that of the random walk S itself. On the other hand, the maximum M_n inherits some of the extreme value behavior of the perturbations. Consequently, the limit theory that appears in this paper combines features of the classical diffusion approximations for conventional random walk with certain characteristics associated with the extreme value behavior of $(\xi_n : n \ge 0)$.

As indicated above, the distribution of M_n arises in several applications settings. In particular:

1.) M_n has the distribution of the time spent by order n in a make-to-order production facility, in which possible delays in the delivery of supplier components are explicitly modeled;

2.) M_n arises as the end-to-end delay for the n'th data packet in a communications network with multiple paths connecting the source node to the destination node;

3.) $\mathrm{P}(M_\infty > x)$ is the ruin probability for an insurer having initial reserve x, in which certain customers do not necessarily pay their premiums on time.

We refer the reader to Araman and Glynn (2004) for further details on these modeling applications. Araman and Glynn (2004) also study the tail probability $\mathrm{P}(M_\infty > x)$ as $x \to \infty$ and develop a number of different asymptotics for both heavy-tailed and light-tailed perturbations.

In contrast, our interest here is on studying approximations for M_n when the random walk S has drift close to zero. This setting is of great importance, in view of our modeling applications. In particular, asserting that S is almost driftless corresponds to a make-to-order facility that is running at close to 100% utilization. In addition, in the insurance risk setting, this assumption is equivalent to a marketplace in which the insurer can set its premiums at a rate only slightly higher than the average pay-out rate. Of course, from a mathematical standpoint, this nearly driftless setting corresponds to the environment in which diffusion approximations for (conventional) random walk are applicable.

This paper presents four different types of results. Section 2 is concerned with finite-time diffusion approximations for M_n when the perturbations are light-tailed, whereas Section 3 describes the corresponding theory for heavy-tailed perturbations. Section 4 provides a diffusion approximation for M_∞ in the light-tailed case, while Section 5 concludes the paper with a description of approximations for M_∞ in the presence of heavy tails.

2. Finite Horizon Limit Theory with Light Tails

To rigorously describe such heavy-traffic limit theory, we need to consider a family of perturbed random walks that is parameterized by the mean of the increment random variable. To this end, consider a perturbed random walk that describes a processing facility with "balanced loading", so that the increment random variables $(X_i : i \geq 1)$ have zero mean. For $\mu \in \mathbb{R}$, let $X_i(\mu) = X_i + \mu$ and put $S_n(\mu) = \sum_{i=1}^n X_i(\mu)$ for $n \geq 0$. Furthermore, we shall permit the distribution of the perturbation to depend upon μ. Specifically, for the perturbed random walk indexed by μ, put $\xi_i(\mu) = \kappa(\mu)\xi_i$, for some appropriately chosen $\kappa(\cdot)$. We shall be concerned with the behavior of the maximum random variable

$$M_n(\mu) = \max_{0 \leq k \leq n} (S_k(\mu) + \xi_k(\mu))$$

for μ close to zero.

The assumption that we impose on the X_i's (in which the X_i's are describing a system with "balanced loading") is standard in the "heavy traffic" literature, namely that $(S_n :$

$n \geq 0$) satisfy a functional central limit theorem or invariance principle.

A1. Suppose that there exist constants σ and $p \geq 2$ and a probability space supporting a sequence of random variables $(X_i' : i \geq 1)$ and a standard Brownian motion $B = (B(t) : t \geq 0)$ for which:

$$\text{(i)} \quad (X_i' : i \geq 1) \stackrel{D}{=} (X_i : i \geq 1)$$

$$\text{(ii)} \quad \sum_{i=1}^{n} X_i' = \sigma B(n) + o(n^{1/p}) \quad \text{a.s. as } n \to \infty$$

The precise form of A1 is that of a "strong approximation" hypothesis. Such an assumption is valid when the X_i's are independent and identically distributed (i.i.d.) with $\mathrm{E}\,|X_i|^p < \infty$ for $p > 2$; see p.107 of Csörgő and Révész (1981). However, this hypothesis is also known to be valid for a large class of dependent X_i's, as well; see, for example, Philipp and Stout (1975) and Csáki and Csörgő (1995). Because the X_i''s appearing in A1 share the same joint distributions as those of the X_i's, we can and will henceforth assume that the probability space supporting the X_i's is that guaranteed by A1.

Turning now to our assumptions on the perturbations, consider:

A2. $(\xi_j : j \geq 0)$ is a stationary sequence for which there exist positive constants γ and β such that

$$\frac{\max_{0 \leq k \leq n} \xi_k}{(\log n)^\gamma} \Rightarrow \beta$$

as $n \to \infty$.

A2 holds, for example, when the ξ_j's are i.i.d. with a right tail satisfying

$$\frac{\log \mathrm{P}(\xi_j > x)}{x^{1/\gamma}} \to -\beta^{-1/\gamma} \tag{1}$$

as $x \to \infty$. However, A2 continues to be valid under quite modest dependency hypotheses on the ξ_j's; see Glynn and Zeevi (2000) for details. It should be noted that the tail condition (1) holds for ξ_j's that have Gaussian, gamma, or Weibull-type (right) tails.

We are now ready to state our first heavy-traffic limit theorem.

Theorem 1. *Assume A1 and A2, and suppose that $\kappa(\mu) \sim c(|\mu|(\log(1/|\mu|))^\gamma)^{-1}$ as μ either decreases or increases to zero, where c is a positive constant. Then,*

$$|\mu| M_{\lfloor t/\mu^2 \rfloor}(\mu) \Rightarrow \max_{0 \leq s \leq t} [\sigma B(s) - s] + 2^\gamma \beta c$$

as $\mu \nearrow 0$ *and*

$$|\mu|M_{\lfloor t/\mu^2 \rfloor}(\mu) \Rightarrow \max_{0 \le s \le t}[\sigma B(s) + s] + 2^\gamma \beta c$$

as $\mu \searrow 0$.

Proof. We consider only the case where $\mu \nearrow 0$, so that μ is negative throughout the proof; the case $\mu \searrow 0$ can be handled analogously. We start by noting that the upper bound

$$|\mu|M_{\lfloor t/\mu^2 \rfloor}(\mu) \le |\mu| \max_{0 \le k \le \lfloor t/\mu^2 \rfloor} S_k(\mu) + |\mu| \max_{0 \le k \le \lfloor t/\mu^2 \rfloor} \xi_k(\mu) \tag{2}$$

clearly holds. For the lower bound, fix $\varepsilon > 0$ and use the path-by-path uniform continuity of Brownian motion to choose l large enough so that

$$\mathrm{P}(\max_{\substack{0 \le |s-u| \le t/l \\ 0 \le s,u \le t}} |(\sigma B(s) - s) - (\sigma B(u) - u)| > \varepsilon/3) < \varepsilon.$$

Let k^* be a maximizer of $\max\left\{S_k(\mu) : 0 \le k \le \lfloor t/\mu^2 \rfloor\right\}$, $t(i) = it/l$, and let $i^* = \lfloor l\mu^2 k^*/t \rfloor$, so that $t(i^*) \le \mu^2 k^* \le t(i^*+1)$. We then have the lower bound

$$\begin{aligned}
|\mu|M_{\lfloor t/\mu^2 \rfloor}(\mu) &\ge |\mu| \max_{t(i^*) \le \mu^2 k \le t(i^*+1)} \{S_k(\mu) + \xi_k(\mu)\} \\
&\ge |\mu| \min_{t(i^*) \le \mu^2 k \le t(i^*+1)} S_k(\mu) + |\mu| \max_{t(i^*) \le \mu^2 k \le t(i^*+1)} \xi_k(\mu) \\
&\ge |\mu| \max_{0 \le k \le \lfloor t/\mu^2 \rfloor} S_k(\mu) \\
&\quad - |\mu| \max_{\substack{\mu^2|k-j| \le t/l \\ 0 \le k,j \le \lfloor t/\mu^2 \rfloor}} |S_k(\mu) - S_j(\mu)| \\
&\quad + |\mu| \min_{0 \le i < l} \max_{it/l\mu^2 \le k < (i+1)t/l\mu^2} \xi_k(\mu).
\end{aligned} \tag{3}$$

It follows from the upper bound (2) and the lower bound (3) that

$$\begin{aligned}
\left| |\mu|M_{\lfloor t/\mu^2 \rfloor}(\mu) - |\mu| \max_{0 \le k \le \lfloor t/\mu^2 \rfloor} S_k(\mu) - 2^\gamma \beta c \right| &\le \left| |\mu| \max_{0 \le k \le \lfloor t/\mu^2 \rfloor} \xi_k(\mu) - 2^\gamma \beta c \right| \\
&\quad + \left| |\mu| \min_{0 \le i < l} \max_{it/l\mu^2 \le k < (i+1)t/l\mu^2} \xi_k(\mu) - 2^\gamma \beta c \right| \\
&\quad + |\mu| \max_{\substack{\mu^2|k-j| \le t/l \\ 0 \le k,j \le \lfloor t/\mu^2 \rfloor}} |S_k(\mu) - S_j(\mu)|
\end{aligned} \tag{4}$$

On the basis of A1, standard arguments imply that

$$|\mu| \max_{0 \le k \le \lfloor t/\mu^2 \rfloor} S_k(\mu) \Longrightarrow \max_{0 \le s \le t}\{\sigma B(s) - s\} \tag{5}$$

and

$$|\mu| \max_{\substack{\mu^2|k-j|\leq t/l \\ 0\leq k,j\leq \lfloor t/\mu^2 \rfloor}} |S_k(\mu) - S_j(\mu)| \Longrightarrow \max_{\substack{0\leq |s-u|\leq t/l \\ 0\leq s,u\leq t}} |(\sigma B(s) - s) - (\sigma B(u) - u)| \tag{6}$$

as $\mu \nearrow 0$; see Glynn (1998), for example. Also A2 shows that

$$|\mu| \max_{0\leq k\leq \lfloor t/\mu^2 \rfloor} \xi_k(\mu) = |\mu|\kappa(\mu) \max_{0\leq k\leq \lfloor t/\mu^2 \rfloor} \xi_k \Longrightarrow 2^\gamma \beta c \tag{7}$$

as $\mu \nearrow 0$. The stationarity of the ξ_j's then implies that

$$|\mu| \max_{it/l\mu^2 \leq k < (i+1)t/l\mu^2} \xi_k(\mu) \Longrightarrow 2^\gamma \beta c$$

as $\mu \nearrow 0$, from which it follows easily that

$$|\mu| \min_{0\leq i<l} \max_{it/l\mu^2 \leq k < (i+1)t/l\mu^2} \xi_k(\mu) \Longrightarrow 2^\gamma \beta c \tag{8}$$

as $\mu \nearrow 0$. Consequently, (4) through (8) establish that

$$\begin{aligned}
&\mathrm{P}\left(\left| |\mu| M_{\lfloor t/\mu^2 \rfloor}(\mu) - |\mu| \max_{0\leq k\leq \lfloor t/\mu^2 \rfloor} S_k(\mu) - 2^\gamma \beta c \right| > \varepsilon \right) \\
&\leq \mathrm{P}\left(\left| |\mu| \max_{0\leq k\leq \lfloor t/\mu^2 \rfloor} \xi_k(\mu) - 2^\gamma \beta c \right| > \varepsilon/3 \right) \\
&\quad + \mathrm{P}\left(\left| |\mu| \min_{0\leq i<l} \max_{it/l\mu^2 \leq k < (i+1)t/l\mu^2} \xi_k(\mu) - 2^\gamma \beta c \right| > \varepsilon/3 \right) \\
&\quad + \mathrm{P}\left(|\mu| \max_{\substack{\mu^2|k-j|\leq t/l \\ 0\leq k,j\leq \lfloor t/\mu^2 \rfloor}} |S_k(\mu) - S_j(\mu)| > \varepsilon/3 \right) \\
&\to \mathrm{P}\left(\max_{\substack{0\leq |s-u|\leq t/l \\ 0\leq s,u\leq t}} |(\sigma B(s) - s) - (\sigma B(u) - u)| > \varepsilon/3 \right) < \varepsilon
\end{aligned}$$

as $\mu \nearrow 0$. Thus we have proved that

$$\left| |\mu| M_{\lfloor t/\mu^2 \rfloor}(\mu) - |\mu| \max_{0\leq k\leq \lfloor t/\mu^2 \rfloor} S_k(\mu) - 2^\gamma \beta c \right| \Longrightarrow 0$$

as $\mu \nearrow 0$. A "converging together" argument together with (5) prove the theorem.

We note that $\max_{0\leq s\leq t} [\sigma B(s) - s] \overset{D}{=} X(t)$, where $X = (X(t) : t \geq 0)$ is a reflecting Brownian motion with drift -1 and variance parameter σ^2, conditional on starting from the origin. As a consequence, the limiting random variable appearing in Theorem 1 is known to have a closed-form distribution; see, for example, Harrison (1985).

The magnitude of $\kappa(\mu)$ is chosen in Theorem 1 so that we obtain a limiting regime in which both the random walk $(S_n : n \geq 0)$ and the perturbation $(\xi_j : j \geq 0)$ influence the limiting random variable. (If $\kappa(\mu)$ is chosen to go to infinity more slowly than is specified in Theorem 1, the limit random variable turns out to be the same reflecting Brownian motion as would appear in the standard "heavy traffic" limit for the unperturbed random walk $(S_n : n \geq 0)$, whereas if $\kappa(\mu)$ is chosen to go to infinity more rapidly, the limit is governed only by the distribution of the ξ_j's.)

Intuitively, we expect better approximations to the distribution of the maximum of perturbed random walk when the approximating limit random variable depends on the distributions of both the underlying random walk and the associated perturbations. For a given perturbed random walk $(S_j + \xi_j : j \geq 0)$, the approximation suggested by Theorem 1 is

$$\max_{0 \leq k \leq n} \{S_k + \xi_k\} \overset{D}{\approx} \max_{0 \leq s \leq n} \{\sigma B(s) + \mu s\} + (\log n)^{\gamma} \beta \tag{9}$$

where $\overset{D}{\approx}$ denotes "has approximately the same distribution as" (and is intended to have no rigorous meaning), and the parameters μ, σ^2, γ and β are obtained from (1) and the relations

$$\mu = \lim_{n \to \infty} \frac{1}{n} \, \mathrm{E} \, S_n \tag{10}$$

and

$$\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \operatorname{var} S_n. \tag{11}$$

We define μ and σ^2 as in (10) and (11) because Theorem 1 permits the increments of the underlying random walk to be dependent, in which event (10) and (11) are the appropriate relations that typically define μ and σ^2. It should be further noted that Theorem 1 permits some dependency in the ξ_j's (see Glynn and Zeevi (2000) for details), and further allows the random walk and the perturbations to be correlated. Permitting such a dependency can be useful. For instance, the processing times at the server (in Example 1 of Araman and Glynn (2004)) may well be correlated with the perturbations. Indeed, supplier delay may be a consequence of a large order, which in turn may also cause the processing time at the production facility for that order to be larger than normal.

With regard to the mathematical validity of the approximation (9), Theorem 1 proves that the approximation should be good over spatial scales of order $1/|\mu|$ when $|\mu|$ is small, the time n is of order $1/\mu^2$, and the perturbations are quite large (of the order of $\kappa(|\mu|)$.

Unlike the conventional heavy-traffic limit theorem for queues, Theorem 1 can not be derived by directly applying a "continuous mapping" argument to a functional limit theorem for the perturbed random walk $(|\mu|(S_{\lfloor t/\mu^2 \rfloor}(\mu) + \xi_{\lfloor t/\mu^2 \rfloor}(\mu) : t \geq 0)$. The reason is that $|\mu| S_{\lfloor t/\mu^2 \rfloor}(\mu)$ is converging (weakly) to a process with continuous paths (namely, Brownian motion), while $(|\mu|\xi_{\lfloor t/\mu^2 \rfloor}(\mu) : t \geq 0)$ takes on both very large and very small values in any time interval of positive length as $\mu \nearrow 0$, precluding the possibility that its limit can live on a function space like $D[0,\infty)$, the space of right continuous functions defined on $[0,\infty)$, with left limits. As a consequence, $(|\mu|(S_{\lfloor t/\mu^2 \rfloor}(\mu) + \xi_{\lfloor t/\mu^2 \rfloor}(\mu) : t \geq 0)$ typically does not obey a functional limit theorem.

Nevertheless, there is a functional analog of Theorem 1. In particular, the same techniques as used in proving Theorem 1 can be used to show that

$$|\mu| M_{\lfloor \cdot/\mu^2 \rfloor}(\mu) \Longrightarrow X(\cdot) + 2^\gamma \beta c$$

in $D[0,\infty)$, as $\mu \nearrow 0$, where $X = (X(t) : t \geq 0)$ is a reflecting Brownian motion with drift -1 and variance parameter σ^2, conditioned on $X(0) = 0$; a corresponding limit theorem holds when $\mu \searrow 0$. Note that the presence of the maximum in the defining relation for the M_n's means that $(M_n : n \geq 0)$ only "sees" large values of the perturbations, so that the presence of the small values causes no difficulties. Thus, the mathematical issues raised in our previous paragraph do not arise here.

3. Finite Horizon Limit Theory with Heavy Tails

We turn now to studying heavy-traffic limit theory when the perturbations have a "heavy tail". In particular, we replace A3 with the following hypothesis.

A3. $(\xi_j : j \geq 0)$ is an i.i.d. sequence of random variables, independent of the sequence $(X_j : j \geq 1)$, having a common distribution given by

$$\mathrm{P}(\xi_j > x) = \begin{cases} 1 & , \ x \leq b \\ (b/x)^\alpha & , \ x \geq b \end{cases}$$

for some positive constants b and α.

Thus, A3 is concerned with the special case in which the heavy-tailed perturbations follow a Pareto distribution. Note also that A3 demands more independence than does A2.

As in Theorem 2.1, our heavy-tailed heavy-traffic limit theorem concerns the parameterized family of perturbed random walks given by $(\sum_{j=1}^n X_j + n\mu + \kappa(\mu)\xi_n : n \geq 0)$.

Theorem 2. *Assume A1 and A3, and suppose that $\kappa(\mu) = |\mu|^{\frac{2-\alpha}{\alpha}}$. Then*

$$\mathrm{P}(|\mu| M_{\lfloor t/\mu^2 \rfloor}(\mu) > x) \to 1 - \\ \mathrm{E}\exp(-b^\alpha \int_0^t (x - \sigma B(s) + s)^{-\alpha} ds) I(\max_{0 \leq s \leq t}[\sigma B(s) - s] \leq x)$$

as $\mu \nearrow 0$ and

$$\mathrm{P}(|\mu| M_{\lfloor t/\mu^2 \rfloor}(\mu) > x) \to 1 - \\ \mathrm{E}\exp(-b^\alpha \int_0^t (x - \sigma B(s) - s)^{-\alpha} ds) I(\max_{0 \leq s \leq t}[\sigma B(s) + s] \leq x)$$

as $\mu \searrow 0$.

Proof. As in the proof of Theorem 2.1, we prove only the case where $\mu \nearrow 0$. We start by observing that A1 implies that

$$\begin{aligned} |\mu| M_{\lfloor t/\mu^2 \rfloor}(\mu) &= \max_{0 \leq k \leq \lfloor t/\mu^2 \rfloor} \{|\mu| S_k(\mu) + |\mu|^{2/\alpha}\xi_k\} \\ &= \max_{0 \leq k \leq \lfloor t/\mu^2 \rfloor} \{|\mu|(\sigma B(k) + k\mu) + |\mu|^{2/\alpha}\xi_k\} + o(1) \text{ a.s.} \end{aligned}$$

as $\mu \nearrow 0$. But the scaling properties of Brownian motion yield the distributional equality

$$\max_{0 \leq k \leq \lfloor t/\mu^2 \rfloor} \{|\mu|(\sigma B(k) + k\mu) + |\mu|^{2/\alpha}\xi_k\} \stackrel{D}{=} \max_{0 \leq k \leq \lfloor t/\mu^2 \rfloor} \{\sigma B(\mu^2 k) - \mu^2 k + |\mu|^{2/\alpha}\xi_k\}.$$

Furthermore, if $\bar{F}_\xi(x) = \mathrm{P}(\xi_j > x)$, the independence of the perturbations and the random walk (and consequently the Brownian motion) show that

$$\begin{aligned}
&\mathrm{P}(\max_{0\le k\le \lfloor t/\mu^2\rfloor}\{\sigma B(\mu^2 k)-\mu^2 k+|\mu|^{2/\alpha}\xi_k\}>x)\\
&=\mathrm{P}(\max_{0\le k\le \lfloor t/\mu^2\rfloor}\{\sigma B(\mu^2 k)-\mu^2 k\}>x-b|\mu|^{2/\alpha}\}\\
&\quad+\mathrm{P}(\max_{0\le k\le \lfloor t/\mu^2\rfloor}\{\sigma B(\mu^2 k)-\mu^2 k\}\le x-b|\mu|^{2/\alpha},\\
&\quad\max_{0\le k\le \lfloor t/\mu^2\rfloor}\{\sigma B(\mu^2 k)-\mu^2 k+|\mu|^{2/\alpha}\xi_k\}>x)\\
&=\mathrm{P}(\max_{0\le k\le \lfloor t/\mu^2\rfloor}\{\sigma B(\mu^2 k)-\mu^2 k\}>x-b|\mu|^{2/\alpha}\})\\
&\quad+\mathrm{E}[1-\prod_{k=0}^{\lfloor t/\mu^2\rfloor}(1-\bar{F}_\xi((x-\sigma B(\mu^2 k)+\mu^2 k)|\mu|^{-2/\alpha}))\\
&\qquad\cdot I(\max_{0\le k\le \lfloor t/\mu^2\rfloor}\{\sigma B(\mu^2 k)-\mu^2 k\}\le x-b|\mu|^{2/\alpha})].
\end{aligned}\tag{12}$$

The path continuity of Brownian motion shows that

$$\max_{0\le k\le \lfloor t/\mu^2\rfloor}\{\sigma B(\mu^2 k)-\mu^2 k\}\to\max_{0\le s\le t}\{\sigma B(s)-s\}\quad \text{a.s.}\tag{13}$$

as $\mu\nearrow 0$ and

$$\begin{aligned}
&\mathrm{E}[\prod_{k=0}^{\lfloor t/\mu^2\rfloor}(1-\bar{F}_\xi((x-\sigma B(\mu^2 k)+\mu^2 k)|\mu|^{-2/\alpha}))\\
&\qquad\cdot I(x-|\mu|^{1/2\alpha}\le\max_{0\le k\le \lfloor t/\mu^2\rfloor}\{\sigma B(\mu^2 k)-\mu^2 k\}\le x-b|\mu|^{2/\alpha})]\\
&\le \mathrm{P}(x-|\mu|^{1/2\alpha}\le\max_{0\le k\le \lfloor t/\mu^2\rfloor}\{\sigma B(\mu^2 k)-\mu^2 k\}\le x-b|\mu|^{2/\alpha})\\
&\to 0
\end{aligned}\tag{14}$$

as $\mu\nearrow 0$. Furthermore, because $\max_{0\le s\le t}\{\sigma B(s)-s\}$ is a random variable possessing a density, it follows that conditional on the maximum of $\sigma B(s)-s$ being less than or equal to x, the maximum is almost surely strictly smaller than x. On a sample path on which the maximum is strictly less than $x - |\mu|^{1/2\alpha}$,

$$\begin{aligned}
&\prod_{k=0}^{\lfloor t/\mu^2 \rfloor} (1 - \bar{F}_\xi((x - \sigma B(\mu^2 k) + \mu^2 k)|\mu|^{-2/\alpha})) \\
&= \exp(\sum_{k=0}^{\lfloor t/\mu^2 \rfloor} \log(1 - \bar{F}_\xi((x - \sigma B(\mu^2 k) + \mu^2 k)|\mu|^{-2/\alpha}))) \\
&= \exp(\sum_{k=0}^{\lfloor t/\mu^2 \rfloor} \log(1 - b^\alpha (x - \sigma B(\mu^2 k) + \mu^2 k)^{-\alpha} \mu^2)) \qquad (15)
\end{aligned}$$

for μ small enough so that the maximum is smaller than $x - 2|\mu|^{1/2\alpha}$. Furthermore, for such values of μ,

$$(x - \sigma B(\mu^2 k) + \mu^2 k)^{-\alpha} \mu^2 \le 2|\mu|^{3/2}$$

uniformly in k. Consequently, for such μ,

$$\log(1 - b^\alpha (x - \sigma B(\mu^2 k) + \mu^2 k)^{-\alpha} \mu^2)) = -b^\alpha (x - \sigma B(\mu^2 k) + \mu^2 k)^{-\alpha} \mu^2) + O(\mu^3)$$

uniformly in k, so that

$$\begin{aligned}
&\sum_{k=0}^{\lfloor t/\mu^2 \rfloor} \log(1 - \bar{F}_\xi((x - \sigma B(\mu^2 k) + \mu^2 k)|\mu|^{-2/\alpha})) \\
&= -b^\alpha \sum_{k=0}^{\lfloor t/\mu^2 \rfloor} (x - \sigma B(\mu^2 k) + \mu^2 k)^{-\alpha} \mu^2 + o(1) \quad \text{a.s.} \qquad (16)
\end{aligned}$$

as $\mu \nearrow 0$. On paths on which the maximum is strictly less than $x - 2|\mu|^{1/2\alpha}$, $(x - \sigma B(s) + \mu s)^{-\alpha}$ is continuous and bounded in s over $[0, t]$. It follows that because the above sum is a Riemann approximation to the integral,

$$\sum_{j=0}^{\lfloor t/\mu^2 \rfloor} (x - \sigma B(\mu^2 k) + \mu^2 k)^{-\alpha} \mu^2 \to \int_0^t (x - \sigma B(s) + s)^{-\alpha} ds \qquad (17)$$

as $\mu \nearrow 0$ for such paths. In view of (15) through (17), we therefore may conclude that

$$\begin{aligned}
&\prod_{k=0}^{\lfloor t/\mu^2 \rfloor} (1 - \bar{F}_\xi((x - \sigma B(\mu^2 k) + \mu^2 k)|\mu|^{-2/\alpha})) \\
&\to \exp(-b^\alpha \int_0^t (x - \sigma B(s) + s)^{-\alpha} ds) \qquad (18)
\end{aligned}$$

for such paths. But the left hand side of (18) is a family of random variables that is bounded in μ (by the constant 1), so the Bounded Convergence Theorem, together with (12), (13) and (14), then yields the desired conclusion.

Theorem 2 suggests an approximation for the maximum of a perturbed random walk that is appropriate when the perturbations have a right tail that is Pareto. In particular, for a given perturbed random walk $(S_j+\xi_j : j \geq 0)$ with Pareto perturbations, the approximation suggested by Theorem 2 is

$$\mathrm{P}(\max_{0\leq k\leq n}\{S_k+\xi_k\} > x) \overset{D}{\approx} 1 - \mathrm{E}[\exp(-b^\alpha \int_0^n (x-\sigma B(s)-\mu s)^{-\alpha} ds) \cdot I(\max_{0\leq s\leq n}[\sigma B(s)+\mu s] \leq x-b)] \tag{19}$$

The parameters μ and σ^2 appearing in (19) are defined as in (10) and (11), while the quantities b and α are the parameters that define the Pareto distribution of the perturbations. As for Theorem 1, we expect the above approximation to be good over spatial scales of order $1/|\mu|$ when $|\mu|$ is small, the time n is of order $1/\mu^2$, and the perturbations are of order $|\mu|^{\frac{2-\alpha}{\alpha}}$.

4. Infinite Horizon Limit Theory with Light Tails

The previous two sections were concerned through Theorems 1 and 2 with the heavy traffic setting, providing approximations to the maximum of perturbed random walk over finite time intervals. However, in view of the discussion in Section 2 of Araman and Glynn (2004), perhaps the most interesting characteristic of the perturbed random walk is the distribution of the all-time maximum M_∞. Since M_∞ involves the "infinite-time" behavior of the perturbed random walk, we can not conclude from Theorems 1 and 2 that our Brownian approximations are appropriate for use as approximations of M_∞. In the remainder of this section, we provide rigorous justification for such Brownian approximations to the distribution of M_∞. We start again by considering light-tailed perturbations by (slightly) strengthening assumption A3:

A4. $(\xi_j : j \geq 0)$ is a stationary sequence for which there exist positive constants γ and β such that

$$\frac{\max_{0\leq k\leq n}\xi_k}{(\log n)^\gamma} \to \beta \quad \text{a.s.}$$

as $n \to \infty$.

See Zeevi and Glynn (1999) for such almost sure convergence results, in the stationary sequence setting.

Clearly, M_∞ is finite-valued only when the underlying random walk has negative drift. We therefore restrict attention, in the following results, to the case where $\mu \nearrow 0$.

Theorem 3. *Assume A1 and A4, and suppose that $\kappa(\mu) \sim c(|\mu|(\log(1/|\mu|))^\gamma)^{-1}$ as $\mu \nearrow 0$, where c is a positive constant. Then,*

$$|\mu| M_\infty(\mu) \Rightarrow \max_{t \geq 0}[\sigma B(t) - t] + 2^\gamma \beta c$$

as $\mu \nearrow 0$.

Proof. Fix $\varepsilon > 0$. We first set $t^* = \max(1, t_1, t_2)$, where t_1 and t_2 are chosen so that

$$\mathrm{P}(\max_{t \geq t_1}[\sigma B(t) - t] \geq -|\xi_0|) < \varepsilon$$

and

$$(\log t + 2)^\gamma / t \leq (16\beta c)^{-1} \quad \text{for } t \geq t_2.$$

Observe that

$$\begin{aligned} &|\,\mathrm{P}(\max_{0 \leq t \leq t^*}[\sigma B(t) - t] + 2^\gamma \beta c \in \cdot) - \mathrm{P}(\max_{t \geq 0}[\sigma B(t) - t] + 2^\gamma \beta c \in \cdot)| \\ &\leq \mathrm{P}(\max_{t \geq t^*}[\sigma B(t) - t] \geq 0) < \varepsilon \end{aligned} \tag{20}$$

Assumption A1 guarantees that there exists a finite-valued random variable L_1 such that

$$|S_k - \sigma B(k)| \leq 1/4 \cdot k^{1/2}$$

for $k \geq L_1$, whereas A4 ensures the existence of L_2 for which

$$\max_{0 \leq k \leq n} \xi_k \leq 2\beta(\log n)^\gamma$$

for $n \geq L_2$. Set $L = \max(L_1, L_2)$. Note that

$$\begin{aligned} &|\,\mathrm{P}(|\mu| M_{\lfloor t^*/\mu^2 \rfloor}(\mu) \in \cdot) - \mathrm{P}(|\mu| M_\infty(\mu) \in \cdot)| \\ &\qquad \leq \mathrm{P}(\max_{k \geq \lceil t^*/\mu^2 \rceil}\{|\mu| S_k(\mu) + |\mu| \xi_k(\mu)\} \geq |\mu| \xi_0(\mu)) \\ &\qquad \leq \mathrm{P}(\max_{k \geq \lceil t^*/\mu^2 \rceil}\{|\mu| S_k(\mu) + |\mu| \xi_k(\mu)\} \geq |\mu| \xi_0(\mu)) \\ &\qquad\qquad ; L \leq \lceil t^*/\mu^2 \rceil) \\ &\qquad\quad + \mathrm{P}(L > \lceil t^*/\mu^2 \rceil). \end{aligned} \tag{21}$$

On $\{L \leq \lceil t^*/\mu^2 \rceil\}$,

$$\begin{aligned} |\mu|\xi_k(\mu) &\leq \frac{2c}{(\log 1/|\mu|)^\gamma} \max_{0\leq j\leq k} \xi_j \\ &\leq 4\beta c \frac{(\log k)^\gamma}{(\log 1/|\mu|)^\gamma} \end{aligned} \tag{22}$$

for $k \geq \lceil t^*/\mu^2 \rceil$ and μ small enough. But $(\log k)^\gamma/k$ is decreasing for k large, so that it follows that for $|\mu| \leq 1/e$ and $k \geq \lceil t^*/\mu^2 \rceil$,

$$\begin{aligned} 4\beta c \frac{(\log k)^\gamma}{(\log 1/|\mu|)^\gamma} &\leq 4\beta c \frac{(\log t^*/\mu^2)^\gamma \cdot k}{t^*/\mu^2 (\log 1/|\mu|)^\gamma} \\ &= 4\beta c (\frac{\log t^*}{\log 1/|\mu|} + 2)^\gamma \frac{1}{t^*} k\mu^2 \\ &\leq 4\beta c (\log t^* + 2)^\gamma \frac{1}{t^*} k\mu^2 \\ &\leq 1/4 \cdot k\mu^2 \end{aligned} \tag{23}$$

where t_2's definition was used in the final inequality. Also, on $\{L \leq \lceil t^*/\mu^2 \rceil\}$,

$$\begin{aligned} |\mu| \cdot |S_k - \sigma B(k)| &\leq |\mu| 1/4 k^{1/2} \leq |\mu| 1/4 k^{1/2} k^{1/2} |\mu| \\ &\leq 1/4 \cdot k\mu^2 \end{aligned} \tag{24}$$

for $k \geq \lceil t^*/\mu^2 \rceil$, where $t^* \geq 1$ was used in the second inequality above.

Combining (22) through (24), we conclude that on $\{L \leq \lceil t^*/\mu^2 \rceil\}$,

$$|\mu| S_k + |\mu|\xi_k(\mu) \leq |\mu|\sigma B(k) - \frac{\mu^2 k}{2}$$

for $k \geq \lceil t^*/\mu^2 \rceil$. Hence, for μ sufficiently small,

$$\begin{aligned} &\mathrm{P}(\max_{k\geq\lceil t^*/\mu^2\rceil} \{|\mu| S_k + |\mu|\xi_k(\mu)\} \geq |\mu|\xi_0(\mu)) \\ &\leq \mathrm{P}(\max_{k\geq\lceil t^*/\mu^2\rceil} \{\mu\sigma B(k) - \mu^2 k\} \geq |\mu|\xi_0(\mu)) \\ &\leq \mathrm{P}(\max_{s\geq t^*} \{\sigma B(s) - s/2\} \geq -|\xi_0|) < \varepsilon \end{aligned} \tag{25}$$

Inequalities (20), (21) and (25) imply that

$$
\begin{aligned}
| \operatorname{P}(|\mu| M_\infty(\mu) \in \cdot) &- \operatorname{P}(\max_{t\ge 0}[\sigma B(t) - t] + 2^\gamma \beta c \in \cdot)| \\
&\le | \operatorname{P}(|\mu| M_\infty(\mu) \in \cdot) - \operatorname{P}(|\mu| M_{\lfloor t^*/\mu^2 \rfloor}(\mu) \in \cdot)| \\
&\quad + | \operatorname{P}(\max_{t\ge 0}[\sigma B(t) - t] + 2^\gamma \beta c \in \cdot) \\
&\quad - \operatorname{P}(\max_{0\le t\le t*}[\sigma B(t) - t] + 2^\gamma \beta c \in \cdot)| \\
&\le 2\varepsilon + \operatorname{P}(L > \lceil t^*/\mu^2 \rceil).
\end{aligned}
$$

Letting $\mu \nearrow 0$ first, and then letting $\varepsilon \searrow 0$, we obtain the desired result.

Theorem 3 suggests an approximation for the distribution of M_∞ that is appropriate to perturbed random walks with negative drift, for which the perturbations have a right tail of the type described in (1). However, in developing the approximation, care must be taken, since naive substitution of $n = \infty$ into the finite-time approximation (9) leads to a right-hand side that is infinite. The key is to take advantage of the "diffusion scaling" that is implicit in all the results of this section. In particular, the results describe the spatial fluctuations of order $1/|\mu|$ that occur over time scales of $1/\mu^2$. Thus, in order that the maximum of perturbed random walk exceeds a level of order $1/|\mu|$, roughly $1/\mu^2$ time units are required. This suggests that the implicit time scale that is relevant in approximating the all-time maximum M_∞ is a time of order $1/\mu^2$, so that the logarithmic term in (9) is then roughly given by $2^\gamma(\log 1/\mu)^\gamma$. This yields the approximation

$$
M_\infty \overset{D}{\approx} \max_{t\ge 0}[\sigma B(t) + \mu t] + 2^\gamma \beta (\log 1/|\mu|)^\gamma, \tag{26}
$$

where μ, σ^2, γ and β are defined as in (9). In view of Theorem 3, (26) should give reasonable approximations over spatial scales of order $1/|\mu|$, when μ is small and negative and the perturbations are roughly of order $\kappa(|\mu|)$.

5. Infinite Horizon Limit Theory with Heavy Tails

Finally, we turn again to heavy-tailed perturbations that satisfy A3. We study the infinite-horizon analog of (19). We note that the term $(x - \sigma B(s) - \mu s)^{-\alpha}$ appearing there is almost surely asymptotic to $(-\mu s)^{-\alpha}$ as $s \to \infty$. Hence, even when the drift is negative, it follows that

$$\int_0^\infty (x - \sigma B(s) - \mu s)^{-\alpha} ds = \infty \quad \text{a.s.}$$

for $0 < \alpha \leq 1$, in which case the right-hand side of (19) describes an improper random variable. Consequently, the approximation described by Theorem 3 has a finite all-time maximum only when the perturbations follow a Pareto distribution with finite mean. This is in fact consistent with Proposition 2 of Araman and Glynn (2004). The latter result states that a necessary and sufficient condition for the all time maximum of a perturbed random walk to be finite almost surely is that the perturbations sequence (assumed i.i.d. and non-negative) admits a finite mean. We therefore henceforth restrict our attention to the case where $\alpha > 1$. It is then easily seen that the right-hand side of (19) defines a proper (finite-valued) random variable.

Theorem 4. *Assume A1 and A3 hold, with $\alpha > 1$. If $\kappa(\mu) = |\mu|^{\frac{2-\alpha}{\alpha}}$, then*

$$\begin{aligned} \mathrm{P}(|\mu| M_\infty(\mu) > x) \rightarrow & 1 - \mathrm{E}[\exp(-b^\alpha \int_0^\infty (x - \sigma B(s) + s)^{-\alpha} ds) \\ & \cdot I(\max_{t \geq 0} [\sigma B(t) - t] \leq x)] \end{aligned} \tag{27}$$

as $\mu \nearrow 0$.

The proof of Theorem 4 follows an argument virtually identical to that of Theorem 3, and is therefore omitted.

The infinite-horizon approximation suggested by Theorem 4 is

$$\begin{aligned} & \mathrm{P}(\max_{k \geq 0} [S_k + \xi_k] > x) \\ & \approx 1 - \mathrm{E}[\exp(-b^\alpha \int_0^\infty (x - \sigma B(s) - \mu s)^{-\alpha} ds) \\ & \quad \cdot I(\max_{t \geq 0} [\sigma B(t) + \mu t] \leq x - b)], \end{aligned} \tag{28}$$

where μ, σ^2, α, and b are defined as in (19). We expect (28) to give reasonable approximations over spatial scales of order $1/|\mu|$ when μ is small and negative and the perturbations are roughly of order $|\mu|^{\frac{2-\alpha}{\alpha}}$.

The remainder of this section is devoted to a discussion of how to compute the expectation appearing in (28). We start by noting that since

$$\begin{aligned} w(y) \triangleq &\mathrm{P}(\max_{t\geq 0}[\sigma B(t)+\mu t]\leq -y-b) \\ =&1-\exp(-2\mu(y+b)/\sigma^2) \end{aligned}$$

is known in closed-form for $y \leq -b$ (and μ negative), the key is to compute

$$\mathrm{E}[\exp(-b^\alpha \int_0^\infty |W(s)|^{-\alpha} ds)|W(0)=-x, \max_{t\geq 0} W(t)\leq -b], \tag{29}$$

where $W=(W(t):t\geq 0)$ is Brownian motion with drift μ and variance parameter σ^2. The process W, conditioned on $\max[W(t):t\geq 0]\leq -b$, is itself a diffusion. To (non-rigorously) compute the generator of the conditioned process Y, let

$$\tilde{A}=\mu\frac{d}{dx}+\frac{\sigma^2}{2}\frac{d^2}{dx^2}$$

be the generator of Brownian motion with drift μ and variance parameter σ^2, and observe that for $y<-b$,

$$\begin{aligned} &\mathrm{E}[f(Y(h))|Y(0)=y] \\ &=\mathrm{E}[f(W(h))|W(0)=y, \max_{t\geq 0} W(t)\leq -b] \\ &\approx \frac{1}{w(y)}\,\mathrm{E}[f(W(h))w(W(h))|W(0)=y] \\ &=\frac{1}{w(y)}[f(y)w(y))+\tilde{A}(fw))(y)h+o(h)]. \end{aligned}$$

This suggests that the generator A of the conditioned process Y is given by

$$\begin{aligned} (Af)(y)&=\mu\frac{d}{dy}(f(y)w(y))+\frac{\sigma^2}{2}\frac{d^2}{dy^2}(f(y)w(y)) \\ &=(\tilde{A}f)(y)+\sigma^2\frac{w'(y)}{w(y)}f'(y) \\ &=\left[\mu+\frac{2\mu}{\exp(\frac{2\mu(y+b)}{\sigma^2})-1}\right]f'(y)+\frac{\sigma^2}{2}f''(y) \end{aligned}$$

for $y<-b$ and f twice continuously differentiable; a related (non-rigorous) computation can be found on pages 271-272 of Karlin and Taylor (1981). Note that the drift decreases to $-\infty$ as y approaches $-b$, making $-b$ an inaccessible boundary for Y (as expected). To make the above computation rigorous, a "change-of-measure" argument, based on Girsanov's formula, can be applied, as in Glynn and Thorisson (2001).

Given the above computation, (29) now requires calculating $v(-x)$, where

$$v(y) = \mathrm{E}[\exp(-b^{\alpha} \int_0^{\infty} |Y(t)|^{-\alpha} dt)|Y(0) = y].$$

The function $v(\cdot)$ can be found by solving an appropriate differential equation.

Proposition 1. *Suppose that there exists a twice continuously differentiable function $v = (v(y) : y < -b)$ such that v is positive and decreasing, with $v(-\infty) = 1$, and satisfies*

$$[\mu + \frac{2\mu}{\exp(2\mu(y+b)/\sigma^2) - 1}]v'(y) + \frac{\sigma^2}{2} v''(y) - b^{\alpha}|y|^{-\alpha} v(y) = 0 \tag{30}$$

for $y < -b$. Then,

$$v(y) = \mathrm{E}[\exp(-b^{\alpha} \int_0^{\infty} |Y(t)|^{-\alpha} dt)|Y(0) = y]$$

for $y < -b$.

Proof. The process Y satisfies the stochastic differential equation

$$dY(t) = [\mu + \frac{2\mu}{\exp(2\mu(Y(t)+b)/\sigma^2) - 1}]dt + \frac{\sigma^2}{2} dB(t).$$

Ito's formula then ensures that

$$\begin{aligned} &d(\exp(-b^{\alpha} \int_0^t |Y(s)|^{-\alpha} ds) \cdot v(Y(t))) \\ &= \exp(-b^{\alpha} \int_0^t |Y(s)|^{-\alpha} ds)((Av)(Y(t)) - b^{\alpha}|Y(t)|^{-\alpha} v(Y(t)))dt \\ &\quad + \exp(-b^{\alpha} \int_0^t |Y(s)|^{-\alpha} ds) \cdot v'(Y(t))\sigma dB(t) \\ &= \exp(-b^{\alpha} \int_0^t |Y(s)|^{-\alpha} ds) \cdot v'(Y(t))\sigma dB(t), \end{aligned}$$

where the second equality relies on the fact that v satisfies the stated differential equation. If $T_n = \inf\{t \geq 0 : Y(t) \leq -n \text{ or } Y(t) \geq -b - 1/n\}$, the boundedness of v' on $[-n, -b-1/n]$ implies that the Optional Sampling Theorem may be applied at time T_n to the stochastic integral, yielding the equality

$$v(y) = \mathrm{E}[\exp(-b^{\alpha} \int_0^{T_n} |Y(s)|^{-\alpha} ds) \cdot v(Y(T_n))|Y(0) = y]. \tag{31}$$

Because $W(t) \to -\infty$ a.s. and the conditioning event $\{\max[W(t) : t \geq 0] \leq -b\}$ has positive probability, it follows that $Y(t) \to -\infty$ a.s. as $t \to \infty$. The path continuity of Y therefore permits us to conclude that $Y(T_n) \to -\infty$ a.s., so that $v(Y(T_n)) \to 1$. The positive and decreasing nature of v implies that $v(Y(T_n))$ is bounded in absolute value by one, so that the Bounded Convergence Theorem applied to (31) yields the desired conclusion.

We are unaware of any closed-form solution to the differential equation. Consequently, it must be solved numerically. An alternative is to develop asymptotics for the solution.

Theorem 5. *Suppose that $\mu < 0$ and $\alpha > 2$. Then*

$$\begin{aligned}
&1 - \mathrm{E}\exp[-b^\alpha \int_0^\infty (x - \sigma B(s) - \mu s)^{-\alpha} ds] I(\max_{t\geq 0}[\sigma B(t) + \mu t] \leq x - b) \\
&= \frac{b^\alpha}{|\mu|\alpha} x^{1-\alpha} + b^\alpha \frac{\sigma^2}{2\mu^2} x^{-\alpha} + o(x^{-\alpha})
\end{aligned}$$

as $x \to \infty$.

Proof. Observing that scaling properties of Brownian motion imply that

$$\begin{aligned}
&\mathrm{E}\exp(-b^\alpha \int_0^\infty (x - \sigma B(s) - \mu s)^{-\alpha} ds) I(\max_{t\geq 0}[\sigma B(t) + \mu t] \leq x - b) \\
&= \mathrm{E}\exp(-b^\alpha x^{1-\alpha} J(x)) I(x)
\end{aligned}$$

where

$$\begin{aligned}
J(x) &= \int_0^\infty (1 - \sigma x^{-1/2} B(r) - \mu r)^{-\alpha} dr, \\
I(x) &= I(\max_{r\geq 0}[\sigma x^{-1/2} B(r) + \mu r] \leq 1 - b/x).
\end{aligned}$$

On $\{I(x) = 1\}$, we can express the integrand of $J(x)$ as

$$\begin{aligned}
(1 - \sigma x^{-1/2} B(r) - \mu r)^{-\alpha} = (1-\mu r)^{-\alpha} &+ \alpha(1 - \mu r)^{-\alpha-1} \sigma B(r) x^{-1/2} \\
&+ \frac{\alpha(\alpha+1)}{2}(1 - \mu r - \xi(r,x))^{-\alpha-2} \sigma^2 B^2(r) x^{-1},
\end{aligned}$$

where $|\xi(r,x)| \leq \sigma x^{-1/2}|B(r)|$. In fact,

$$\begin{aligned}
(1 - \mu r - \xi(r,x)) &\geq (1 - \mu r) \wedge ((1 - \mu r) - x^{-1/2}\sigma B(r)) \\
&\geq (1 - \mu r) \wedge (1 - \mu r(1 - x^{-1/2}) - x^{-1/2} R) \\
&= (1 - \mu r(1 - x^{-1/2}) - x^{-1/2} R),
\end{aligned}$$

where $R = \max\{\sigma B(r) + \mu r : r \geq 0\}$. It follows that if $L = \sup\{t \geq 0 : |\sigma B(t)| > |\mu| t/2\}$

then for $x \geq 1$,

$$\begin{aligned}
&x|J(x) - \int_0^\infty (1-\mu r)^{-\alpha} dr - \sigma x^{-1/2}\alpha \int_0^\infty (1-\mu r)^{-\alpha-1} B(r) dr | I(x) \\
&\leq \frac{\alpha(\alpha+1)}{2}\sigma^2 \int_0^\infty (1-\mu r - \xi(r,x))^{-\alpha-2} B^2(r) dr I(x) I(R > x^{1/2}/2) \\
&\quad + \frac{\alpha(\alpha+1)}{2}\sigma^2 \int_0^\infty (1/2-\mu r)^{-\alpha-2} B^2(r) dr I(x) I(R \leq x^{1/2}/2) \qquad (32)\\
&\leq \frac{\alpha(\alpha+1)}{2}\sigma^2 \int_0^L (1-\mu r - \xi(r,x))^{-\alpha-2} B^2(r) dr I(x) I(R > x^{1/2}/2) \\
&\quad + \frac{\alpha(\alpha+1)}{2}\sigma^2 \int_L^\infty (1-\mu r/2)^{-\alpha-2} B^2(r) dr + \frac{\alpha(\alpha+1)}{2}\sigma^2 \int_0^\infty (1/2-\mu r)^{-\alpha-2} B^2(r) dr \\
&\leq \frac{\alpha(\alpha+1)}{2}\sigma^2 2^{\alpha+2} b^{-\alpha-2} R^{\alpha+2} \int_0^L B^2(r) dr \\
&\qquad + \alpha(\alpha+1)\sigma^2 \int_0^\infty (1/2-\mu r/2)^{-\alpha-2} B^2(r) dr.
\end{aligned}$$

Here, we used the fact that $(1-\mu r - \sigma x^{-1/2} B(r))^{-\alpha-2} \leq (x/b)^{\alpha+2}$ when $I(x) = 1$ to obtain the third inequality above.

Recall that for $w \geq 0$,

$$1 - w \leq \exp(-w) \leq 1 - w + w^2/2.$$

Since $J(x) \geq 0$ on $\{I(x) = 1\}$,

$$x^\alpha |1 - \exp(-b^\alpha x^{1-\alpha} J(x)) - b^\alpha x^{1-\alpha} J(x)| \leq x^{2-\alpha} J^2(x)/2. \qquad (33)$$

Because L and R have exponential tails (for L see Robbins et al. (1968), and for R see Harrison (1985)), evidently $R^{\alpha+2}\int_0^\infty B^2(r) I(L > r) dr$ is integrable. The Dominated Convergence Theorem, together with (32) and (33), shows that

$$x^\alpha (1 - \mathrm{E}\exp(-b^\alpha x^{1-\alpha} J(x)) I(x)) = b^\alpha x\, \mathrm{E}\, J(x) I(x) + o(1)$$

as $x \to \infty$.

A path by path application of the Dominated Convergence Theorem, taking advantage of the bound (32), yields

$$\begin{aligned}
xJ(x)I(x) = x\int_0^\infty (1-\mu r)^{-\alpha} dr + \sigma x^{-1/2}\alpha \int_0^\infty (1-\mu r)^{-\alpha-1} B(r) dr \\
+ \frac{\alpha(\alpha+1)}{2}\sigma^2 \int_0^\infty (1-\mu r)^{-\alpha-2} B^2(r) dr + o(1) \quad \text{a.s.} \qquad (34)
\end{aligned}$$

as $x \to \infty$. The integrability of $R^{\alpha+2}\int_0^L B^2(r) dr$ allows us to again apply the Dominated Convergence Theorem, this time to pass expectations through (34). This proves the theorem.

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