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Stochastic Simulation

Homework 3 & 4

due on April 21, 2005

Exercise 1 This problem involves writing a program to simulate an $M/M/\infty$ queue-length process $\{Q(t) : t \geq 0\}$. We let $\rho = \lambda/\mu < 1$. Assume that $\mu = 1$. The performance measure of interest is $\mathbb{P}(Q(T) > N)$. Consider two cases: $\lambda = 5, N = 4, T = 4$ and $\lambda = 10, N = 9, T = 10$. Determine the number of replications needed and both a point estimate and a confidence interval. The process $\{Q(t) : t \geq 0\}$ has a Poisson distribution at time t given by: $Q(t) \sim Poisson(\lambda(1 - \exp(-\mu t))/\mu)$.

Exercise 2 This problem is a simulation of the $M/M/1$ queueing system with $\rho = \lambda/\mu < 1$. Assume that $\mu = 1$. Let $Q(t)$ denote the number of jobs in the system at time t. Our interest here is in estimating the steady state mean $\mathbb{E}Q$, where $Q(t) \Rightarrow Q$ as $t \to \infty$. Treat two cases $\rho = 0.5$ and $\rho = 0.9$ (the latter case represents a system in so-called "heavy-traffic" as the utilization, ρ becomes close to 1).

1. Use the standard regenerative method to estimate $\mathbb{E}Q$, with $Q(0) = 0$. Find both a point estimate and a confidence interval. Explain how do you obtain the number of regenerative cycles to simulate and how do you estimate the standard deviation.

2. Repeat part 1. assuming you know the relevant σ^2. Knowing that the theoretical values of $\mathbb{E}Q = \rho/(1-\rho)$ and $\sigma(Z_1)^2/\mathbb{E}\tau_1 = \frac{2\rho(1+\rho)}{\mu(1-\rho)^4}$ where $Z_1 = \int_0^{\tau_1} Q(s)ds$ and τ_1 is the length of a cycle. The steady state probabilities are given by $\pi_i = (1-\rho)\rho^i$ for $i \geq 0$.

3. Analyze your results comparing simulations with *known* or *unknown* standard deviations. Discuss both cases of $\rho = 0.5$ and $\rho = 0.9$.

Exercise 3 A *terminating* simulation has been run and i.i.d. observations $X_1, X_2, ..., X_n$ have been generated. Our goal is to estimate the third central moment of X_1,

$$\alpha = \mathbb{E}[(X_1 - \mathbb{E}X_1)^3].$$

1. Produce a point estimate, α_n of α
2. Develop a central limit theorem for $\alpha_n - \alpha$. Make sure to define any parameters contained in your result.
3. Produce a 90% confidence interval for α.

Exercise 4 Let $\{(Y_i, \mathbf{C_i}) : 1 \leq i \leq n\}$ be an i.i.d. sequence. The control vector $\mathbf{C_i} = (C_1, ..., C_k)$ with $k > 1$. Assume that $\mathbb{E}Y_1 = \mu, \mathbb{E}\mathbf{C_1} = 0, \sigma^2(Y_1) = \sigma^2$ and $\mathbb{E}\mathbf{C_1}\mathbf{C_1^t} = \Sigma$. Our goal is to compute μ using the $\mathbf{C_i}$'s as control variables. Find the optimal regression vector α and compute the amount of variance reduction, R^2 that results.

Exercise 5 This problem concerns the use of importance sampling to estimate the tail distribution of the waiting time, W, $\mathbb{P}(W > x)$ for the $M/M/1$ queue with $\rho < 1$. As usual take $\mu = 1$.

1. For different values of ρ ($\rho = 0.2, 0.5, 0.9, 0.95, 0.99$), find the values of x such that $\mathbb{P}(W > x(\rho) = 10^{-2}$. We recall that for $\rho < 1$ and $x > 0$,

$$\mathbb{P}(W > x) = 1 - \rho \exp(-\mu(1 - \rho)x).$$

2. Use an “optimal” importance sampling method to estimate $\mathbb{P}(W > x(\rho))$ for the same values of ρ. Generate 1000 values of the relevant random variable and then construct a point estimate and 90% confidence interval.

   Hint: Use an exponential change-of-measure.

3. Estimate $\mathbb{P}(W > x(\rho))$ using the regenerative method. Use the same run length that was used in part 2. (the amount of computer resource should be the same. Maybe need of taking the same seeds as well.) Again construct a point estimate and a confidence interval.

4. Compare the variances of the estimates in parts 2. and 3. How much variance reduction is achieved as a function of ρ? What can you also say about the efficiency of your results?