THE IMPACT OF PERFORMANCE-BASED ADVERTISING ON THE PRICES OF ADVERTISED GOODS

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This draft: August 7, 2009

Abstract. An important current trend in advertising is the replacement of traditional *pay-per-exposure* (aka *pay-per-impression*) pricing models with performance based mechanisms in which advertisers pay only for measurable *actions* by consumers. Such *pay-per-action* mechanisms are becoming the predominant method of selling advertising on the Internet. Well-known examples include pay-per-click, pay-per-call and pay-per-sale. This work highlights an important, and hitherto unrecognized, side-effect of pay-per-action advertising. I find that, if the prices of advertised goods are endogenously determined by advertisers to maximize profits net of advertising expenses, pay-per-action mechanisms induce firms to distort the prices of their goods (usually upwards) relative to the prices that would maximize profits in settings where advertising is sold under pay-per-exposure methods. Upward price distortions reduce consumer surplus and one or both of advertiser profits and publisher revenues, leading to a net reduction in social welfare. They persist in current quality-weighted pay-per-action schemes, such as the ones used by Google and Yahoo. In the latter settings they *always* reduce publisher revenues relative to pay-per-exposure methods. I propose enhancements to today's quality-weighted pay-per-action schemes that resolve these problems and show that the steady state limit of my enhanced mechanisms has identical allocation and revenue properties to those of an optimal pay-per-exposure mechanism.

Acknowledgments: Many thanks to Jason Hartline, Lorin Hitt, De Liu, Michael Schwarz, Siva Viswanathan and seminar participants at Wharton, Tepper, Toulouse IDEI and Boston University for valuable feedback on an earlier version of this work.

Half the money I spend on advertising is wasted;
the trouble is, I don't know which half.
John Wanamaker, owner of America's first department store

1. Introduction

John Wanamaker's famous quote has been haunting the advertising industry for over a century. It now serves as the motivation behind much of the innovation taking place in Internet-based advertising. From Google, Yahoo and Microsoft, to Silicon Valley upstarts, some of the best and brightest technology firms are focusing a significant part of their energies on new mechanisms to reduce advertising waste. These come in many forms but have one thing in common: a desire to replace traditional *pay-per-exposure* (also known as *pay-per-impression*) pricing models, in which advertisers pay a lump sum for the privilege of exposing an audience of uncertain size and interests to their message, with performance-based mechanisms in which advertisers pay only for measurable *actions* by consumers. Pay-per-click sponsored search, invented by Overture and turned into a multi-billion dollar business by Google, Yahoo and other online advertising agencies, is perhaps the best known of these approaches: advertisers bid in an online auction for the right to have their link displayed next to the results for specific search terms and then pay only when a user actually clicks on that link, indicating her likely intent to purchase. Pay-per-call, pioneered by firms such as Ingenio (acquired by AT&T in 2007), is a similar concept: the advertiser pays only when she receives a phone call from the customer, usually initiated through a web form. Pay-per-click and pay-per-call are viewed by many as only an intermediate step towards what some in the industry consider to be the "holy grail of advertising": the pay-per-sale approach where the advertiser pays only when exposure to an advertising message leads to an actual sale.[1] All of these approaches are attempting to reduce all or part of Wanamaker's proverbial waste by tying advertising expenditures to consumer actions that are directly related or, at least, correlated with the generation of sales. In the rest of the paper I will refer to them collectively as *pay-per-action* (PPA) pricing models.

[1] See, for example, "Pay per sale", Economist magazine, Sep. 29, 2005.

The current surge in pay-per-action advertising methods has generated considerable interest from researchers in a variety of fields including economics, marketing, information systems and computer science. This is important and timely since most of these methods have been invented by practitioners and their properties and consequences are not yet fully understood. Although the literature (surveyed in Section 2) has made significant advances in a number of areas, an important area that, so far, has received almost no attention is the impact of various forms of pay-per-action advertising on the prices of the advertised products. With very few exceptions (also discussed in Section 2), papers in this stream of research have made the assumption that the prices of the goods being advertised are set exogenously and independently of the advertising payment method.

In this paper I make the assumption that the prices of the goods being advertised are an endogenous decision variable of firms bidding for advertising resources. I find that, if such prices are endogenously determined by advertisers to maximize profits net of advertising expenses, pay-per-action advertising mechanisms induce firms to distort the prices of their goods (usually upwards) relative to the prices that would maximize their profits in settings where there is no advertising or where advertising is sold under pay-per-exposure methods. Upward price distortions reduce consumer surplus and one or both of publisher revenues and advertiser profits, leading to a net reduction in social welfare.

The intuition behind this result is the following: In pay-per-exposure schemes a firm pays a lump sum for leasing an advertising resource (e.g. space on a popular web page). Its willingess to pay for advertising is perfectly correlated with the total value that it expects to receive from that resource. That value is usually equal to the incremental demand that the firm expects to generate through advertising, times the profit per sale. When several firms bid for a scarce advertising resource, competition for the resource is perfectly aligned with the firms' incentive to maximize the total value each obtains from the resource. In both cases firms have an incentive to price their products at the point that maximizes the total incremental revenue they obtain from advertising. In contrast, in pay-per-action (e.g. pay-per-sale) mechanisms advertisers only pay the publisher every time a payment triggering action (e.g. a sale) takes place. Throughout this paper I make the intuitive assumption that price increases reduce demand but increase the advertiser's profit per sale. Under this assumption there are two reasons why pay-per-sale schemes induce advertisers to raise the price of their products above the value maximizing price: First, a reduction in demand reduces the frequency of payment to the publisher, and thus the advertiser's net advertising expenditures. Second, an increase in the profit per sale increases the advertiser's (per-sale) willingness to pay for the resource, and thus its probability of obtaining it. Both forces lead to an equilibrium where all competing firms increase the price of their advertised products above the profit maximizing levels, even though such price increases end up reducing the total (per-exposure) value of obtaining the advertising resource and, in many cases, the advertisers' net profits.

I show that such price and revenue distortions also arise in equilibrium in quality-weighted pay-per-action schemes, such as the ones currently used by Google and Yahoo. In the latter settings they always lead to lower publisher revenues relative to pay-per-exposure methods. I propose a simple enhancement to today's quality-weighted pay-per-action schemes that removes an advertiser's incentive to distort the prices of her products. My enhancement is based in the idea of making an advertiser's quality weight a function of both her history of triggering action frequencies (which is what Google and Yahoo currently do) *and* her current product price. I show that the steady state limit of my enhanced mechanism has identical allocation and revenue properties to those of an optimal pay-per-exposure mechanism.

The rest of the paper is organized as follows. Section 2 discusses related work. Section 3 introduces the setting. Section 4 presents the key results in a single period pay-per-action (PPA) setting. Section 5 shows that the paper's main results also apply in quality-adjusted PPA mechanisms where the publisher dynamically updates each advertiser's quality weight on the basis of past observations. It also proposes mechanism enhancements that eliminate the incentives to distort prices. Finally, Section 6 concludes.

2. Related Work

This work relates to a number of important streams of marketing, economics and computer science literature. Nevertheless, the phenomenon discussed herein has so far not been addressed by any of these literatures.

Sponsored search advertising. Pay-per-click online advertising, such as sponsored search links, is one of the most successful and highly publicized methods of performance-based advertising. It is the main source of revenue for sites like Google and Yahoo and one of the fastest growing sectors of the advertising industry. Not surprisingly, this field has experienced an explosion of interest by both researchers and practitioners. Important advances have been made on understanding the properties of the *generalized second price* (GSP) auction mechanisms that are currently the prevalent method of allocating advertising resources in such spaces (see, for example, Athey and Ellison 2008; Edelman et al. 2007; Varian 2007). A related stream of research has proposed several extensions to baseline GSP auctions that aim to improve their properties. The following is an illustrative subset: Aggarwal et al. (2006) propose an alternative advertising slot auction mechanism that is revenue-equivalent to GSP but induces truthful bidding (GSP does not); Feng et al. (2007), Lahaie and Pennock (2007) and Liu and Chen (2006) explore the allocative efficiency and publisher revenue implications of alternative methods for ranking bidders, including “rank by bid” and “rank by revenue”; Katona and Sarvary (2008) explore the equilibrium behavior of keyword auctions under more sophisticated assumptions about users’ search behavior; Ashlagi et al. (2008) and Liu et al. (2008b) explore auction design in the presence of competing publishers.

Growing attention is also being given to the perspective of advertisers bidding on such auctions; the most important problems here include how to identify appropriate keywords (Abhishek and Hosanagar 2007; Joshi and Motwani 2006; Rutz and Bucklin 2007) and how to dynamically allocate one’s budget among such keywords (Borgs et al. 2007; Cary et al. 2007; Feldman et al. 2007; Rusmevichientong et al. 2006). Finally, researchers have paid attention to incentive issues that are inherent in pay-per-action advertising, most important among them being the potential for click fraud, i.e. the situation where a third party maliciously clicks on an advertiser’s sponsored link without any intention of purchasing her product (Immorlica et al. 2006; Wilbur and Zhu 2008) as well as the advertiser’s incentive to misreport the frequency of her triggering action in order to avoid paying the publisher (Agarwal et al. 2009; Nazeradeh et al. 2008).

The above theoretical and algorithmic contributions are complemented by a growing number of empirical works (e.g. Animesh et al. 2009; Ghose and Yang 2009; Goldfarb and Tucker 2007; Rutz and Bucklin 2008; Yao and Mela 2009b). For comprehensive overviews of current research and open questions in sponsored search auctions the reader is referred to excellent chapters by Feldman et al. (2008), Lahaie et al. (2007), Liu et al. (2008a) and Yao and Mela (2009a).

Interestingly, almost all papers on this burgeoning field assume that an advertiser's value per sale is exogenously given and do not consider how the performance-based nature of advertising affects the advertiser's pricing of the products being sold. The only two exceptions I am aware of are Chen and He (2006) and Feng and Xie (2007). Chen and He (2006) study seller bidding strategies in a paid-placement position auction setting with endogenous prices and explicit consumer search. However, they only assume a pay-per-exposure mechanism and derive results that are essentially identical to my Proposition 3, Part 1, i.e. (using the language of my paper) that advertisers price their product at the point that maximizes their per-exposure value function. Feng and Xie (2007) focus on the quality signaling aspects of advertising and propose a model that is in many ways orthogonal to mine. I discuss their paper later in this section.

Performance-based contracting. Performance-based advertising is a special case of performance-based contracting. Contract theory has devoted significant attention to such contracts, as they can help balance incentives in principal-agent settings where moral hazard exists or where the sharing of risk between the two parties is a concern (Holmstrom 1979; Holmstrom and Milgrom 1987, 1991). In the context of information goods, Sundararajan (2004) studies optimal pricing under incomplete information about the buyers' utility. He finds that the optimal pricing usually involves a combination of fixed-fee and usage-based pricing. Closer to the context of this work, Hu (2006) and Zhao (2005) study how performance-based advertising contracts that optimally balance the incentives of both the publisher and the advertise can be constructed. They both find that the optimal contract must have both a fixed (i.e. PPE) and a performance-based (i.e. PPA) component. Once again, however, both of these studies consider the prices of advertised products as fixed and not as an endogenous decision variable under the control of advertisers.

Advertising and Product Prices. The relationship between product prices and advertising has received quite a bit of attention in the economics and marketing literature. These literatures have primarily focused on the quality signaling role of prices in conjunction with advertising. The main result is that the simultaneous presence of prices and advertising improves a firm's ability to successfully signal its quality to consumers because firms can partially substitute quality-revealing price distortions with quality-revealing advertising expenditures (see, for example, Fluet and Garella 2002; Hertzendorf and Overgaard 2001; Milgrom and Roberts 1986; Zhao 2000). Almost all works in this stream of literature assume that advertising is sold under a traditional pay-per-exposure model.

The only exception I am aware of is Feng and Xie (2007). They study how the move from exposure-based to performance-based advertising affects the ability of price and advertising to signal product quality. Their main result is that such a move generally reduces the number of situations where advertising expenditures can be used to signal quality and increases the prices charged to consumers, since firms must now rely harder on the price signal to reveal their quality. Their result relies on the assumption that higher quality firms are more likely to have a higher proportion of repeat customers who would be clicking and purchasing the product anyway, but who nevertheless induce incremental advertising charges in a pay-per-performance (PPP) model. Therefore, PPP advertising is relatively more wasteful for high quality vs. low quality firms and this moderates a high quality firm's incentive to spend more on PPP advertising.

My results are orthogonal to this work since in my model price distortions are unrelated to the advertisers' desire to signal their quality and to the presence of repeat customers and occur even in settings where consumers have perfect knowledge of each advertiser's quality or where repeat customers do not exist. A key aspect of my model is the presence of a profit-maximizing publisher who controls access to advertising resources and prices them according to what the market will bear. Competition among advertisers increases the unit price of access in the PPA case. In contrast, Feng and Xie do not model the publisher as a separate actor and assume that the unit price of advertising is independent of the payment model.

In summary, the traditional literature on advertising has examined various aspects of the relationship between product prices and advertising expenditures in settings that essentially correspond to what I call PPE. On the other hand, the rapidly growing literature on performance-based advertising has largely assumed that product prices are exogenous to the choice of advertisement payment mechanism. This work breaks new ground by showing that when one endogenizes product prices, performance-based advertising mechanisms create incentives for price distortions that in most cases have negative consequences for most stakeholders.

3. The setting

A monopolist *publisher* owns an advertising resource and leases it on a per-period basis to a heterogeneous population of N firms (*advertisers*). Examples of such a resource include a billboard located at a busy city square, a time slot in prime time TV, or space at the top of a popular web page. Advertisers are characterized by a privately known unidimensional type $q \in [\underline{q}, \overline{q}]$, independently drawn from a distribution with CDF $F(q)$. An advertiser's type relates to the attractiveness of her products or services to consumers; I assume that *ceteris paribus* higher types are, on average, more attractive. In the rest of the paper I will refer to q as the advertiser's quality, even though other interpretations are possible.[2] An advertiser's quality affects the *ex-ante* value she expects to obtain from leasing the advertising resource for one period. In most real-life settings this value will be equal to the expected profit from additional sales that the advertiser expects to realize by leasing the resource and can thus be expressed as:

$$V(p, q) = D(p, q)(p - c(q)) \tag{1}$$

where p is the unit price of the advertised product, $c(q)$ is the corresponding unit cost and $D(p, q)$ is the increase in demand due to advertising. The analysis that follows will be based directly on the value function $V(p, q)$ and will not rely on (1) or any other specific

[2] For example, in settings with network effects (e.g. when the advertisers are social networks) q can be the size of the advertiser's user base.

interpretation of this function. My intention is to make the specification as general as possible, avoiding any assumptions regarding the market structure (e.g. monopoly, oligopoly, etc.) or any other details of the game (e.g. quality signaling, awareness building, etc.) that advertisers play after they acquire the resource.

The following are assumed to hold for all $p \in \mathbb{R}^+$ and $q \in [\underline{q}, \overline{q}]$:

A1 $V(p,q)$ is unimodal in p, attaining its unique maximum at some $p^*(q) > 0$

A2 $\lim_{p \to \infty} V(p,q) = 0$

A3 $V_2(p,q) > 0$

A1 and A2 are common and intuitive consequence of treating $V(p,q)$ as a sales profit function. A3 implies that *some* information about an advertiser's type becomes available to consumers at some point during the advertising-purchasing process, but still allows for a fairly general range of settings (for example settings where this information might be noisy, where only a subset of consumers are informed, where firms might attempt to obfuscate their true types, etc.).

Throughout the paper I assume that the advertiser has full control of the prices of advertised goods and will set such prices to optimize her profits, taking into account any advertising expenditures. Even though the advertisers' risk aversion is an often-cited motivator for pay-per-performance mechanisms (see, for example, Mahdian and Tomak 2008), to isolate the price and revenue distortion effects that form the focal point of this paper, I assume that the publisher and all advertisers are risk-neutral.

The effects of interest to this work are orthogonal to the specifics of the mechanism used by the publisher to allocate the resource, as long as the mechanism strives to maximize the publisher's revenue. For simplicity I assume that the publisher allocates the resource to one of the competing firms using a Vickrey auction. Auction-based allocation of advertising resources is the norm in sponsored search advertising and is also not uncommon in offline settings (e.g superbowl ads). Furthermore, the effects I discuss are orthogonal to whether the publisher offers one or several (identical or vertically-differentiated) resources. This allows us to ignore the multi-unit mechanism design complications present, say, in sponsored search position auctions (Athey and Ellison 2008; Edelman et al. 2006; Varian 2007) and focus on a single-unit auction. Finally, even though in most real-life settings allocation of an advertising resource takes place repeatedly on a per-period basis, our baseline results do not rely on the dynamic nature of the game. I will, therefore, initially focus our attention on a static one-period game, deferring the discussion of dynamic settings until Section 5.

Traditional *pay-per-exposure* (PPE) methods charge advertisers a fee that is levied upfront and is independent of the ex-post value that advertisers obtain by leasing the resource. Assuming that every other bidder of type y bids an amount equal to $\beta_E(y)$ and that, as I will later show, it is $\beta_E'(y) \geq 0$, at a symmetric Bayes-Nash equilibrium an advertiser of type q bids $b_E(q)$ and sets the price of her product at $p_E(q)$ to maximize her net expected profit:

$$\Pi_E(q; b_E(q), p_E(q), \beta_E(\cdot)) = \int_{\underline{q}}^{\beta_E^{-1}(b_E(q))} \left(V(p_E(q), q) - \beta_E(y)\right) G'(y)dy \tag{2}$$

where $G(y) = F^{N-1}(y)$ is the probability that the second highest bidder's type is less than or equal to y and $G'(y)$ is the corresponding density.[3] At equilibrium it must also be $\beta_E(q) = b_E(q)$. The above specification subsumes the special case where product prices $p(q)$ are given exogenously. In the latter case, a bidder of type q simply chooses a bid $b_E(q; p(\cdot))$ that maximizes $\Pi_E(q; b_E(q; p(\cdot)), p(q), \beta_E(\cdot))$ subject to $\beta_E(q) = b_E(q; p(\cdot))$.

I use the following shorthand notation:[4]

$\Pi_E(q)$	advertisers' PPE equilibrium profit function (endogenous product prices)
$\Pi_E(q; p(\cdot))$	advertisers' PPE equilibrium profit function (exogenous product prices)

According to standard auction theory (e.g. Riley and Samuelson 1981) the expected publisher revenue associated with bids $\beta(y)$ is equal to:

$$R_E(\beta(\cdot)) = N \int_{\underline{q}}^{\overline{q}} \left(\int_{\underline{q}}^{z} \beta(y)G'(y)dy \right) F'(z)dz \tag{3}$$

I use the following shorthand notation:

[3] Throughout this paper I restrict my attention to symmetric Bayes-Nash equilibria. Unless specified otherwise, all subsequent references to "equilibrium" thus imply "symmetric Bayes-Nash equilibrium".

[4] See Appendix I for a summary of key notation used throughout the paper.

$R_E = R_E(b_E(\cdot))$	publisher's PPE equilibrium revenue (endogenous product prices)
$R_E(p(\cdot)) = R_E(b_E(\cdot; p(\cdot)))$	publisher's PPE equilibrium revenue (exogenous product prices)

Pay-per-action (PPA) approaches make payment to the publisher contingent on a *triggering action* that is either a sale, or some other consumer action (e.g. click, call) that has the following properties:

(1) It is linked to the advertising, i.e. only consumers who have been exposed to this particular advertising message can perform the triggering action.

(2) It is a *necessary* step of a consumer's purchase decision process. This means that even though not all consumers who perform the triggering action may buy the product, consumers cannot purchase the product without performing the triggering action.

The above assumptions allow us to uniquely express the advertiser's ex-ante value function as a product $V(p, q) = U(p, q)W(p, q)$ where:

$U(p, q)$	is the expected *triggering action frequency* (TAF) and
$W(p, q)$	is the expected *value-per-action* (VPA).

The precise meanings of $U(p, q)$ and $W(p, q)$ depend on the specifics of the payment mechanism. For example:

- In pay-per-sale mechanisms $U(p, q)$ is equal to the incremental per-period demand due to advertising while $W(p, q)$ is equal to the unit profit.
- In pay-per-click mechanisms $U(p, q)$ is equal to the per-period audience size times the probability of a click (the *clickthrough rate*) and $W(p, q)$ is equal to the conditional probability of a purchase given a click (the *conversion rate*) times the unit profit.
- Traditional per-per-exposure mechanisms are a special case of the above framework where $U(p, q) = 1$ and $W(p, q) = V(p, q)$.

I am assuming that the publisher has a reliable way of obtaining correct estimates of $U(p,q)$ and, therefore, that strategic misreporting of triggering actions from the part of the advertiser is not an issue.[5]

Assuming that every other bidder of type y bids an amount equal to $\beta_A(y)$ and that, as I will later show, it is $\beta'_A(y) \geq 0$, at equilibrium an advertiser of type q bids $b_A(q)$ and sets the price of her product at $p_A(q)$ to maximize her net expected profit:

$$\Pi_A(q; b_A(q), p_A(q), \beta_A(\cdot)) = \int\limits_0^{\beta_A^{-1}(b_A(q))} U(p_A(q), q)\left[W(p_A(q), q) - \beta_A(y)\right] G'(y)dy \tag{4}$$

subject to the equilibrium condition $b_A(q) = \beta_A(q)$. As before, the above specification subsumes the special case where product prices $p(q)$ are given exogenously. In the latter case, a bidder of type q simply chooses a bid $b_A(q; p(\cdot))$ that maximizes $\Pi_A(q; b_A(q; p(\cdot)), p(q), \beta_A(\cdot))$ subject to $\beta_A(q) = b_A(q; p(\cdot))$.

I use the following shorthand notation:

$\Pi_A(q)$	advertisers' PPA equilibrium profit function (endogenous prices)
$\Pi_A(q; p(\cdot))$	advertisers' PPA equilibrium profit function (exogenous prices)

The publisher's PPA revenue associated with bids $\beta(\cdot)$ and product prices $p(\cdot)$ is given by:

$$R_A(\beta(\cdot), p(\cdot)) = N \int\limits_{\underline{q}}^{\overline{q}} U(p(z), z) \left(\int\limits_{\underline{q}}^{z} \beta(y) G'(y) dy \right) F'(z) dz \tag{5}$$

I use the following shorthand notation:

$R_A = R_A(b_A(\cdot), p_A(\cdot))$	publisher's PPA equilibrium revenue (endogenous prices)
$R_A(p(\cdot)) = R_A(b_A(\cdot; p(\cdot)), p(\cdot))$	publisher's PPA equilibrium revenue (exogenous prices)

In the rest of this section I shall assume that:

A4 $\quad \frac{\partial}{\partial q} W(p_A(q), q) \geq 0$

[5] Addressing an advertiser's incentive to misreport the frequency of payment triggering action to the publisher is an important consideration in pay-per-action schemes but orthogonal to the focus of this paper. See Agarwal et al. (2009) and Nazerzadeh et al. (2008) for discussion and proposed solutions.

When assumption A4 does not hold, the simple auction mechanism discussed here may not always allocate the resource to the advertiser that maximizes the publisher's revenue. Appropriately designed *quality-adjusted* PPA mechanisms can often restore allocative efficiency in such cases. I discuss this important case in Section 5.

The objective of the analysis that follows is to study how the move from pay-per-exposure (i.e. $U(p, q) = 1$) to pay-per-action (arbitrary $U(p, q)$) payment mechanisms affects the prices of the advertised products, the publisher's revenue, the advertiser's profits and social welfare.

4. Baseline analysis

4.1. **Exogenous product prices.** To better appreciate how the move from PPE to PPA mechanisms affects publisher revenues and advertiser profits, it is instructive to begin our analysis by considering a setting where product prices are set exogenously to the advertisement payment mechanism. The vast majority of prior academic work on sponsored search and other forms of performance-based advertising have made this assumption. Equilibrium bidding strategies are straightforward in such cases:

Proposition 1. *Consider a setting where the prices $p(\cdot)$ of advertised products are set exogenously:*

(1) *If advertising is sold on a pay-per-exposure (PPE) basis, all advertisers bid their expected ex-ante value of acquiring the resource, given their price:*

$$b_E(q; p(\cdot)) = V(p(q), q)$$

(2) *If advertising is sold on a pay-per-action (PPA) basis, all advertisers bid their expected ex-ante value-per-action, given their price:*

$$b_A(q; p(\cdot)) = W(p(q), q)$$

The impact of moving from PPE to PPA on publisher revenues is more interesting. The key property is the relationship of the triggering action frequency $U(p(q), q)$ with the advertiser's type.

Proposition 2. *Consider a setting where the prices $p(q)$ of advertised products are set exogenously and satisfy:*

$$\frac{\partial}{\partial q}V(p(q),q) \geq 0 \quad \textit{and} \quad \frac{\partial}{\partial q}W(p(q),q) \geq 0 \quad \textit{for all } q \in [\underline{q}, \overline{q}] \tag{6}$$

(1) *If $\frac{\partial}{\partial q}U(p(q),q) \geq 0$ for all q, with the inequality strict for at least some q, then:*

$$R_A(p(\cdot)) > R_E(p(\cdot)) \textit{ and } \Pi_A(q;p(\cdot)) \leq \Pi_E(q;p(\cdot)) \quad \textit{for all } q \in [\underline{q}, \overline{q}]$$

(2) *If $\frac{\partial}{\partial q}U(p(q),q) \leq 0$ for all q, with the inequality strict for at least some q, then:*

$$R_A(p(\cdot)) < R_E(p(\cdot)) \textit{ and } \Pi_A(q;p(\cdot)) \geq \Pi_E(q;p(\cdot)) \quad \textit{for all } q \in [\underline{q}, \overline{q}]$$

(3) *If $\frac{\partial}{\partial q}U(p(q),q) = 0$ for all q, then:*

$$R_A(p(\cdot)) = R_E(p(\cdot)) \textit{ and } \Pi_A(q;p(\cdot)) = \Pi_E(q;p(\cdot)) \quad \textit{for all } q \in [\underline{q}, \overline{q}]$$

The intuition behind the above result is the following: In the case of PPE, publisher revenue (3) is equal to the product of the *second highest* bidder's triggering action frequency times the second highest bidder's value per action. In contrast, PPA publisher revenue (5) is equal to the product of the *highest* bidder's triggering action frequency times the second highest bidder's value per action. If the triggering action frequency is a monotonically increasing (decreasing) function of the advertiser's type then PPA results in higher (lower) expected publisher revenues compared to PPE.

The result about advertiser profits is a simple corollary of the fact that:

$$(BuyerProfits) = (PrivateValue)(ProbWin) - (ExpectedPayment)$$

In settings with exogenous product prices $(PrivateValue)$ is independent of the choice of payment mechanism. Furthermore, if (6) holds then, by Proposition 1, bids are monotone in q so $(ProbWin)$ is a function of the advertiser's type, and thus identical in both PPE and PPA. Therefore, when $(ExpectedPayment)$ increases $(BuyerProfits)$ decline and vice versa.

4.2. **Endogenous product prices.** The situation becomes considerably more interesting if we assume that advertisers set the prices of the advertised products endogenously to maximize their profits net of advertising. Product prices then become a function of the advertising payment mechanism. This scenario has been almost completely ignored in the literature so far.

The first important result shows that in settings where some information about the price of advertised products is conveyed to at least a subset of consumers before the action (click, call, sale) that triggers payment from the advertiser to the publisher, PPA payment mechanisms induce advertisers to distort the price of the advertised products relative to the case where there are no advertising expenses or where advertising is sold on a PPE basis.

Proposition 3. *In settings where advertisers endogenously set the prices of their advertised products to maximize profits net of advertising expenditures:*

(1) *If advertising is sold on a PPE basis, then:*
   (a) *Advertisers bid their expected ex-ante value of acquiring the resource, given their price:*
   $$b_E(q) = V(p_E(q), q)$$
   (b) *Advertisers set the price of their products at the point that maximizes their ex-ante expected value of acquiring the resource:*
   $$p_E(q) = \arg\max_p V(p, q) = p^*(q)$$
   (c) *Equilibrium PPE publisher revenues are equal to or higher to publisher revenues obtained in any PPE setting where the products of advertised products are set exogenously:*
   $$R_E = R_E(p^*(\cdot)) \geq R_E(p(\cdot))$$
   with the inequality strict if and only if $p^*(q) \neq p(q)$ *for at least one* $q \in [\underline{q}, \overline{q}]$.

(2) *If advertising is sold on a pay-per-action basis then:*

(a) *Advertisers of type* q *set the price of their products at a point* $p_A(q)$ *that has the following properties:*

$$
\begin{aligned}
p_A(q) > p^*(q) & \quad \textit{if } U_1(p,q) < 0 \textit{ for all } p \\
p_A(q) < p^*(q) & \quad \textit{if } U_1(p,q) > 0 \textit{ for all } p \\
p_A(q) = p^*(q) & \quad \textit{if } U_1(p,q) = 0 \textit{ for all } p
\end{aligned}
$$

(b) *In settings that admit interior solutions, at all symmetric equilibria:*

(i) *Advertisers bid their expected ex-ante value-per-action, given their price:*

$$b_A(q) = W(p_A(q), q)$$

(ii) *Product prices* $p_A(q)$ *satisfy:*

$$V_1(p_A(q), q)G(q) - U_1(p_A(q), q)J_A(q) = 0 \tag{7}$$

where $J_A(q) = \int_{\underline{q}}^{q} W(p_A(y), y)G'(y)dy$ *is the advertiser's expected per-action payment to the publisher.*

In most practical settings of interest where some information about price is available to at least a subset of consumers before they perform the triggering action, for a given quality level, the triggering action frequency (e.g. demand, clickthrough rate) will be monotonically decreasing with price. In the rest of the paper I will therefore make the assumption that:

A5 $\quad U_1(p, q) < 0$

which, by Proposition 3, implies $p_A(q) > p^*(q)$, i.e. an *upward* price distortion.

Let us first discuss the intuition behind the price distortion result of Proposition 3. Consider a hypothetical setting where the payment mechanism has recently been changed from PPE to PPA. Assume that every bidder, except our focal bidder, still prices her products at $p^*(q)$ and bids her corresponding VPA $\beta_A(q) = W(p^*(q), q)$. Under these assumptions our focal bidder's profit function (4) becomes:

$$\Pi_A(q; b, p, \beta_A(\cdot)) = V(p, q)G(\beta_A^{-1}(b)) - U(p, q)J(b)$$

where $J(b) = \int_{\underline{q}}^{\beta_A^{-1}(b)} W(p^*(y), y)G'(y)dy$ is the expected per-action payment to the publisher.

At $p = p^*(q)$ it is $V_1(p^*(q), q) = U_1(p^*(q), q)W(p^*(q), q) + U(p^*(q), q)W_1(p^*(q), q) = 0$. The assumption $U_1(p, q) < 0$ then implies that $W_1(p^*(q), q) > 0$. A small increase in the focal bidder's product price above $p^*(q)$ then decreases her triggering action frequency $U(p, q)$ and increases her value-per-action $W(p, q)$. This has the following consequences:

(1) Since $V_1(p^*(q), q) = 0$ and $V_{11}(p^*(q), q) < 0$ the net effect on the advertiser's value function is negative.

(2) The total expected payment to the publisher $U(p, q)J(b)$ decreases since the publisher gets paid less often.

(3) The optimal bid amount $b = W(p, q)$ increases. This increases the probability of winning the auction but also the expected per-action payment to the publisher $J(b)$. At equilibrium these two effects cancel out.

Since $V_1(p^*(q), q) = 0$, for prices that are sufficiently close to $p^*(q)$ effect 1 is always smaller than effect 2. Therefore, our focal bidder has a unilateral incentive to increase the price of her products up to the point where the marginal decrease in her value function becomes equal to the marginal decrease in her expected payment to the publisher. Since every advertiser has the same incentive the situation leads to a symmetric equilibrium where everyone prices their products above PPE levels and places correspondingly higher (per-action) bids.

It is important to note that the advertisers' incentive to increase the price of their products is not driven by the fact that they compete for a scarce advertising resource via an auction. Specifically, effects 1 and 2 and the ensuing advertiser's incentive to increase product prices are also present in settings where the number of available advertising resources is unlimited and a monopolist publisher sets a fixed per-action price J for each. In such settings, profit maximizing publishers will respond to the advertisers' tendency to increase product prices by correspondingly increasing the (fixed) per-action price of each resource. The resulting equilibria have qualitatively similar properties to those analyzed in this paper.

Observe that there are no price distortions when $U_1(p, q) = 0$ for all p, i.e. when the triggering action frequency is not a function of price. This condition can occur in settings where no consumer knows (or can guess) the price of the advertised good before performing the action that triggers payment to the publisher. I argue that this condition is not likely to hold in the majority of real-life PPA settings. It is clearly incompatible with pay-per-sale methods or with any other method where advertisers willingly disclose the price of their products at a point that precedes the triggering action (e.g. list their prices on a sponsored link ad and invite consumers to click the ad). It will also be violated in settings where at least a subset of consumers has access to product price information through separate channels, such as online product reviews.

The following corollary distills the most important result of this section:

Corollary 1. *In settings where the triggering action frequency is a monotonically decreasing function of product price, PPA payment mechanisms induce all advertisers to raise the price of their products so that they make fewer sales (and, thus, pay the publisher less often) but realize higher profit per sale relative to the case where advertising is sold using traditional PPE methods.*

This hitherto unrecognized side effect of PPA methods has important implications for all stakeholders: consumers, advertisers and the publisher.

4.3. **Revenue, surplus and welfare implications.** This section explores the implications of PPA price distortions for consumers, the publisher, advertisers and social welfare.

Implications for consumers. The most straightforward implication of the above price distortion is for consumers: Higher product prices unambiguously reduce the surplus of all consumers.

Corollary 2. *In settings where the triggering action frequency is a monotonically decreasing function of product price, PPA advertising methods always reduce consumer surplus relative to PPE methods.*

Implications for publisher revenues. Next I discuss the implications for publisher revenues. The important observation here is that the shift from PPE to PPA payment methods has two coupled consequences:

(1) The publisher's expected revenue changes by an amount equal to the difference of the first and second highest bidder's triggering action frequency times the second highest bidder's value-per-action (Proposition 2)

(2) Price distortions change every advertiser's triggering action frequency and value-per-action (Proposition 3).

Since price distortions always reduce every advertiser's value function relative to its optimum value and auction revenue is a function of the bidders' private values, the impact of effect 2 on publisher revenues is always negative. From Proposition 2 we know that the impact of effect 1 is positive if $\frac{\partial}{\partial q}U(p(q), q) \geq 0$ and negative if $\frac{\partial}{\partial q}U(p(q), q) \leq 0$. The cumulative impact on publisher revenues is the sum of these two effects: this is always negative if $\frac{\partial}{\partial q}U(p(q), q) \leq 0$ and can be positive or negative if $\frac{\partial}{\partial q}U(p(q), q) \geq 0$. The following proposition formalizes this intuition.

Proposition 4. *In settings where advertisers endogenously set the prices of their advertised products to maximize profits net of advertising expenditures:*

(1) *If, for all* $q \in [\underline{q}, \overline{q}]$*, it is* $\frac{\partial}{\partial q}U(p_A(q), q) \leq 0$*, then* $R_A < R_E$.

(2) *If, for all* $q \in [\underline{q}, \overline{q}]$*, it is* $\frac{\partial}{\partial q}U(p_A(q), q) \geq 0$ *with the inequality strict for at least some* q*, then:*

$R_A > R_E$ *if the price distortion* $|p_A(q) - p^*(q)|$ *is sufficiently small for all* q

$R_A < R_E$ *otherwise*

It is interesting to further explore Case 2 of the above proposition. Specifically, I will show that, for given $U(p, q)$ and $W(p, q)$, the magnitude of the price distortion induced by a PPA mechanism has a positive relationship with the ratio of a bidder's expected (per-action) payment relative to her value-per-action. The latter ratio, in turn, has a negative relationship with the dispersion of valuations among the bidder population.

Proposition 5. *Let:*

$$\zeta_A(q) = \frac{\int_{\underline{q}}^{q} W(p_A(y), y)G'(y)dy}{W(p_A(q), q)G(q)} \tag{8}$$

denote the expected payment-to-valuation ratio of an advertiser of type q conditional on that advertiser winning the publisher's auction. Fixing $U(p,q)$ and $W(p,q)$, the following statements summarize how the magnitude of $\zeta_A(q)$ impacts equilibrium PPA prices and the value of the advertising resource:

(1) $\frac{\partial p_A(q;\zeta_A(q))}{\partial \zeta_A(q)} \geq 0$

(2) *If* $W_1(p,q) > 0$ *for all* p *then it is* $\lim_{\zeta_A(q)\to 1} V(p_A(q;\zeta_A(q)),q) = 0$

The intuition behind this result is the following: The higher the per-action payment to the publisher, the higher the advertisers' marginal gain from increasing p_A and thus reducing the triggering action frequency (i.e. the frequency of paying the publisher). At the limit where the per-action expected payment approaches a bidder's value-per-action an advertiser's losses from the reduction in demand that results from price increases are almost exactly compensated by the corresponding reduction in the payment to the publisher. At the same time, if $W_1(p,q) > 0$, higher product prices result in a higher value-per-action, which allows the advertiser to place a higher bid. Competition among bidders for the advertising resource then pushes product prices upwards to the point where the triggering action frequency (and thus the value of the resource to the advertiser) goes to zero. This is a rat-race situation that, clearly, has negative consequences for all parties involved.

Integrating (8) by parts gives:

$$\zeta_A(q) = 1 - \frac{\int_{\underline{q}}^{q} \frac{\partial}{\partial y}\left[W(p_A(y),y)\right] G(y)dy}{W(p_A(q),q)G(q)} \tag{9}$$

From (9) it follows that $\zeta_A(q)$ is inversely related to the variability of the bidder population's *equilibrium* value-per-action $W(p_A(y),y)$ as a function of the bidders' type. The more homogeneous the VPA across bidders, the smaller the distance between the valuations of any two consecutive bidders and thus the higher the expected payment relative to the winning bidder's VPA. At the limit where $\frac{\partial}{\partial y}\left[W(p_A(y),y)\right] \to 0$, it is $\zeta_A(q) \to 1$. Intuitively, if the bidder population is homogeneous with respect to its value per action, the bidding competition for obtaining the resource becomes more intense and drives product prices up to the point where demand drops to zero.

Corollary 3. *Price distortions associated with PPA advertising are more severe in settings where the bidder population's equilibrium value-per-action is more homogeneous.*

Implications for advertiser profits. I now examine the implications of the shift from PPE to PPA for advertiser profits. Consider the advertiser profit functions under PPE and PPA rewritten as follows for easier comparison:

$$\Pi_E(q) = V(p_E(q), q)G(q) - \int_{\underline{q}}^{q} U(p_E(y), y)W(p_E(y), y)G'(y)dy$$
$$\Pi_A(q) = V(p_A(q), q)G(q) - U(p_A(q), q)\int_{\underline{q}}^{q} W(p_A(y), y)G'(y)dy$$

The shift from PPE to PPA has three consequences for the advertiser:

(1) The form of the total payment to the publisher changes from the product of the *second highest* bidder's triggering action frequency times the second highest bidder's value per action to the product of the *highest* bidder's triggering action frequency times the second highest bidder's value per action. As previously discussed (see Proposition 2), keeping the prices of advertised products constant, if $\frac{\partial}{\partial q}U(p(q), q) > 0$ (< 0) this results in a higher (lower) payment to the publisher, thus a reduction (increase) to net advertiser profits.

(2) The prices of advertised products increase from $p_E(q) = p^*(q)$ to $p_A(q)$. This reduces advertiser revenues $V(\cdot, \cdot)$ but also the frequency of payment to the publisher $U(\cdot, \cdot)$. If every other bidder's value per action stays constant, per (7), at equilibrium these two opposite effects balance out so the net effect is zero (Proposition 3).

(3) Every other bidder's value-per-action $W(\cdot, \cdot)$ increases as a result of the higher equilibrium product prices. Keeping the frequency of payment to the publisher constant, this increases the payment to the publisher and decreases net advertiser profits (this is the rat-race effect).

At equilibrium, effect 2 nets to zero, effect 3 is negative, whereas effect 1 may be negative or positive depending on the sign of $\frac{\partial}{\partial q}U(p(q), q)$. The overall effect is summarized in the following proposition. The result is symmetrical to that of Proposition 4.

Proposition 6. *In settings where advertisers endogenously set the prices of their advertised products to maximize profits net of advertising expenditures:*

(1) *If, for* $q \in [\underline{q}, \overline{q}]$*, it is* $\frac{\partial}{\partial q} U(p_A(q), q) \geq 0$*, then* $\Pi_A(q) < \Pi_E(q)$

(2) *If, for* $q \in [\underline{q}, \overline{q}]$*, it is* $\frac{\partial}{\partial q} U(p_A(q), q) \leq 0$ *with the inequality strict for at least some* q*, then:*

$$\Pi_A(q) > \Pi_E(q) \quad \textit{if the price distortion } |p_A(q) - p^*(q)| \textit{ is sufficiently small for all } q$$
$$\Pi_A(q) < \Pi_E(q) \quad \textit{otherwise}$$

Implications for social welfare. Finally, I explore the implications of moving from PPE to PPA for social welfare. Social welfare in this setting is equal to the value $V(p, q)$ generated by the resource plus consumer surplus from purchasing the advertised product. The payment from the advertiser to the publisher is a net transfer that does not affect social welfare. Recall that Assumption A5 implies that $p_A(q) > p_E(q) = p^*(q)$. This, in turn, implies the following:

(1) $V(p_A(q), q) < V(p_E(q), q)$, i.e. the value generated by the resource is always lower under PPA than PPE

(2) As discussed above, consumer surplus is always lower under PPA than PPE

The following corollary immediately ensues:

Corollary 4. *In settings where the triggering action frequency is a monotonically decreasing function of product price, PPA advertising methods always reduce social welfare relative to PPE methods.*

4.4. **An illustrative example.** This section illustrates the price and revenue implications of replacing a PPE mechanism with a PPA mechanism by analyzing a simple example that admits a closed-form solution. Consider a setting where there are two advertisers competing for a single resource. Each advertiser's quality $q \in [0, 1]$ is drawn independently from a uniform distribution. If an advertiser acquires the resource she gains incremental demand for her products equal to $D(p, q) = 1 + \kappa q - p$. The unit cost is $c(q) = \lambda q$, which implies that the unit profit is $p - \lambda q$ and the advertiser's expected benefit from acquiring the resource equal to $V(p, q) = (1 + \kappa q - p)(p - \lambda q)$. I assume throughout that $\kappa \geq \lambda \geq 0$.

If the publisher auctions the advertising resource using a PPE mechanism then the preceding analysis implies that each advertiser will set her price at the point that maximizes $V(p, q)$ and will bid her expected valuation, given her price. Let $p_{max}(q) = 1 + \kappa q$ denote the price for which demand falls to zero. The price that maximizes $V(p,q)$ is $p^*(q) = \frac{1}{2} + \frac{\kappa+\lambda}{2}q$. For uniformly distributed q this leads to:

demand	$D(p^*(q),q) = \frac{1}{2} + \frac{\kappa-\lambda}{2}q$
unit profit	$\frac{1}{2} + \frac{\kappa-\lambda}{2}q$
bids	$b_E(q) = V(p^*(q),q) = \left(\frac{1}{2} + \frac{\kappa-\lambda}{2}q\right)^2$
consumer surplus	$K(p^*(q),q) = \frac{(p_{max}(q)-p^*(q))D(p^*(q),q)}{2} = \frac{1}{2}\left(\frac{1}{2} + \frac{\kappa-\lambda}{2}q\right)^2$
expected payment	$J_E(q) = \int_0^q V(p^*(z),z)dz = q\frac{(\kappa-\lambda)^2q^2+3(\kappa-\lambda)q+3}{12}$
advertiser's profit	$\Pi_E(q) = V(p^*(q),q)q - J_E(q) = q^2\frac{(\kappa-\lambda)(3+2(\kappa-\lambda)q)}{12}$
publisher's revenue	$R_E = 2\int_0^1 J_E(q)dq = \frac{1}{4} + \frac{1}{6}(\kappa-\lambda) + \frac{1}{24}(\kappa-\lambda)^2$

If the publisher auctions the advertising resource using a *pay-per-sale* mechanism then $U(p,q) = D(p,q)$ and $W(p,q) = p - \lambda q$. Proposition 3 predicts that, at any interior solution, each advertiser will bid its expected value-per-action $b_A(q) = W(p_A(q),q)$ and will set the price $p_A(q)$ of its products to solve:

$$V_1(p_A(q),q)q - D_1(p_A(q),q)J_A(q) = 0 \tag{10}$$

where $J_A(q)$ is the expected per-action payment to the publisher. At equilibrium it will be $J_A(q) = \int_0^q W(p_A(y),y)dy = \int_0^q (p_A(y) - \lambda y)dy$. Substituting $J_A(q)$, $V(p,q)$ and $D(p,q)$ into (10) and differentiating with respect to q I obtain the following differential equation:

$$(-2p_A'(q) + \kappa + \lambda)\, q - p_A(q) + \kappa q + 1 = 0$$

whose solution is $p_A(q) = 1 + \frac{2\kappa+\lambda}{3}q$, leading to:

demand	$D(p_A(q),q) = \frac{\kappa-\lambda}{3}q$
bids (=unit profit)	$b_A(q) = W(p_A(q),q) = 1 + \frac{2}{3}(\kappa-\lambda)q$
consumer surplus	$K(p_A(q),q) = \frac{(p_{max}(q)-p_A(q))D(p_A(q),q)}{2} = \frac{1}{2}\left(\frac{\kappa-\lambda}{3}q\right)^2$
expected payment	$J_A(q) = \int_0^q b_A(y)dy = q(1 + \frac{\kappa-\lambda}{3}q)$
advertiser's profit	$\Pi_A(q) = V(p_A(q),q)q - D(p_A(q),q)J_A(q) = q^3\frac{(\kappa-\lambda)^2}{9}$
publisher's revenue	$R_A = 2\int_0^1 D(p_A(q),q)J_A(q)dq = \frac{2}{9}(\kappa-\lambda) + \frac{1}{18}(\kappa-\lambda)^2$

	PPE	PPA	Difference: PPA-PPE	Difference sign
Price of advertised product	$\frac{1}{2}+\frac{\kappa+\lambda}{2}q$	$1+\frac{2\kappa+\lambda}{3}q$	$\frac{1}{2}+\frac{\mu q}{6}$	+
Demand	$\frac{1}{2}+\frac{\mu q}{2}$	$\frac{\mu q}{3}$	$-\left(\frac{1}{2}+\frac{\mu q}{6}\right)$	-
Profit per sale	$\frac{1}{2}+\frac{\mu q}{2}$	$1+\frac{2\mu q}{3}$	$\frac{1}{2}+\frac{\mu q}{6}$	+
Value of advertising resource	$\left(\frac{1}{2}+\frac{\mu q}{2}\right)^2$	$\frac{\mu q}{3}\left(1+\frac{2\mu q}{3}\right)$	$-\left(\frac{1}{2}+\frac{\mu q}{6}\right)^2$	-
Net expected advertiser's profit	$\frac{\mu q^2}{4}+\frac{\mu^2 q^3}{6}$	$\frac{\mu^2 q^3}{9}$	$-\left(\frac{\mu q^2}{4}+\frac{\mu^2 q^3}{18}\right)$	-
Average publisher's revenue	$\frac{1}{4}+\frac{1}{6}\mu+\frac{1}{24}\mu^2$	$\frac{2}{9}\mu+\frac{1}{18}\mu^2$	$\frac{1}{72}\mu^2+\frac{1}{18}\mu-\frac{1}{4}$	+ if $\mu > 2.69$ − otherwise
Average consumer surplus	$\frac{1}{8}+\frac{1}{8}\mu+\frac{1}{24}\mu^2$	$\frac{1}{54}\mu^2$	$-\left(\frac{5}{216}\mu^2+\frac{1}{8}\mu+\frac{1}{8}\right)$	-

TABLE 1. Illustrative example of how bidding behavior and revenues are affected by the choice of payment method ($\mu = \kappa - \lambda$).

Table 1 summarizes the relevant quantities setting $\mu = \kappa - \lambda$. From a simple comparison it is easy to see that, for $\mu > 0$, the move from a PPE to a PPA mechanism has the following consequences:

- Consistent with theoretical predictions, product prices increase, whereas demand and the value of the advertising resource to all advertisers decrease.
- Net advertiser profits decrease for all q. Given that $\frac{\partial}{\partial q}U(p_A(q), q) = \frac{\partial}{\partial q}D(p_A(q), q) = \mu/3 > 0$ this is consistent with Proposition 6.
- The publisher's revenues decrease for small μ and increase if $\mu > 2.69$.
- Consumer surplus (substantially) decreases.

The limiting behavior of the system for very small and very large μ is also of interest:

- As $\mu \to 0$ consumer demand, advertiser profits and publisher revenue all go to zero.
- As $\mu \to \infty$ the PPA-to-PPE ratios of advertiser profits, publisher revenue, consumer surplus and social welfare (averaged over all q) monotically increase and asymptotically approach the values 2/3 and 4/3, 4/9 and 20/27 respectively (Figure 1).

The magnitude of $\mu = \kappa - \lambda$ is the key parameter in this setting. μ captures the difference between the consumers' marginal demand for quality (κ) and the marginal cost of producing quality (λ). When $\mu \to 0$, demand gains from higher quality are completely offset by the higher cost of producing quality. The equilibrium profit-per-sale is then identical for all advertiser types. This implies that all advertisers have the same value-per-action, which, in

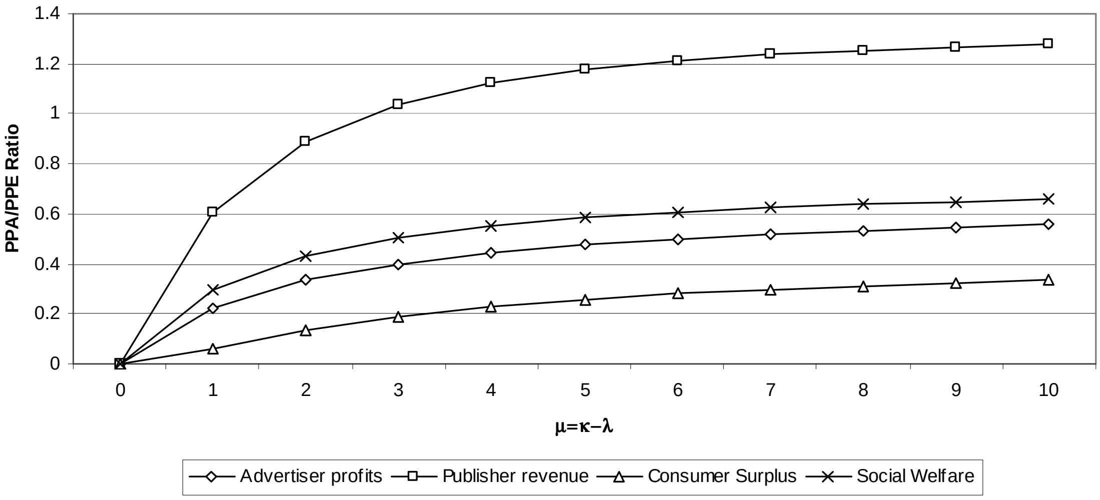

FIGURE 1. PPA-to-PPE ratio of key model quantities as $\mu = \kappa - \lambda$ increases.

turn, implies that the expected payment to the publisher is equal to the advertiser's value-per-action. By Proposition 5 competition for the resource among advertisers then drives prices up to the point where demand drops to zero.

Higher values of μ represent situations where the marginal demand for quality exceeds the marginal cost of producing quality. Higher quality advertisers then enjoy higher demand and higher profits per sale. Furthermore, the higher the μ the higher the difference between both the demand and the profit per sale of any two consecutive bidders. Recall that, under PPA, publisher revenues are equal to the triggering action frequency (in this setting, the demand) of the winning bidder times the value-per-action of the second highest bidder, whereas under PPE revenues are equal to the triggering action frequency times the value-per-action of the second highest bidder. The higher the μ, the higher the publisher revenue gains from capturing the demand of the first highest bidder under PPA (as opposed to the second highest bidder under PPE). In our setting, when $\mu > 2.69$, these gains offset the revenue losses due to the demand losses caused by distorted prices and result in net revenue gains for the publisher.

As our example illustrates, in settings where the population of advertisers is more highly differentiated with respect to their valuation of the advertising resource the consequences of pay-per-action advertising are less severe overall and might become positive for the publisher. In all cases, however, if firms set the prices of the advertised products endogenously to maximize profits net of advertising, replacing a PPE mechanism with a PPA scheme reduces the value generated by the advertising resource. If we also take into consideration the corresponding decline in consumer surplus, the adverse social impact of selling advertising using a pay-per-action mechanism becomes even more pronounced.

5. Quality-adjusted pay-per-action

In settings where $W(p_A(q), q)$ is not monotonically increasing with q, replacing a PPE mechanism with a PPA mechanism might result in allocative inefficiencies since the highest quality advertiser may no longer be the bidder with the highest value-per-action. Furthermore, as shown in Proposition 2, in settings where the triggering action frequency $U(p_A(q), q)$ is not monotonically increasing with q, moving from PPE to PPA may decrease publisher revenues even in the absence of price distortions. It is for such reasons that many practical PPA mechanism implementations are using a quality-adjusted winner determination rule (Athey and Ellison 2008; Varian 2007).

The idea behind *quality-adjusted pay-per-action* (QPPA) is straightforward: The publisher computes a quality weight u_i for each advertiser. The quality weight is typically based on past performance data and attempts to approximate that advertiser's expected triggering action frequency. Once bidders submit their bids b_i the publisher computes a *score* $s_i = u_i b_i$ for each bidder. The publisher allocates the resource to the bidder with the highest score and charges the winning bidder an amount $u_2 b_2 / u_1$ equal to the second highest score divided by the winning bidder's quality weight. Both Google and Yahoo use variants of this mechanism in their sponsored link auctions.[6]

In this section I will show that in settings where advertisers endogenously set the prices of their products, the price distortions identified in the previous section persist in the current generation of QPPA mechanisms. Furthermore, I show that in QPPA mechanisms price distortions *always* reduce publisher profits relative to those attainable in a PPE mechanism. I propose a mechanism enhancement that solves these problems and show that the steady

[6] Some researchers (e.g. Lahaie and Pennock 2007) have used the terms "rank by bid" and "rank by revenue" to refer to the winner determination rules in PPA and QPPA respectively.

state limit of my enhanced dynamic QPPA mechanism has identical allocation and revenue properties to those of an optimal PPE mechanism.

5.1. **Static settings.** Let $\Phi(q,s)$ denote an advertiser's beliefs about every other bidder's joint quality (q) and score (s) distribution. At equilibrium these beliefs must be consistent with bidding and publisher behavior. Let $\Phi(q) \equiv F(q)$ and $\Phi(s)$ be the corresponding marginal distributions and let $\Psi(s) = [\Phi(s)]^{N-1}$ denote the advertiser's belief that every other bidder's score will be less than s . Denote the advertiser's current quality weight as u. The single period specification of the advertiser's QPPA bidding problem is to choose a bid $b_Q(q,u)$ and a price $p_Q(q,u)$ that maximize:

$$\Pi_Q(q,u;b_Q(q,u),p_Q(q,u)) = \int_0^{ub_Q(q,u)} U(p_Q(q,u),q)\left(W(p_Q(q,u),q) - \frac{s}{u}\right)\Psi'(s)ds \tag{11}$$

The corresponding single period QPPA publisher revenue is equal to:

$$R_Q(b_Q(\cdot,\cdot),p_Q(\cdot,\cdot)) = N\int_{\underline{q}}^{\overline{q}} E_{u|q}\left[U(p_Q(q,u),q)\left(\int_0^{ub_Q(q,u)} \frac{s}{u}\Psi'(s)ds\right)\right]F'(q)dq \tag{12}$$

where $E_{u|q}[\cdot]$ denotes expectation with respect to u conditional on an advertiser's type being q.

The publisher's objective is to use u as an approximation of an advertiser's triggering action frequency. Of particular interest, therefore, is the behavior of the system at the limit where the publisher has obtained "correct" estimates of all quality weights, i.e. where each quality weight is equal to the respective advertiser's equilibrium triggering action frequency*:*

$$u_i = U(p_Q(q_i,u_i),q_i)$$

I use the following shorthand notation to refer to equilibrium quantities in such "correct quality weight" equilibria:

$p_Q(q) = p_Q(q, U(p_Q(q), q))$	equilibrium product prices
$b_Q(q) = b_Q(q, U(p_Q(q), q))$	equilibrium bids
$\Pi_Q(q) = \Pi_Q(q, U(p_Q(q), q); b_Q(q), p_Q(q))$	equilibrium advertiser's profits
$R_Q = R_Q(b_Q(q), p_Q(q))$	equilibrium publisher's revenue

The following proposition summarizes equilibrium bidding behavior and revenues in a static QPPA:

Proposition 7. *If advertising is sold on a quality-adjusted per-action (QPPA) basis and the publisher sets every advertiser's quality weight to her respective equilibrium triggering action frequency, the following hold:*

(1) *Advertisers set the price of their products at a point $p_Q(q)$ that has the following properties:*

$$\begin{aligned} p_Q(q) &> p^*(q) && \text{if } U_1(p,q) < 0 \text{ for all } p \\ p_Q(q) &< p^*(q) && \text{if } U_1(p,q) > 0 \text{ for all } p \\ p_Q(q) &= p^*(q) && \text{if } U_1(p,q) = 0 \text{ for all } p \end{aligned}$$

(2) *In settings that admit interior solutions:*

(a) *Advertisers bid their expected ex-ante value per action given their price:*

$$b_Q(q) = W(p_Q(q), q)$$

(b) *Product prices $p_Q(q)$ satisfy:*

$$V_1(p_Q(q), q)G(q) - U_1(p_Q(q), q)J_Q(q) = 0 \tag{13}$$

where $J_Q(q) = \left(\int_{\underline{q}}^{q} V(p_Q(y), y)G'(y)dy\right) / U(p_Q(q), q)$ is the expected per-action payment to the publisher.

Proposition 7 shows that price distortions persist in static QPPA settings where the publisher has perfect knowledge of each advertiser's triggering action frequency. Intuitively, if an advertiser's quality weight is predetermined and does not rely in any way on her current actions, the reasoning of Section 4.2 qualitatively applies here as well. Therefore, the advertiser's incentives to raise the price of her products persist even in quality-weighted mechanisms. Note however that, even though (7) and (13) are almost identical, the definition of $J_A(q)$ and $J_Q(q)$ is different. Therefore, in general it will be $p_A(q) \neq p_Q(q)$.

The next result shows that, if quality weights are fixed and exactly equal to equilibrium triggering action frequencies, QPPA is allocation and revenue equivalent to a PPE setting where product prices are exogenously set to $p_Q(\cdot)$.

Proposition 8. *If advertising is sold on a quality-adjusted per-action (QPPA) basis and the publisher sets every advertiser's quality weight to her respective equilibrium triggering action frequency then:*

(1) *Advertiser profits are identical to her equilibrium profits in a PPE setting where prices are exogenously set to* $p_Q(\cdot)$*:*

$$\Pi_Q(q) = \Pi_E(q; p_Q(\cdot))$$

(2) *Publisher revenues are identical to his equilibrium revenues in a PPE setting where every advertiser exogenously prices her products at* $p_Q(\cdot)$*:*

$$R_Q = R_E(p_Q(\cdot))$$

5.2. **Revenue, surplus and welfare implications.** This section explores the implications of QPPA price distortions for consumers, the publisher, advertisers and social welfare at the limit where all quality weights are equal to each advertiser' equilibrium triggering action frequency. It is the counterpart of Section 4.3 that performs the same analysis for the simple PPA mechanism.

Consumer surplus and social welfare. The situation is qualitatively identical to the simple PPA setting (Section 4.3): Product prices increase, reducing consumer surplus, and the value $V(p_Q(q), q)$ of the resource to the advertiser goes down. The following corollary immediately ensues:

Corollary 5. *Consumer surplus and social welfare are strictly lower in a static QPPA mechanism with endogenous product prices and perfectly estimated quality weights than in a corresponding PPE mechanism.*

Publisher revenue. From Proposition 8 we know that $R_Q = R_E(p_Q(\cdot))$ and that in general it will be $p_Q(\cdot) \neq p^*(\cdot)$. From Proposition 3 (Part 1(c)) it will then be $R_E(p^*(\cdot)) > R_E(p_Q(\cdot))$. The following important corollary ensues:

Corollary 6. *Equilibrium publisher revenues are strictly lower in a static QPPA mechanism with endogenous product prices and perfectly estimated quality weights than in a corresponding PPE mechanism.*

The reader should compare this result with the corresponding result in simple PPA settings. Proposition 4 shows that in such settings publisher revenues may be either lower or higher to those attained in a PPE mechanism. In contrast, static QPPA publisher revenues are always lower than PPE publisher revenues.

Advertiser profits. From Proposition 8 we know that $\Pi_Q(q) = \Pi_E(q; p_Q(\cdot))$. From standard auction theory, PPE advertising profits are equal to the difference between the first and second bidder's value function evaluated at the corresponding prices. Therefore, the impact of moving from PPE to QPPA on advertiser profits will depend on whether, as prices move from $p^*(\cdot)$ to $p_Q(\cdot)$, the corresponding value functions $V(p(q), q)$ of adjacent advertiser types q converge or diverge. In general this will be context-specific; advertiser profits may increase or decrease and the impact can be different for different advertiser types. The following result provides a *sufficient* condition for profits to increase or decrease for all types.

Proposition 9. *If advertising is sold on a quality-adjusted per-action (QPPA) basis and the publisher sets every advertiser's quality weight to her respective equilibrium triggering action frequency, then:*

(1) *If*

$$V_1(p_Q(q), q)\frac{\partial p_Q(q)}{\partial q} + V_2(p_Q(q), q) > V_2(p^*(q), q) \text{ for all } q \in [\underline{q}, \overline{q}]$$

the profits of all advertisers are higher under QPPA than under PPE.

	$PPE \rightarrow PPA$	$PPE \rightarrow QPPA$
Impact on consumer surplus	-	-
Impact on publisher revenue	+ or -	-
Impact on advertiser profits	+ or -	+ or -
Impact on social welfare	-	-

TABLE 2. Impact of moving from PPE to PPA and QPPA advertising methods for key stakeholders.

(2) *If*

$$V_1(p_Q(q), q)\frac{\partial p_Q(q)}{\partial q} + V_2(p_Q(q), q) < V_2(p^*(q), q) \text{ for all } q \in [\underline{q}, \overline{q}]$$

the profits of all advertisers are lower under QPPA than under PPE.

Table 2 summarizes the results of Sections 4.3 and 5.2. Observe that the impact of moving from PPE to either PPA or QPPA is negative for the majority of stakeholders and always negative for consumers and social welfare.

5.3. **Dynamic settings.** Most current implementations of QPPA involve a dynamic process whereby advertisers repeatedly bid for (and occasionally acquire) the resource and the publisher iteratively learns an advertiser's quality weight from observations of the advertiser's triggering action frequencies in past periods (Pandey and Olston 2006). This section shows that the price distortions that form the focus of this paper persist at the steady state limit of such processes but can be eliminated by a mechanism enhancement that makes an advertiser's current period quality weight also a function of her current period product price.

Let us assume an infinite horizon repeated game in which a set of advertisers bid for the resource at each round. Assume, further that the publisher maintains a quality weight u_t for each advertiser and updates it in every round according to the formula:

$$u_{t+1} = \begin{cases} h(u_t, U) & \text{if the advertiser acquires the resource in round } t \\ u_t & \text{otherwise} \end{cases} \tag{14}$$

where U is the observed triggering action frequency in round t and $h(\cdot,\cdot) \geq 0$ is an updating function that satisfies $h_1(u, \cdot) \geq 0$ and $h_2(\cdot, U) \geq 0$. In such a setting acquisition of the resource results both in current period gains $V(p, q) = U(p, q)W(p, q)$ as well as in future gains (or losses) due to the publisher's updating of the advertiser's quality weight. The advertiser's dynamic decision problem is to choose a sequence of bids b_t and prices p_t that satisfy the following Bellman equation:

$$\Omega(q,u_t)=\max_{b_t,p_t}\left\{\Pi_Q(q,u_t)+\delta\left[\Omega(q,h(u_t,U(p_t,q)))\Psi(u_t b_t)+\Omega(q,u_t)(1-\Psi(u_t b_t))\right]\right\} \quad (15)$$

The above dynamic specification affects the advertisers' bidding strategies relative to the static case as each bidder's value per action now incorporates both current and future payoffs. The following result holds:

Proposition 10. *Consider a repeated game where advertising is sold on a QPPA basis, an advertiser's quality weight is iteratively adjusted according to (14) and converges to a steady state that is characterized by* $u_i = U(p_Q(q_i,u_i),q_i)$.

(1) *Advertisers bid:*

$$b_t(p_t,u_t,q)=W(p_t,q)+\delta\frac{\Omega(q,h(u_t,U(p_t,q)))-\Omega(q,u_t)}{U(p_t,q)}$$

(2) *It is* $\Omega_2(q,u)>0$ *for all* q,u

Proposition 10 (Part 2) implies that profit-maximizing advertisers will strive to maintain a high quality weight u. If $U_1(p,q)<0$, any increases in current-round product prices p_t result in a lower observed $U(p_t,q)$ and therefore (given the assumption $h_2(\cdot,U)\geq 0$) in lower future quality weight estimates. Intuition then suggests that an advertiser's desire to maintain a high quality weight in future periods will moderate her incentive to increase her products' price during the current period. The following result confirms this intuition:

Proposition 11. *Let* $p_Q(q,u,\delta)$ *denote the price function that solves (15). This function satisfies:*

$$\frac{\partial p_Q(q,u,\delta)}{\partial\delta}\leq 0 \text{ for all } \delta\in[0,1]$$

Since the static case is equivalent to a setting where $\delta=0$ the above result shows that, in dynamic settings, the "shadow of the future" helps moderate price distortions relative to the static case.

The preceding discussion invites the question of whether one can design an updating function $h(u_t, U)$ that *exactly* balances an advertiser's current-round and continuation incentives and completely eliminates an advertiser's incentive to distort the prices of her products away from $p^*(q)$. Such a scheme would be similar in spirit to the click-fraud resistant clickthrough rate learning algorithms proposed by Immorlica et al. (2005).

Unfortunately, the following Proposition provides a negative answer.

Proposition 12. *Consider a QPPA mechanism that uses a quality weight updating process of the general form (14). Let $p_Q(q)$ denote the advertiser's product price at the steady-state limit where process (14) converges to a true assessment of each advertiser's triggering action frequency. At that limit it must be:*

$$p_Q(q) \neq p^*(q)$$

Otherwise stated, the above result shows that it is impossible to choose a function $h(\cdot, \cdot)$ that simultaneosuly achieves convergence of process (14) to a true assessment of each advertiser's triggering action frequency *and* induces advertisers to price their products at the PPE profit-maximizing level. In conjunction with Propositions 3 and 8 the above result implies that:

Corollary 7. *Steady-state publisher revenues in a dynamic QPPA mechanism that uses a quality weight updating process of the general form (14) are strictly lower than publisher revenues in a corresponding PPE mechanism.*

5.4. **A proposed solution.** The preceding analysis shows that the current generation of QPPA mechanisms induces advertisers to distort the prices of their products in a way that reduces consumer surplus, publisher revenues and social welfare relative to a more traditional PPE mechanism. The key to all previous results is the non-reliance of an advertiser's current period quality weight on the *current period* price of her products.

In this section I propose an enhanced QPPA mechanism that asymptotically induces advertisers to price their products at the per-exposure profit-maximizing level. The enhanced mechanism is based on the standard QPPA mechanism with the following modifications:

(1) Each period advertisers disclose to the publisher *both* their current period bid b_t as well as their current period product price p_t

(2) The advertiser's current period quality weight u_t is a function of both an advertiser's history of past prices and observed triggering action frequencies *and her current period price.* The quality weight attempts to predict the advertiser's current period triggering action frequency at price p_t

(3) The publisher uses the above quality weight as an input to the standard QPPA mechanism to determine the winner of the current period auction and the price the winner pays to the publisher.

From an implementation perspective the enhanced mechanism requires the publisher to maintain estimates of each advertiser's triggering action frequency *function* $U_i(p)$ from observations of past prices p_{it} and observed triggering action frequencies $U_{it}(p_{it})$. Although functions are infinite-dimensional objects, in the majority of practical settings (and especially if publishers have some domain knowledge) fairly accurate estimates can be obtained using finite-dimensional models and an appropriate iterative parameter updating method, such as maximum likelihood estimation. Such models can usually be easily extended to allow for non-deterministic settings where the observed triggering action frequencies have a random component. The model's parameter vector would then also include parameters that relate to the distribution of the random error.

A detailed analysis of the statistical and convergence properties of such schemes is outside of the scope of this paper.[7] My focus is to show that, provided that such schemes do converge to correct estimates of each advertiser's triggering action frequency function, the enhanced QPPA mechanism proposed above converges to a steady state that has identical allocation and revenue properties to those of an optimal PPE mechanism with endogenous prices. This is stated more formally below:

Proposition 13. *Consider an enhanced QPPA (EQPPA) mechanism that maintains estimates $\hat{U}_{it}(p)$ of each advertiser's triggering frequency function and sets her current period quality weight to $u_{it} = \hat{U}_{it}(p_t)$. At the limit where the publisher's estimate becomes exactly*

[7] See Cary et al. (2007) and Kominers (2008) for examples of such an analysis.

equal to the advertiser's true triggering action frequency function $U(p, q_i)$ the system reaches a steady state where the following hold:

(1) $b_{EQ}(q) = W(p_{EQ}(q), q)$

(2) $p_{EQ}(q) = p_E(q) = p^*(q)$

(3) $\Pi_{EQ}(q) = \Pi_E(q)$

(4) $R_{EQ} = R_E$

The preceding sections have shown that, in settings with endogenous product prices, a PPE mechanism results in higher consumer surplus, higher social welfare, higher publisher revenue and (sometimes) higher advertiser profits than a QPPA mechanism. It is therefore expected that, in most practical settings, the above mechanism enhancement will improve the economic attractiveness of current implementations of QPPA advertising for all classes of stakeholders.

6. Concluding Remarks

Technological advances have made it increasingly feasible to track the impact of individual advertising messages on consumer behavior. Accordingly, pay-per-performance advertising mechanisms, whereby the publisher is only paid when consumers perform certain actions (e.g. clicks, calls, purchases) that are tied to a specific advertising stimulus, have been gaining ground. Such pay-per-action (PPA) mechanisms are proving popular with advertisers because they help limit their risk when investing in new and often untested advertising technologies as well as allow them to better estimate their advertising ROI.

This paper highlights an important, and previously unnoticed, side-effect of PPA advertising. I show that, in settings where at least a subset of consumers receives price information before performing the action (click, call, purchase, etc.) that triggers payment to the publisher, PPA mechanisms induce advertisers to distort the prices of their products - usually upwards - as it is more beneficial to them to pay the publisher fewer times but realize a higher net profit per sale. Unfortunately, since every advertiser has the incentive to do the same, such behavior leads to rat-race equilibria where all advertisers end up paying more for access to advertising resources. Such equilibria always reduce social welfare and often reduce the payoffs of all stakeholders involved: consumers are always left with a lower surplus (because they pay higher prices) and one or both of advertiser profits and publisher revenues decline.

Price distortions persist in the quality-weighted variants of PPA advertising currently practiced by Google and Yahoo. Interestingly, in the latter settings they *always* reduce publisher revenues relative to more traditional pay-per-exposure methods. Fortunately, a relatively simple enhancement of quality-weighted PPA can eliminate the incentive to distort product prices. Specifically, if the publisher asks advertisers bidding for an advertising resource to disclose their current period product prices and makes each advertiser's quality weight a function of both past history and current product prices, I show that it is possible to construct payment mechanisms whose steady state has identical allocation and revenue properties to those of an optimal pay-per-exposure method.

To keep my models tractable but also to better highlight the phenomena that form the focus of the paper I made a number of simplifying assumptions. I am arguing that these assumptions do not detract from the essence of the phenomenon.

First, I have assumed that the resource is allocated to advertisers using a second-price auction. As I discuss in Section 4.2, the price distortions that form the core interest of this paper are not driven either by the scarcity of the resource or by the specifics of the auction mechanism. For example, they also occur in settings where an unlimited number of advertising resources are made available to advertisers at a fixed per-action price.

Second, I assumed that the publisher is a monopolist. Again, I argue that the core of the phenomenon I study is orthogonal to the publisher market structure. Price distortions would also occur in settings where several publishers are competing for advertisers: Once an advertiser has chosen a publisher and as long as there is a positive per-action payment from the advertiser to the publisher, the discusion of Section 4.2 shows that there will be incentives to distort prices (relative to the corresponding pay-per-exposure case) so as to reduce the total payment to the publisher.

Internet technologies have spurred a tremendous wave of innovation in advertising methods. New ideas are being continuously invented and tried out by ambitious entrepreneurs, often without being rigorously analyzed. Given the fast pace of competition and innovation in the Internet arena it is only natural that some of these ideas might have shortcomings or side-effects that are not immediately obvious to their inventors. One role for academia in this fast-changing environment is to place these new ideas under a rigorous theoretical lens, identify their shortcomings and propose economically sound improvements. This work is very much in this spirit.

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APPENDIX

I. Summary of key notation

Key Notation (listed alphabetically).

Symbol	*Description*
$b_\mu(q)$	Equilibrium bid amount of type q under payment mechanism $\mu^\dagger$
$F(q)$	Probability CDF of type q
$G(q)$	Probability CDF of second highest bidder's type
$J_\mu(q)$	Type q's expected per-period payment to publisher under mechanism $\mu^\dagger$
$p^*(q)$	Value-maximizing product price of type q
$p_\mu(q)$	Equilibrium product price of type q under mechanism $\mu^\dagger$
$\Pi_\mu(q)$	Type q's expected equilibrium profit under mechanism $\mu^\dagger$
q	Advertiser's type
R_μ	Publisher's expected revenue under mechanism $\mu^\dagger$
s	Advertiser's score = bid amount × quality weight
u	Advertiser's quality weight
$U(p,q)$	Triggering action frequency (TAF) function
$V(p,q)$	Per period value advertiser obtains by leasing the resource = TAF × VPA
$W(p,q)$	Value per action (VPA) function
$\Phi(q,s)$	Joint CDF of every other bidder's quality and score
$\Psi(s)$	CDF of second highest bidder's score
$\Omega(q,u)$	Infinite horizon advertiser's QPPA Bellman equation if current quality weight is u

†Subscript μ denotes the advertising payment mechanism to which the relevant quantity corresponds (see below).

Payment Mechanism Abbreviations.

Subscript (μ) in notation	*Abbreviation in text*	*Description*
E	PPE	Pay per exposure
A	PPA	Pay per action
Q	QPPA	Quality-adjusted pay per action
EQ	EQPPA	Enhanced quality-adjusted pay per action

II. Proofs

Proposition 1. It is well-known (see, e.g. Krishna 2002) that in Vickrey auctions with private values a bidder's optimal bid is equal to her expected valuation of the good she is trying to obtain. In PPE the ex-ante per-exposure value of the advertising resource is $V(p(q),q)$, hence $b_E(q;p(\cdot)) = V(p(q),q)$. In PPA the ex-ante per-action value of the advertising resource is equal to the value per action $W(p(q),q)$, hence $b_A(q;p(\cdot)) = W(p(q),q)$.

Proposition 2. Under the assumption $\frac{\partial V(p(q),q)}{\partial q} \geq 0$, substituting $\beta_E(q) = V(p(q),q)$ into (2) and (3) I obtain:

$$\Pi_E(q;p(\cdot)) = V(p(q),q)G(q) - \int_{\underline{q}}^{q} U(p(y),y)W(p(y),y)G'(y)dy$$

$$R_E(p(\cdot)) = N\int_{\underline{q}}^{\overline{q}}\left(\int_{\underline{q}}^{z} U(p(y),y)W(p(y),y)G'(y)dy\right)F'(z)dz$$

Similarly, under the assumption $\frac{\partial W(p(q),q)}{\partial q} \geq 0$, substituting $\beta_A(q) = W(p(q),q)$ into (4) and (5) I obtain:

$$\Pi_A(q;p(\cdot)) = V(p(q),q)G(q) - U(p(q),q)\int_{\underline{q}}^{q} W(p(y),y)G'(y)dy$$

$$R_A(p(\cdot)) = N\int_{\underline{q}}^{\overline{q}}\left(U(p(z),z)\int_{\underline{q}}^{z} W(p(y),y)G'(y)dy\right)F'(z)dz$$

Straightforward algebra produces:

$$\Pi_E(q;p(\cdot)) - \Pi_A(q;p(\cdot)) = \int_{\underline{q}}^{q} \left[U(p(q),q) - U(p(y),y)\right] W(p(y),y)G'(y)dy$$

$$R_E(p(\cdot)) - R_A(p(\cdot)) = N\int_{\underline{q}}^{\overline{q}}\left(\int_{\underline{q}}^{z} \left[U(p(y),y) - U(p(z),z)\right] W(p(y),y)G'(y)dy\right)F'(z)dz$$

It is now easy to see the following:

(1) If $U(p(q),q) \geq U(p(y),y)$ for all $q \in [\underline{q},\overline{q}]$ and all $\underline{q} \leq y \leq q$ with the inequality strict for at least some y, q then $\Pi_E(q;p(\cdot)) - \Pi_A(q;p(\cdot)) > 0$ and $R_E(p(\cdot)) - R_A(p(\cdot)) < 0$. The above *sufficient* condition is equivalent to $\frac{\partial U(p(q),q)}{\partial q} \geq 0$ for all $q \in [\underline{q},\overline{q}]$ with the inequality strict for at least some q.

(2) If $U(p(q),q) \leq U(p(y),y)$ for all $q \in [\underline{q},\overline{q}]$ and all $\underline{q} \leq y \leq q$ with the inequality strict for at least some y, q then $\Pi_E(q;p(\cdot)) - \Pi_A(q;p(\cdot)) < 0$ and $R_E(p(\cdot)) - R_A(p(\cdot)) > 0$. The above *sufficient* condition is equivalent to $\frac{\partial U(p(q),q)}{\partial q} \leq 0$ for all $q \in [\underline{q},\overline{q}]$ with the inequality strict for at least some q.

(3) If $U(p(q),q) = U(p(y),y)$ for all $q \in [\underline{q},\overline{q}]$ and all $\underline{q} \leq y \leq q$ then $\Pi_E(q;p(\cdot)) - \Pi_A(q;p(\cdot)) = 0$ and $R_E(p(\cdot)) - R_A(p(\cdot)) = 0$. The above *sufficient* condition is equivalent to $\frac{\partial U(p(q),q)}{\partial q} = 0$ for all $q \in [\underline{q},\overline{q}]$.

Proposition 3.

Part 1. (a) and (b).

Assume that every other bidder bids $\beta_E(y)$ and that (as I will show) $\beta_E'(y) > 0$. An advertiser of type q will choose bid $b_E(q)$ and price $p_E(q)$ that maximize:

$$\Pi_E(q; b_E(q), p_E(q), \beta_E(\cdot)) = \int_{\underline{q}}^{\beta_E^{-1}(b_E(q))} \left(V(p_E(q),q) - \beta_E(y)\right) G'(y) dy$$

First-order conditions with respect to bid and price give:

$$\frac{\partial \beta_E^{-1}(b_E(q))}{\partial b_E(q)} \left(V(p_E(q),q) - b_E(q)\right) G'(\beta_E^{-1}(b_E(q))) = 0 \quad \text{and} \quad V_1(p_E(q),q) G(\beta_E^{-1}(b_E(q))) = 0$$

At a symmetric equilibrium it must be $b_E(q) = \beta_E(q)$ which implies that $G(\beta_E^{-1}(b_E(q))) = G(q) > 0$, $G'(\beta_E^{-1}(b_E(q))) = G'(q) > 0$ and $\frac{\partial \beta_E^{-1}(b_E(q))}{\partial b_E(q)} = \frac{1}{b_E'(q)} > 0$. The above then reduces to:

$$b_E(q) = V(p_E(q),q) \qquad \text{and} \qquad V_1(p_E(q),q) = 0$$

Assumption A1 implies that $p_E(q) = p^*(q)$ is uniquely defined for all q and also that $V_{11}(p_E(q),q) < 0$. Assumption A3 and the envelope theorem further imply that $\frac{\partial V(p^*(q),q)}{\partial q} > 0$ and hence that $b_E'(q) > 0$, as originally assumed. The corresponding Hessian matrix is:

$$H_E(b_E(q), p_E(q), q) = \begin{bmatrix} -\frac{G'(q)}{V_2(p_E(q),q)} & 0 \\ 0 & V_{11}(p_E(q),q)G(q) \end{bmatrix}$$

It is straightforward to show that H_E is negative definite and, therefore, that the above pair $(b_E(q), p_E(q))$ corresponds to a local maximum of $\Pi_E(q; b_E(q), p_E(q), \beta_E(\cdot))$ for all q.

(c). With reference to Proposition 2 it is:

$$R_E(p(\cdot)) = N\int_{\underline{q}}^{\overline{q}}\left(\int_{\underline{q}}^{z} U(p(y),y)W(p(y),y)G'(y)dy\right)F'(z)dz = N\int_{\underline{q}}^{\overline{q}}\left(\int_{\underline{q}}^{z} V(p(y),y)G'(y)dy\right)F'(z)dz$$

Because $V(p^*(y),y) \geq V(p(y),y)$ for all y (with equality iff $p^*(y) = p(y)$), it is $R_E = R_E(p^*(\cdot)) \geq R_E(p(\cdot))$ with equality iff $p^*(\cdot) = p(\cdot)$.

Part 2. Assume that every other bidder bids $\beta_A(y)$ such that $\beta_A'(y) > 0$. An advertiser of type q will choose bid $b_A(q)$ and price $p_A(q)$ that maximize (4). The latter can be equivalently rewritten as:

$$\Pi_A(q; b_A(q), p_A(q), \beta_A(q)) = V(p_A(q),q)G(q) - U(p_A(q),q)J_A(b_A(q))$$

where $J_A(b_A(q)) = \int_{\underline{q}}^{\beta_A^{-1}(b_A(q))} \beta_A(y)G'(y)dy$. Differentiating with respect to $p_A(q)$ gives:

$$\frac{\partial \Pi_A}{\partial p_A(q)} = V_1(p_A(q),q)G(q) - U_1(p_A(q),q)J_A(b_A(q)) \tag{16}$$

Assumption A1 implies that:

$$\begin{aligned} &V_1(p,q) > 0 \text{ for all } p < p^*(q) \\ &V_1(p,q) = 0 \text{ for } p = p^*(q) \\ &V_1(p,q) < 0 \text{ for all } p > p^*(q) \end{aligned} \tag{17}$$

- If $U_1(p,q) < 0$ for all p then (16) and (17) imply that $\frac{\partial \Pi_A}{\partial p_A(q)} > 0$ for all $p_A(q) \leq p^*(q)$ and, therefore, that the advertiser can strictly increase net profits if she raises the price of her products above $p^*(q)$.
- If $U_1(p,q) > 0$ for all p then (16) and (17) imply that $\frac{\partial \Pi_A}{\partial p_A(q)} < 0$ (which is equivalent to $\frac{\partial \Pi_A}{\partial(-p_A(q))} > 0$) for all $p_A(q) \geq p^*(q)$ and, therefore, that the advertiser can strictly increase net profits if she reduces the price of her products below $p^*(q)$.
- Finally, if $U_1(p,q) = 0$ for all p then (16) and (17) imply that $\frac{\partial \Pi_A}{\partial p_A(q)} = V_1(p_A(q),q)G(q)$ and, therefore, that the price that maximizes $V(\cdot)$ also maximizes $\Pi_A(\cdot)$

Note that the above hold for any positive $b_A(q)$. In cases that admit interior solutions, first-order conditions with respect to bid and price give:

$$\frac{\partial \beta_A^{-1}(b_A(q))}{\partial b_A(q)} U(p_A(q),q)\left(W(p_A(q),q) - b_A(q)\right)G'(\beta_A^{-1}(b_A(q))) = 0$$

$$V_1(p_A(q),q)G(\beta_A^{-1}(b_A(q))) - U_1(p_A(q),q) \int\limits_{\underline{q}}^{\beta_A^{-1}(b_A(q))} \beta_A(y)G'(y)dy = 0 \tag{18}$$

At a symmetric equilibrium it must be $b_A(q) = \beta_A(q)$ which implies that $G(\beta_A^{-1}(b_A(q))) = G(q) > 0$, $G'(\beta_A^{-1}(b_A(q))) = G'(q) > 0$ and $\frac{\partial \beta_A^{-1}(b_A(q))}{\partial b_A(q)} = \frac{1}{b_A'(q)} > 0$. Therefore:

$$b_A(q) = W(p_A(q), q)$$

Substituting into (18) I obtain:

$$V_1(p_A(q),q)G(q) - U_1(p_A(q),q) \int\limits_{\underline{q}}^{q} W(p_A(y),y)G'(y)dy = 0$$

Proposition 4.

Part 1. With reference to Proposition 2 it is:

$$R_E(p(\cdot)) = N\int_{\underline{q}}^{\overline{q}} \left(\int_{\underline{q}}^{z} U(p(y),y)W(p(y),y)G'(y)dy \right) F'(z)dz = N\int_{\underline{q}}^{\overline{q}} \left(\int_{\underline{q}}^{z} V(p(y),y)G'(y)dy \right) F'(z)dz$$

Because $V(p^*(y),y) \geq V(p_A(y),y)$ it is $R_E = R_E(p^*(\cdot)) \geq R_E(p_A(\cdot))$. Furthermore, for $\frac{\partial U(p(q),q)}{\partial q} \leq 0$ (with the inequality strict for some q) it is $R_E(p(\cdot)) - R_A(p(\cdot)) > 0$ for any exogenous function $p(\cdot)$. Combining the two I obtain:

$$R_E(p^*(\cdot)) - R_A(p_A(\cdot)) \geq R_E(p_A(\cdot)) - R_A(p_A(\cdot)) > 0$$

which implies the first result.

Part 2. Again, with reference to Proposition 2, for $\frac{\partial U(p(q),q)}{\partial q} \geq 0$ (with the inequality strict for some q) it is $R_E(p(\cdot)) - R_A(p(\cdot)) < 0$ for any exogenous function $p(\cdot)$. Therefore:

$$R_E(p^*(\cdot)) - R_A(p^*(\cdot)) < 0$$

Furthermore, it is:

$$R_A(p(\cdot)) = N\int_{\underline{q}}^{\overline{q}}\left(\int_{\underline{q}}^{z} U(p(z),z)W(p(y),y)G'(y)dy\right)F'(z)dz \leq N\int_{\underline{q}}^{\overline{q}} V(p(z),z)G(z)F'(z)dz$$

By assumption A2 it is $\lim_{p\to\infty} V(p,q) = 0$ for all q. Let $p_\infty(\cdot)$ be a function defined by $p_\infty(q) = \infty$ for all q. It is then $R_A(p_\infty(\cdot)) \leq 0$ and

$$R_E(p^*(\cdot)) - R_A(p_\infty(\cdot)) > 0$$

As $p_A(\cdot)$ continuously ranges from $p^*(\cdot)$ to $p_\infty(\cdot)$ continuity implies the result.

Proposition 5.

Part 1. Let:

$$\Pi_A(q; p(q), \zeta(q)) = V(p(q),q)G(q) - U(p(q),q)J(q) = V(p(q),q)G(q)(1-\zeta(q)) \tag{19}$$

Let $p_A(q)$ be the price that maximizes $\Pi_A(q; p(q), \zeta(q))$. From Proposition 3 we know that, if $U_1(p,q) < 0$, it is $p_A(q) > p^*(q)$, which, by Assumption A1 implies that $V_1(p_A(q),q) < 0$. From monotone comparative statics theory (Milgrom and Shannon 1994) a sufficient condition for $\partial p_A(q)/\zeta(q) > 0$ is that $\partial^2\Pi_A(q; p(q), \zeta(q))/\partial p(q)\partial\zeta(q) > 0$. Differentiating (19) I obtain:

$$\frac{\partial^2\Pi_A(q; p_A(q), \zeta(q))}{\partial p_A(q)\partial\zeta(q)} = -V_1(p_A(q),q)G(q) > 0$$

Part 2. The price $p_A(q)$ that maximizes $\Pi_A(q; p(q), \zeta(q))$ must satisfy the first-order condition:

$$\begin{aligned} &V_1(p_A(q),q)G(q) - U_1(p_A(q),q)J_A(q) \\ &\quad = U_1(p_A(q),q)\left(W(p_A(q),q)G(q) - J_A(q)\right) + U(p_A(q),q)W_1(p_A(q),q)G(q) = 0 \end{aligned} \tag{20}$$

At the limit where $\zeta(q) \to 1$ it is $W(p_A(q),q)G(q) - J_A(q) \to 0$ and (20) simplifies to:

$$U(p_A(q),q)W_1(p_A(q),q)G(q) = 0$$

If $W_1(p,q) > 0$ the above implies $U(p_A(q),q) = 0$, which also implies $V(p_A(q),q) = 0$.

Proposition 6.

Case 1: $\frac{\partial}{\partial q}U(p_A(q), q) \geq 0$. Let $\upsilon(p, q, \lambda)$ be *any* family of triggering action frequency functions with the property $\lim_{\lambda \to -\infty} \upsilon(p, q, \lambda) = 1$ and $\lim_{\lambda \to +\infty} \upsilon(p, q, \lambda) = U(p, q)$ where $U(p, q)$ satisfies $U_1(p, q) \leq 0$ and $\frac{\partial}{\partial q}U(p_A(q), q) \geq 0$. Let $\Pi_A(q, \lambda)$ denote the advertiser's equilibrium PPA profit function corresponding to triggering action frequency $\upsilon(p, q, \lambda)$:

$$\Pi_A(q, \lambda) = V(p_A(q, \lambda), q)G(q) - \upsilon(p_A(q, \lambda), q, \lambda) \int_{\underline{q}}^{q} \frac{V(p_A(y, \lambda), y)}{\upsilon(p_A(y, \lambda), y, \lambda)} G'(y)dy \tag{21}$$

and let

$$V(p_A(q, \lambda), q) = \upsilon(p_A(q, \lambda), q, \lambda)w(p_A(q, \lambda), q, \lambda) \tag{22}$$

The integral:

$$\Delta\Pi = \int_{-\infty}^{+\infty} \frac{\partial \Pi_A(q, \lambda)}{\partial \lambda} d\lambda$$

is equal to the difference between the advertiser's profits in a PPA setting with triggering action frequency $U(p, q)$ and her profits in a PPE setting. I will show that $\Delta\Pi < 0$.

From Proposition 3, $p_A(q, \lambda)$ satisfies:

$$V_1(p_A(q, \lambda), q)G(q) - \upsilon_1(p_A(q), q, \lambda) \int_{\underline{q}}^{q} \frac{V(p_A(y, \lambda), y)}{\upsilon(p_A(y, \lambda), y, \lambda)} G'(y)dy = 0 \tag{23}$$

for all q, λ. Differentiating (21) with respect to λ and substituting (22) and (23) I obtain:

$$\begin{aligned} \tfrac{\partial \Pi_A(q,\lambda)}{\partial \lambda} &= -\upsilon(p_A(q, \lambda), q, \lambda) \textstyle\int_{\underline{q}}^{q} w(p_A(y, \lambda), y, \lambda) \left(\tfrac{\upsilon_3(p_A(q,\lambda),q,\lambda)}{\upsilon(p_A(q,\lambda),q)} - \tfrac{\upsilon_3(p_A(y,\lambda),y,\lambda)}{\upsilon(p_A(y,\lambda),y)} \right) G'(y)dy \\ &\quad -\upsilon(p_A(q, \lambda), q, \lambda) \textstyle\int_{\underline{q}}^{q} w_1(p_A(y, \lambda), y, \lambda) \tfrac{\partial p_A(y,\lambda)}{\partial \lambda} G'(y)dy \end{aligned} \tag{24}$$

It is easy to see that $\Delta\Pi < 0$ if both terms of (24) are non-positive for all q, λ and strictly negative for at least some q, λ. The first term of (24) satisfies this condition if the expression in parentheses is non-negative for all $q > y, \lambda$ and strictly positive for at least some $q > y, \lambda$. The latter holds true if:

$$\frac{\upsilon_3(p_A(q, \lambda), q, \lambda)}{\upsilon(p_A(q, \lambda), q)} \geq \frac{\upsilon_3(p_A(y, \lambda), y, \lambda)}{\upsilon(p_A(y, \lambda), y)} \text{ for all } q > y, \lambda$$

with the inequality strict for at least some y, λ. A sufficient condition for this to hold is:

$$\frac{\partial}{\partial q} \left(\frac{\upsilon_3(p_A(q, \lambda), q, \lambda)}{\upsilon(p_A(q, \lambda), q)} \right) \geq 0 \text{ for all } q, \lambda \tag{25}$$

with the inequality strict for at least some q, λ.

The second term of (24) is non-positive for all all q, λ if:

$$w_1(p_A(q,\lambda), q, \lambda)\frac{\partial p_A(q,\lambda)}{\partial \lambda} \geq 0 \text{ for all } q, \lambda \tag{26}$$

with the inequality strict for at least some q, λ.

Consider now the special family of triggering action frequencies:

$$\upsilon(p, q, \lambda) = H(\lambda)U(p, q) + (1 - H(\lambda)) \tag{27}$$

where $H(\lambda)$ is the Heaviside (step) function with $H(0) = 1$ and $H'(\lambda) = \delta(\lambda)$ where $\delta(\lambda)$ is Dirac's delta (unit impulse) function. It is easy to see that:

$$p_A(q, \lambda) = H(\lambda)p_U(q) + (1 - H(\lambda))p^*(q) \tag{28}$$

where $p^*(q)$ satisfies $V_1(p^*(q), q) = 0$ and $p_U(q) > p^*(q)$ solves:

$$V_1(p_U(q), q)G(q) - U_1(p_U(q), q)\int_{\underline{q}}^{q} \frac{V(p_U(y), y)}{U(p_U(y), y)}G'(y)dy = 0 \tag{29}$$

It is therefore:

$$\frac{\partial p_A(q,\lambda)}{\partial \lambda} = \delta(\lambda)\left[p_U(q) - p^*(q)\right] \tag{30}$$

Substituting (27) into (25) I obtain:

$$\frac{\partial}{\partial q}\left(\frac{\upsilon_3(p_A(q,\lambda), q, \lambda)}{\upsilon(p_A(q,\lambda), q)}\right) = \frac{\delta(\lambda)\frac{\partial}{\partial q}U(p_A(q,\lambda), q)}{\upsilon(p_A(q,\lambda), q, \lambda)^2} = \begin{cases} \frac{\delta(0)\frac{\partial}{\partial q}U(p_U(q),q)}{U(p_U(q),q)^2} & \text{if } \lambda = 0 \\ 0 & \text{otherwise} \end{cases}$$

If $\frac{\partial}{\partial q}U(p_U(q), q) \geq 0$ for all q with the inequality strict for at least some q, the above expression is positive at $\lambda = 0$ for at least some q and thus satisfies condition (25).

It is also:

$$w(p, q, \lambda) = H(\lambda)W(p, q) + (1 - H(\lambda))V(p, q)$$

where $W(p, q) = V(p, q)/U(p, q)$ and thus

$$w_1(p, q, \lambda) = H(\lambda)W_1(p, q) + (1 - H(\lambda))V_1(p, q)$$

Substituting the above and (30) into (26) and taking into account the properties of the Heaviside and Dirac functions gives:

$$w_1(p_A(q,\lambda),q,\lambda)\frac{\partial p_A(q,\lambda)}{\partial \lambda} = \begin{cases} \delta(0)W_1(p_U(q),q)\left[p_U(q)-p^*(q)\right] & \text{if } \lambda = 0 \\ 0 & \text{otherwise} \end{cases} \tag{31}$$

Substituting $W(p,q) = V(p,q)/U(p,q)$ into equation (29), under assumptions A4 and A5 I obtain:

$$W_1(p_U(q),q) = -\frac{U_1(p_U(q),q)}{U(p_U(q),q)}\left(W(p_U(q),q) - \int_{\underline{q}}^{q} W(p_U(y),y)\frac{G'(y)}{G(q)}dy\right) \geq 0$$

with the inequality strict for at least some q. Expression (31) then becomes positive at $\lambda = 0$ for at least some q and thus satisfies condition (26).

Since conditions (25) and (26) are both satisfied, this implies that $\Delta\Pi < 0$

Case 2: $\frac{\partial}{\partial q}U(p_U(q),q) \leq 0$. The result is an immediate corollary of the following observations:

(1) From Proposition 2 it is $\Pi_A(q;p^*(\cdot)) > \Pi_E(q;p^*(\cdot))$

(2) By assumption A2 it is $\lim_{p\to\infty} V(p,q) = 0$ for all q. Let $p_\infty(\cdot)$ be a function defined by $p_\infty(q) = \infty$ for all q. It is then $\Pi_A(q,p_\infty(\cdot)) \leq V(p_\infty(q),q) = 0$ and

$$\Pi_A(q,p_\infty(\cdot)) < \Pi_E(q,p^*(\cdot))$$

As $p_A(\cdot)$ continuously ranges from $p^*(\cdot)$ to $p_\infty(\cdot)$ continuity implies the result.

Proposition 7. The proof is similar to the Proof of Proposition 3, Part 2, the only difference being the definition of $J_Q(q,u;b_Q(\cdot)) = \int_0^{ub_Q(q,u)} \frac{s}{u}\Psi'(s)ds$. For $u = U(p_Q(q),q)$ and $b_Q(q) = W(p_Q(q),q)$ this becomes:

$$\begin{aligned} J_Q(q) &= \tfrac{1}{U(p_Q(q),q)} \int_0^{U(p_Q(q),q)W(p_Q(q),q)} s\Psi'(s)ds \\ &= \tfrac{1}{U(p_Q(q),q)} \int_0^{V(p_Q(q),q)} s\Psi'(s)ds = \tfrac{1}{U(p_Q(q),q)} \int_{\underline{q}}^{q} V(p_Q(y),y)G'(y)dy \end{aligned}$$

Proposition 8.

Part 1. Substituting $u = U(p_Q(q), q)$ and $b_Q(q) = W(p_Q(q), q)$ into (11) I obtain:

$$\Pi_Q(q, U(p_Q(q), q); b_Q(q), p_Q(q)) = \int_0^{U(p_Q(q),q)W(p_Q(q),q)} U(p_Q(q), q) \left(W(p_Q(q), q) - \frac{s}{U(p_Q(q),q)} \right) \Psi'(s)ds$$
$$= \int_0^{V(p_Q(q),q)} (V(p_Q(q), q) - s) \Psi'(s)ds = \int_{\underline{q}}^{q} (V(p_Q(q), q) - V(p_Q(y), y)) G'(y)dy$$
$$= V(p_Q(q), q)G(q) - \int_{\underline{q}}^{q} V(p_Q(y), y)G'(y)dy = \Pi_E(p_Q(\cdot))$$

Part 2. Substituting $u = U(p_Q(q), q)$ and $b_Q(q) = W(p_Q(q), q)$ into (12) I obtain:

$$R_Q(b_Q(\cdot, \cdot), p_Q(\cdot, \cdot)) = N \int_{\underline{q}}^{\overline{q}} U(p_Q(q), q) \left(\int_0^{U(p_Q(q),q)W(p_Q(q),q)} \frac{s}{U(p_Q(q),q)} \Psi'(s)ds \right) F'(q)dq$$
$$= N \int_{\underline{q}}^{\overline{q}} \left(\int_0^{V(p_Q(q),q)} s\Psi'(s)ds \right) F'(q)dq$$
$$= N \int_{\underline{q}}^{\overline{q}} \left(\int_{\underline{q}}^{q} V(p_Q(y), y)G'(y)dy \right) F'(q)dq = R_E(p_Q(\cdot))$$

Proposition 9. The proof is based on the following Lemma:

Lemma. *Consider two second-price auction settings E and Q, characterized by the same second-highest bidder distribution $G(q)$ for $q \in [\underline{q}, \overline{q}]$ and monotonically increasing private values $V_E(q)$ and $V_Q(q)$ respectively. A sufficient condition for the expected bidder surplus to be higher (resp. lower) in setting Q relative to setting E for all q is $V'_Q(q) > V'_E(q)$ (resp. $V'_Q(q) < V'_E(q)$) for all q.*

Proof. Let $\Pi_E(q), \Pi_Q(q)$ denote bidder surplus in the two settings. Also, let $\Delta\Pi(q) = \Pi_Q(q) - \Pi_E(q)$, $\Delta V(q) = V_Q(q) - V_E(q)$. It is:

$$\Pi_E(q) = V_E(q)G(q) - \int_{\underline{q}}^{q} V_E(y)G'(y)dy$$
$$\Pi_Q(q) = V_Q(q)G(q) - \int_{\underline{q}}^{q} V_Q(y)G'(y)dy$$
$$\Delta\Pi(q)/G(q) = \Delta V(q) - \int_{\underline{q}}^{q} \Delta V(y) \frac{G'(y)}{G(q)} dy$$

For monotonically increasing values, it is:

$$\Delta\Pi(q)/G(q) = \Delta V(q) - \Delta V(z(q)) \text{ for some } \underline{q} < z(q) < q$$

The above implies that $\Delta V(q) > \Delta V(z(q)) \Leftrightarrow \Pi_Q(q) > \Pi_E(q)$. A sufficient condition for $\Delta V(q) > \Delta V(z)$ for all $z < q$ is $\Delta V'(q) = V'_Q(q) - V'_E(q) > 0$ or, equivalently,$V'_Q(q) > V'_E(q)$ for all q. Reversing the signs of the inequalities one can similarly show that $V'_Q(q) < V'_E(q)$ for all q implies $\Pi_Q(q) < \Pi_E(q)$. □

Setting:

$$V_E(q) = V(p^*(q), q)$$
$$V_Q(q) = V(p_Q(q), q)$$

and applying the above lemma using:

$$V_E'(q) = V_2(p^*(q), q)$$
$$V_Q'(q) = V_1(p_Q(q), q)\tfrac{\partial p_Q(q)}{\partial q} + V_2(p_Q(q), q)$$

produces the result. The simpler expression for $V_E'(q)$ is due to the fact that $V_1(p^*(q), q) = 0$.

Proposition 10. To simplify notation, in this proof I omit the dependency of most quantities on the advertiser's type q.

Part 1. From (15):

$$(b_t, p_t) = \arg\max_{(b,p)} \left\{V(p)\Psi(u_t b) - U(p)J(u_t, b) + \delta\left(\Omega(h(u_t, U(p)))\Psi(u_t b) + \Omega(u_t)(1 - \Psi(u_t b))\right)\right\}$$

where:

$$J(u, b) = \frac{1}{u}\int_0^{ub} s\Psi'(s)ds$$

In all interior solutions b_t must satisfy the first-order condition:

$$\left[V(p_t) - U(p_t)b_t + \delta\left(\Omega(h(u_t, U(p_t))) - \Omega(u_t)\right)\right]\Psi'(u_t b_t)u_t = 0 \tag{32}$$

which implies:

$$b_t(p_t, u_t) = W(p_t) + \delta\frac{\Omega(h(u_t, U(p_t))) - \Omega(u_t)}{U(p_t)} \tag{33}$$

Part 2. Applying the envelope theorem to (15) I obtain:

$$\begin{aligned}\Omega'(u) = {} & V(p_Q(u))\Psi'(ub_Q(u))b_Q(u) - U(p_Q(u))J_1(u, b_Q(u)) \\ & +\delta\left(\Omega(h(u, U(p_Q(u)))) - \Omega'(u)\right)\Psi'(ub_Q(u))b_Q(u) \\ & +\delta\left(\Omega'(h(u, U(p_Q(u))))h_1(u, U(p_Q(u)))\Psi(ub_Q(u)) + \Omega'(u)(1 - \Psi(ub_Q(u)))\right)\end{aligned}$$

where $(b_Q(u), p_Q(u))$ is the bid-price pair that maximizes the right hand side expression of (15). Substituting (33) and:

$$J_1(u,b) = -\frac{1}{u^2}\int_0^{ub} s\Psi'(s)ds + \Psi'(ub)b^2$$

I obtain:

$$\begin{array}{l} \Omega'(u) = \frac{U(p_Q(u))}{u^2}\int_0^{ub_Q(u)} s\Psi'(s)ds \\ \quad +\delta\left[\Omega'(h(u,U(p_Q(u))))h_1(u,U(p_Q(u)))\Psi(ub_Q(u)) + \Omega'(u)(1-\Psi(ub_Q(u)))\right] \end{array} \tag{34}$$

I will now show that the above expression is positive for all u.

Step 1: At the steady-state limit u_0 defined by $u_0 = U(p_Q(u_0)) = h(u_0, U(p_Q(u_0))) = h(u_0, u_0)$ it is:

$$\Omega'(u_0) = \frac{\frac{U(p_Q(u_0))}{u^2}\int_0^{u_0 b_Q(u_0)} s\Psi'(s)ds}{1-\delta\left[h_1(u_0,U(p_Q(u_0)))\Psi(u_0 b_Q(u_0)) + (1-\Psi(u_0 b_Q(u_0)))\right]}$$

The steady state condition $h(x,x) = x$ must be valid for all $x \in \{u_0(q)|\underline{q} \leq q \leq \overline{q}\}$. This implies that $h_1(u_0,u_0) + h_2(u_0,u_0) = 1$, which, together with the assumptions $h_1(\cdot,\cdot) \geq 0$ and $h_2(\cdot,\cdot) \geq 0$ implies that $h_1(u_0, U(p_Q(u_0))) \leq 1$. Therefore $h_1(u_0, U(p_Q(u_0)))\Psi(u_0 b_Q(u_0)) + (1-\Psi(u_0 b_Q(u_0))) \leq 1$ and $\Omega'(u_0) > 0$.

Step 2: For any other u, (34) can be equivalently written as:

$$\Omega'(u) = \frac{\frac{U(p_Q(u))}{u^2}\int_0^{ub_Q(u)} s\Psi'(s)ds + \delta\Omega'(h(u,U(p_Q(u))))h_1(u,U(p_Q(u)))\Psi(ub_Q(u))}{1-\delta(1-\Psi(ub_Q(u)))}$$

The above expression is positive if $\Omega'(h(u,U(p_Q(u)))) > 0$. Since $\Omega'(u_0) > 0$, for any u from which the iterative process (14) converges to u_0, use of a backward induction argument implies that $\Omega'(u) > 0$.

Proposition 11. Let $p \equiv p_Q(q,u,\delta)$ denote the price that maximizes the value function of (15):

$$\Upsilon(b,p,q,u,\delta) = \Pi_Q(q,u) + \delta\left[\Omega(q,h(u,U(p,q)))\Psi(ub) + \Omega(q,u)(1-\Psi(ub))\right] \tag{35}$$

From monotone comparative statics theory (Milgrom and Shannon 1994) a sufficient condition for $\partial p/\partial\delta < 0$ is that $\partial^2\Upsilon(b,p,q,u,\delta)/\partial p\partial\delta < 0$. Differentiating (35) I obtain:

$$\frac{\partial^2\Upsilon(p,q,u,\delta)}{\partial p\partial\delta} = \Omega_2(q,h(u,U(p,q)))h_2(u,U(p,q))U_1(p,q)\Psi(ub) < 0$$

(It is $\Omega_2(\cdot,\cdot) > 0$ by Proposition 10, $h_2(\cdot,\cdot) > 0$ by assumption and $U_1(\cdot,\cdot) < 0$ by A5).

Proposition 12. To simplify notation, in this proof I omit the dependency of most quantities on the advertiser's type q.

Let us assume that the publisher maintains a quality weight u_t for each advertiser and updates it every time that advertiser acquires the resource according to (14). I will explore the behavior of this system at a steady-state limit where the advertiser's quality weight stabilizes into a correct assessment of her current period triggering action frequency. At that limit it will be $u_0 = U(p_Q(u_0))$ where u_0 is a fixed point of (14), defined by:

$$u_0 = h(u_0, U(p_Q(u_0)))$$

From Proposition 8 we know that, at the limit where $u_0 = U(p_Q(u_0))$, advertiser profits and publisher revenues become identical to those of a PPE setting where product prices are exogenously set to $p_Q(u_0)$. From Proposition 3, PPE publisher revenues are maximized when every advertiser chooses $p_Q(u_0, q) = p^*(q)$, such that $V_1(p^*(q), q) = 0$, and are strictly lower for any other price.

I will show that there can be no steady-state limit where $p_Q(u_0, q) = p^*(q)$. The proof will be by contradiction.

An advertiser whose current quality weight is u chooses bid $b_Q(u)$ and price $b_Q(u)$ that solve:

$$(b_Q(u), p_Q(u)) = \arg\max_{(b,p)} \{V(p)\Psi(ub) - U(p)J(u,b) + \delta\left[\Omega(h(u, U(p)))\Psi(ub) + \Omega(u)(1 - \Psi(ub))\right]\}$$

where:

$$J(u,b) = \frac{1}{u}\int_0^{ub} s\Psi'(s)ds$$

In all interior solutions $b_Q(u), p_Q(u)$ must satisfy the first-order conditions:

$$[V(p_Q(u)) - U(p_Q(u))b_Q(u) + \delta\left(\Omega(h(u, U(p_Q(u)))) - \Omega(u)\right)]\,\Psi'(ub_Q(u))u = 0 \tag{36}$$

$$\begin{aligned} &V'(p_Q(u))\Psi(ub_Q(u)) \\ &\quad -U'(p_Q(u))\left[J(u, b_Q(u)) - \delta\Omega'(h(u, U(p_Q(u))))h_2(u, U(p_Q(u)))\Psi(ub_Q(u))\right] = 0 \end{aligned} \tag{37}$$

At a fixed point u_0 it is $h(u_0, U(p_Q(u_0))) = u_0$ and (36) gives:

$$b_Q(u_0) = \frac{V(p_Q(u_0))}{U(p_Q(u_0))} = W(p_Q(u_0))$$

Substituting into (37) I obtain:

$$\begin{aligned} &V'(p_Q(u_0))\Psi(u_0 W(p_Q(u_0))) \\ &\quad -U'(p_Q(u_0))\left[J(u_0, W(p_Q(u_0))) - \delta\Omega'(u_0)h_2(u_0, U(p_Q(u_0)))\Psi(u_0 W(p_Q(u_0)))\right] = 0 \end{aligned} \tag{38}$$

Examining (38), if we assume that $\Psi(u_0 W(p_Q(u_0))) > 0$ and $U'(p_Q(u_0)) < 0$ it will be $p_Q(u_0) = p^*$, such that $V'(p^*) = 0$, if and only if the second term of the equation goes to zero at $p_Q(u_0) = p^*$ and $u_0 = U(p^*)$ where, additionally, u_0 is the limit of the iterative process (14). The problem reduces to finding an updating function $h(x, y)$ that satisfies the following system of constraints for all q:

$$\begin{aligned} &U(p^*) = h(U(p^*), U(p^*)) \\ &J(U(p^*), W(p^*)) - \delta\Omega'(U(p^*))h_2(U(p^*), U(p^*))\Psi(V(p^*)) = 0 \\ &\text{where } p^* \text{ is defined by } V'(p^*) = 0 \end{aligned} \tag{39}$$

I will show that no such function exists.

Because the first constraint must apply for all q, it implies that:

$$h(x, x) = x \text{ for all } x \in \{U(p^*(q)) | \underline{q} \leq q \leq \overline{q}\}$$

which, in turn, implies that:

$$h_1(U(p^*), U(p^*)) + h_2(U(p^*), U(p^*)) = 1 \tag{40}$$

Differentiating

$$\begin{aligned} \Omega(u) = {} & V(p_Q(u))\Psi(ub_Q(u)) - U(p_Q(u))J(u, b_Q(u)) \\ & +\delta\left(\Omega(h(u, U(p_Q(u))))\Psi(ub_Q(u)) + \Omega(u)(1 - \Psi(ub_Q(u)))\right) \end{aligned}$$

and substituting $p_Q(u) = p^*$, $u = h(u, U(p^*)) = U(p^*)$, $b_Q(u) = W(p^*)$, $V'(p^*) = 0$ I obtain:

$$\Omega'(U(p^*)) = \frac{J(U(p^*), W(p^*))\left(1 - U'(p^*)p'_Q(U(p^*))\right)}{1 - \delta\left[\Psi(V(p^*))\left(h_1(U(p^*), U(p^*)) + h_2(U(p^*), U(p^*))U'(p^*)p'_Q(U(p^*))\right) + (1 - \Psi(V(p^*)))\right]} \tag{41}$$

Substituting (41) into the second constraint (39) I get:

$$\frac{1-\delta\left[\Psi(V(p^*))\left(h_1(U(p^*),U(p^*))+h_2(U(p^*),U(p^*))\right)+(1-\Psi(V(p^*)))\right]}{1-\delta\left[\Psi(V(p^*))\left(h_1(U(p^*),U(p^*))+h_2(U(p^*),U(p^*))U'(p^*)p'_Q(U(p^*))\right)+(1-\Psi(V(p^*)))\right]}=0$$

which is equivalent to:

$$h_1(U(p^*),U(p^*))+h_2(U(p^*),U(p^*))=1+\frac{1}{\Psi(V(p^*))}\frac{1-\delta}{\delta}>1 \tag{42}$$

Conditions (40) and (42) lead to a contradiction. Therefore all quality weight updating functions that converge to an accurate assessment of each advertiser's triggering action frequency induce steady state price distortions relative to a PPE scheme.

Proposition 13. To simplify notation, in this proof I omit the dependency of most quantities on the advertiser's type q.

Let us assume that the publisher uses a statistical model to iteratively learn each advertiser's triggering action frequency function $U(p)$ from observations of prices p_t and triggering action frequencies $U_t(p_t)$. Assume, further, that the model is based on a parameter vector λ that the publisher estimates using some sound statistical method, e.g. maximum likelihood estimation. Let λ_t be the current publisher's estimate of an advertiser's parameter vector and let $u_t(\lambda_t, p)$ be the corresponding estimate of the advertiser's triggering action frequency function (as a function of her current price). Finally, let $\lambda_{t+1} = h_t(\lambda_t, p_t, U_t)$ be the parameter vector updating function if the current round price is p_t and the observed triggering action frequency is equal to U_t.

In such a setting a profit-maximizing advertiser chooses bid $b_t(\lambda_t)$ and price $p_t(\lambda_t)$ that solve:

$$\begin{aligned}(b_t(\lambda_t),p_t(\lambda_t)) = \arg\max_{(b,p)} V(p)\Psi(u_t(\lambda_t,p)b) - U(p)J(u_t(\lambda_t,p),b)\\ +\delta\left(\Omega(h_t(\lambda_t,p,U(p)))\Psi(u_t(\lambda_t,p)b)+\Omega(u_t(\lambda_t,p))(1-\Psi(u_t(\lambda_t,p)b))\right)\end{aligned} \tag{43}$$

where:

$$J(u,b)=\frac{1}{u}\int_0^{ub} s\Psi'(s)ds$$

In all interior solutions, $b_t(\lambda_t), p_t(\lambda_t)$ must satisfy the first-order conditions:

$$V(p_t(\lambda_t)) - U(p_t(\lambda_t))b_t(\lambda_t) + \delta\left(\Omega(h_t(\lambda_t,p_t(\lambda_t),U(p_t(\lambda_t)))) - \Omega(\lambda_t)\right) = 0 \tag{44}$$

$$
\begin{aligned}
V'(p_t(\lambda_t))\Psi(u_t(\lambda_t, p_t(\lambda_t))b_t(\lambda_t)) - U'(p_t(\lambda_t))J(u_t(\lambda_t, p_t(\lambda_t)), b_t(\lambda_t)) \\
+ (V(p_t(\lambda_t))\Psi'(u_t(\lambda_t, p_t(\lambda_t))b_t(\lambda_t))b_t(\lambda_t) - U(p_t(\lambda_t))J_1(u_t(\lambda_t, p_t(\lambda_t)), b_t(\lambda_t))) \tfrac{\partial u_t(\lambda_t, p_t(\lambda_t))}{\partial p_t(\lambda_t)} \\
+\delta\Omega'(h_t(\lambda_t, p_t(\lambda_t), U(p_t(\lambda_t)))) \tfrac{\partial h_t(\lambda_t, p_t(\lambda_t), U(p_t(\lambda_t)))}{\partial p_t(\lambda_t)} \Psi(u_t(\lambda_t, p_t(\lambda_t))b_t(\lambda_t)) \\
+\delta\left(\Omega(h_t(\lambda_t, p_t(\lambda_t), U(p_t(\lambda_t)))) - \Omega(\lambda_t)\right) \Psi'(u_t(\lambda_t, p_t(\lambda_t))b_t(\lambda_t)) \tfrac{\partial u_t(\lambda_t, p_t(\lambda_t))}{\partial p_t(\lambda_t)} b_t(\lambda_t)
\end{aligned} \tag{45}
$$

In the rest of the proof I use the notation $h(\cdot,\cdot,\cdot) = \lim_{t\to\infty} h_t(\cdot,\cdot,\cdot)$, $u(\cdot,\cdot) = \lim_{t\to\infty} u_t(\cdot,\cdot)$, $b(\cdot) = \lim_{t\to\infty} b_t(\cdot)$ and $p(\cdot) = \lim_{t\to\infty} p_t(\cdot)$.

If the statistical process $\lambda_{t+1} = h_t(\lambda_t, p_t(\lambda_t), U(p_t(\lambda_t)))$ converges to the true λ, i.e. if

$$\lim_{t\to\infty} h_t(\lambda_t, p_t(\lambda_t), U(p_t(\lambda_t))) = \lambda \tag{46}$$

such that $u(\lambda, p) = U(p)$ for all p, at the limit $t \to \infty$ it must be:

$$\lim_{t\to\infty} \frac{\partial h_t(\lambda_t, p, U(p))}{\partial p} = \frac{\partial h(\lambda, p, U(p))}{\partial p} = 0 \tag{47}$$

The above equation simply states the fact that at the limit where the publisher has perfect knowledge of λ, the advertiser's prices and triggering action frequencies are exactly as predicted by the model and there is no more updating of the parameter vector λ_t.

At the limit $t \to \infty$ (44) and (46) give:

$$b(\lambda) = \frac{V(p(\lambda))}{U(p(\lambda))} = W(p(\lambda))$$

Substituting the above $b(\lambda)$ and (46), at the limit $t \to \infty$ (45) simplifies to:

$$\left[V'(p(\lambda)) - \delta\Omega'(\lambda)\frac{\partial h(\lambda, p(\lambda), U(p(\lambda)))}{\partial p(\lambda)}\right] \Psi(V(p(\lambda))) = 0$$

Substituting (47) the above expression further simplifies to:

$$V'(p(\lambda)) = 0$$

which implies that $p(\lambda) = p^*$. The above together with Propositions 3 (Part 1) and 8 imply the result.