# News Trading and Speed*

Thierry Foucault† Johan Hombert‡ Ioanid Roşu§

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**Abstract**

Adverse selection occurs in financial markets because certain investors have either (a) more precise information, or (b) superior speed in accessing or exploiting information. To disentangle the effects of precision and speed on market performance, we compare two models in which a dealer and a more precisely informed trader continuously receive news about the value of an asset. In the first model the trader and the dealer are equally fast, while in the second model the trader receives the news one instant before the dealer. Compared with the first model, in the second model: (1) the fraction of trading volume due to the informed investor increases from near zero to a large value; (2) liquidity decreases; (3) short-term price changes are more correlated with asset value changes; (4) informed order flow autocorrelation decreases to zero. Our results suggest that the speed component of adverse selection is necessary to explain certain empirical regularities from the world of high frequency trading.

Keywords: Insider trading, Kyle model, noise trading, trading volume, algorithmic trading, informed volatility, price impact.

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†HEC Paris, foucault@hec.fr.

‡HEC Paris, hombert@hec.fr.

§HEC Paris, rosu@hec.fr.

# 1 Introduction

The recent advent of high frequency trading (HFT) in financial markets has raised numerous questions about the role of high frequency traders and their strategies.[1] Because of the proprietary nature of HFT and its extraordinary speed, it is difficult to characterize HFT strategies in general.[2] Nevertheless, there is increasing evidence that at least one category of high frequency traders exploits very quick access to public information in an attempt to analyze the news and trade before everyone else. For example, in their online advertisement for real-time data processing tools, Dow Jones states: "Timing is everything and to make lucrative, well-timed trades, institutional and electronic traders need accurate real-time news available, including company financials, earnings, economic indicators, taxation and regulation shifts. Dow Jones is the leader in providing high-frequency trading professionals with elementized news and ultra low-latency news feeds for algorithmic trading."[3] This category of HFT can also use public market data to infer information from related securities. We call this category high frequency news trading (HFNT) or, in short, *news trading*.

Clearly, news trading generates adverse selection.[4] In general, adverse selection occurs because some investors have either (a) more precise information, or (b) superior speed in accessing or exploiting information. Traditionally, the market microstructure literature, e.g., Kyle (1985), has mainly focused on the first type of adverse selection. In contrast, the speed component of adverse selection has received little attention. Our paper focuses on this second type of adverse selection in the context of news trading.

To separate the role of precision and speed, we consider two models of trading under

[1] In many markets around the world, high frequency trading currently accounts for a majority of trading volume. Hendershott, Jones, and Menkveld (2011) report that in 2009 as much as 73% of trading volume in the United States was due to high frequency trading. A similar result is obtained by Brogaard (2011) for NASDAQ, and Chaboud, Chiquoine, Hjalmarsson, and Vega (2009) for various Foreign Exchange markets. High frequency trading has been questioned espectially after the U.S. "Flash Crash" on May 6, 2010. See, e.g., Kirilenko, Kyle, Samadi, and Tuzun (2011).

[2] SEC (2010) attributes the following characteristics to HFT: (1) the use of extraordinarily high-speed and sophisticated computer programs for generating, routing, and executing orders; (2) use of co-location services and individual data feeds offered by exchanges and others to minimize network and other types of latencies.

[3] See http://www.dowjones.com/info/HighFrequencyTrading.asp.

[4] Hendershott and Riordan (2011) find that on NASDAQ the marketable orders of high frequency traders have a significant information advantage and are correlated with future price changes.

asymmetric information. In both models, a risk-neutral informed trader and a competitive dealer (or market maker) continuously learn about the value of an asset. In both models, the informed trader receives a more precise stream of news than that received by the dealer. The only difference lies in the timing of access to the stream of news. In the first model, the *benchmark model*, the informed trader and the dealer are equally fast.[5] In the second model, the *fast model*, the informed trader receives the news one instant before the dealer. We show that even an infinitesimal speed advantage leads to large differences in the predictions of the two models.

We further argue that the fast model is better suited to describe the world of high frequency trading. For example, consider the recent increase in trading volume observed in various exchanges around that world, which in part has been attributed to the rise of HFT. At high frequencies, traditional models such as Kyle (1985), or extensions such as our benchmark model have difficulty in generating a large trading volume of investors with superior information. To see, this, consider Figure 1. As it is apparent from the plot, the fast model can account for a significant participation rate of informed trading at higher frequencies, while the informed trader in the benchmark model is essentially invisible at high frequencies.[6] Thus, accounting for adverse selection due to speed is important if we want to explain the large observed trading volume due to HFT.

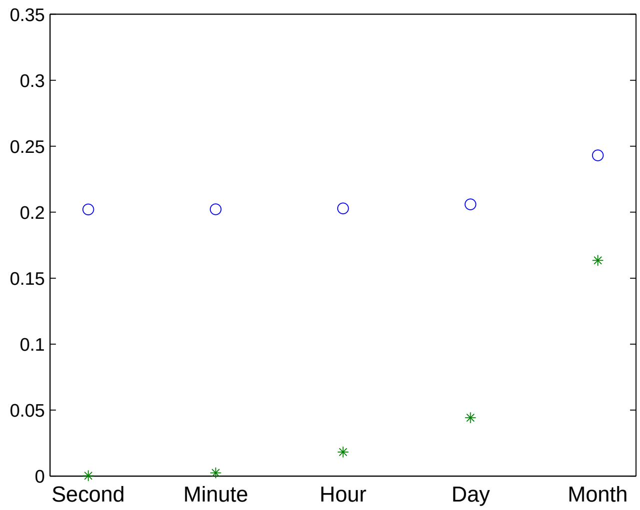
Why would a small speed advantage for the informed trader translate into such a large different in outcomes? For this, we need to understand the difference in optimal strategies of the informed trader in the two models. In principle, when the asset value changes over time, there are two components of the optimal strategy:

(1) *Level Trading* (or the low-frequency, drift, or deterministic component). This is a multiple of the difference between the asset value and the price, and changes slowly over time. Also, it is proportional to the time interval between two trades, thus it is small relative to the other component.

[5]The benchmark model is similar to that of Back and Pedersen (1998), except that in our model the dealer also receives news about the asset value.

[6]In our benchmark model, as in Kyle (1985), there is a single informed trader. We have checked that the pattern shown in the figure can be obtained in models with multiple informed traders, such as Back, Cao, and Willard (2000).

**Figure 1: Informed participation rate at various trading frequencies.** The figure plots the fraction of the trading volume due to the informed trader in a discrete time model for various lengths of time between trading periods (second, minute, hour, day, month) in (a) the benchmark model, marked with "$*$"; and (b) the fast model, marked with "$\circ$". The parameters used are $\sigma_u = \sigma_v = \sigma_e = \Sigma_0 = 1$ (see Theorem 1). The liquidation date $t = 1$ corresponds to 10 calendar years.




(2) *Flow Trading* (or the high-frequency, volatility, or stochastic component). This is a multiple of the new signal, i.e., the innovation in asset value, and changes every instant. This component is relatively much larger than the level trading component.

With no asymmetry in speed, the informed trader in the benchmark model does not have any incentive to trade on the asset value innovation, and trades only on the level of the asset value: the price impact of flow trading would otherwise be too high. By contrast, in the fast model the informed trader also engages in flow trading, in anticipation of a price move in the next instant due to the incorporation of news by the public.

These two components of the optimal strategy of the informed trader drive all the comparisons between the benchmark model and the fast model. To begin with, trading volume is higher in the fast model: in addition to the noise trading which is assumed the same in the two models, there is the large flow trading component from the informed trader (the level component is too small to matter at high frequencies). As observed in Figure 1, the fraction of trading volume due to the informed trader is much larger at high frequencies, due to the large flow trading component.

Liquidity is smaller in the fast model: besides the usual adverse selection coming from the superior precision of the informed trader, anticipatory trading generates additional adverse selection.

The comparison of price informativeness in the two models is more subtle. In the fast model, trades are more correlated with current innovations in asset value because of the flow trading component. Therefore, price changes are also more correlated with innovations in asset value. However, the variance of the pricing error is the same in both models. The reason is that there is a substitution between level trading and flow trading: there is flow trading in the fast model, but level trading is less intense than in the benchmark model. Therefore, in the fast model, trades are more correlated with current innovations in asset value, but also less correlated with past innovations. These two effects exactly offset and leave the variance of the pricing error identical in both models.

The effect of fast trading on price volatility is similarly complex. Price volatility arises from both trading and quote revisions, since the dealer also learns about the asset value and updates quotes. In the fast model, the contribution of trades to price volatility is larger, because of the volatile flow trading component of informed trading. The flip side is that when the market maker receives information, part of it has already been revealed through trading. Therefore, quote revisions are of a smaller magnitude, and price volatility unrelated to trading is lower in the fast model. These two effect on volatility exactly offset each other so that total price volatility is the same in both models and equal to the volatility of the asset value.

In the benchmark model, the informed order flow is autocorrelated: there is only the level trading component, which changes direction only very slowly over time. In the fast model, the informed order flow has zero autocorrelation: at high frequencies, flow trading dominates level trading, and the innovations in asset value are uncorrelated. Our results suggest that the fast model is better suited than the benchmark model to describe the strategies of high frequency traders: Brogaard (2011) observes that their order flow is indeed volatile, and there is little evidence of autocorrelation.

To discuss empirical implications of our paper, we start by arguing that the informed trader of the fast model fits certain stylized facts about high frequency traders: (i) large trading volume: the informed investor in the fast model trades in large quantities, while in the benchmark model informed trading volume is essentially zero at high frequencies; (ii) low order flow autocorrelation: the fast informed investor's trades have low serial correlation, compared to a large autocorrelation in the benchmark model; (iii) anticipatory trading: the order flow of the fast investor has a significant correlation with current price changes, compared to a low correlation in the benchmark model.

We stress that our model applies to the specific category of high frequency traders who engage in flow trading, but not necessarily to other types of high-frequency trading strategies such as high-frequency market making.[7] Recognizing this distinction is important for testing the predictions of our model.

We have two types of empirical predictions: (i) the effect of HFNT on various market outcomes; and (ii) the effect of various market characteristics on HFNT activity. For (i), we analyze the causal effect of HFNT by comparing the equilibrium outcomes when one moves from the benchmark model to the fast model. In the fast model, the informed trader is able to access information before the public does. This can occur, for example, by purchasing access to various high frequency news feeds, by co-location services offered by the exchange, by increasing automation, etc. The converse move from the fast model to the benchmark model is also of interest: it can represent, e.g., the effect of regulation aimed at dampening high frequency trading. From the discussion above, we see that eliminating the speed advantage of the informed trader (a) reduces trading volume; (b) reduces overall adverse selection, and thus increases market liquidity.

The second type of empirical prediction can be obtained in the context of the fast model, by studying the effect of various parameters on informed trading activity. For example, we find that an increase in the precision of public news increases the amount of flow trading, yet improves liquidity. To understand why, recall that flow trading arises because the informed trader is willing to trade based on his signal just before the market maker updates the quotes based on a correlated signal. The more precise

[7]See Jovanovic and Menkveld (2011) for a theoretical and empirical analysis of liquidity provision by fast traders.

the public news, the higher the correlation between the informed trader's signal and the market maker's signal. This increases the benefits of trading in anticipation of the quotes updates. Therefore, flow trading increases.[8] At the same time, more public news also improves liquidity. The reason is simple: having more precise public news reduces adverse selection. Interestingly, it implies that if the amount of public news changes (over time or across securities) then flow trading and liquidity move in the same direction. This is not because flow trading improves liquidity; indeed, we saw that the opposite is true when the informed trader acquires a speed advantage. Instead, this is because more public news increases *both* flow trading and liquidity.

Another example is the effect of price volatility. Holding constant the relative precision of public news, an increase in price volatility can be modeled as an increase in the volatility of the innovation of the asset value. Then, an increase in price volatility causes both an increase in flow trading activity, and a reduction in liquidity. The intuition is straightforward. When the asset is more volatile, the anticipation effect is stronger, and thus the flow trading increases. Because flow trading is more intense, there is more adverse selection due to speed, and liquidity is negatively affected.

Our paper is part of a growing theoretical literature on trading and speed. Biais, Foucault, and Moinas (2011) analyze the welfare implications of the speed advantage of HFTs in a 3-period model: HFTs raise trading volume and gains from trade, but increase adverse selection. In a search model with symmetric information, Pagnotta and Philippon (2011) show that trading platforms seeking to attract order flow have an incentive to relax price competition by differentiating along the speed dimension. Previously, the market microstructure literature has focused on the precision component of adverse selection, e.g., Kyle (1985), Back, Cao, and Willard (2000). In all these models, the behavior of the informed traders is similar to that of the informed trader in our benchmark model. In fact, we can describe our benchmark model as a mixture of Back and Pedersen (1998) and Chau and Vayanos (2008). From Back and Pedersen (1998) our benchmark model borrows the moving asset value; and from Chau and Vayanos

[8]This prediction can be tested in the cross-section of securities, if one has a proxy for the amount of public news that is released over time. It can also be tested in the time-series of a specific security, if there is time-variation in the amount of public news.

(2008) it borrows the periodic release of public information. In neither of these models the informed trader has a speed advantage. Our fast model contributes to the literature by showing that even an infinitesimal speed advantage for the informed trader results in a large difference in outcomes, e.g., speed causes a large participation rate of the informed trader, and an uncorrelated informed order flow.

The paper is organized as follows. Section 2 describes our two models: the benchmark model, and the fast model. The models are set in continuous time, but in Appendix A we present the corresponding discrete versions. Section 3 describes the resulting equilibrium price process and trading strategies, and compares the various coefficients involved. Section 4 discusses empirical implications of the model. Section 5 concludes.

# 2 Model

Trading occurs over the time interval $[0,1]$. The risk-free rate is taken to be zero. During $[0,1]$, a single informed trader ("he") and uninformed noise traders submit market orders to a competitive market maker ("she"), who sets the price at which the trading takes place. There is a risky asset with liquidation value $v_1$ at time 1. The informed trader learns about $v_1$ over time, and the expectation of $v_1$ conditional on his information available until time $t$ follows a Gaussian process given by

$$v_t \;=\; v_0 + \int_0^t \mathrm{d}v_\tau, \quad \text{with} \quad \mathrm{d}v_t \;=\; \sigma_v \, \mathrm{d}B_t^v, \tag{1}$$

where $v_0$ is normally distributed with mean 0 and variance $\Sigma_0$, and $B_t^v$ is a Brownian motion.[9] We refer to $v_t$ as the asset value or the fundamental value, and to $\mathrm{d}v_t$ as the innovation in asset value. Thus, the informed trader observes $v_0$ at time 0 and, at each

[9]This assumption can be justified economically as follows. First, define the asset value $v_t$ as the full information price of the asset, i.e., the price that would prevail at $t$ if all information until $t$ were to become public. Then, assume that (i) $v_t$ is a martingale (true, if the market is efficient), and (ii) $v_t$ is continuous (technically, it has continuous sample paths). Then, $v_t$ can be represented as an Itô integral with respect to a Brownian motion, by the Martingale Representation Theorem (see, e.g., Karatzas and Shreve (1991, Theorem 3.4.2)); our representation (1) is a simple particular case, with zero drift and constant volatility. But, even if $v_t$ has jumps (e.g., at Poisson-distributed random times), we conjecture that our key result of a non-zero $\mathrm{d}v_t$ component in the optimal trading strategy of the informed trader stays the same.

time $t + \mathrm{d}t \in [0, 1]$ observes $\mathrm{d}v_t$.

The aggregate position of the informed trader at $t$ is denoted by $x_t$. The informed trader is risk-neutral and chooses $x_t$ to maximize expected utility at $t = 0$ given by

$$U_0 \; = \; \mathsf{E}\left[\int_0^1 (v_1 - p_{t+\mathrm{d}t})\,\mathrm{d}x_t\right] \; = \; \mathsf{E}\left[\int_0^1 (v_1 - p_t - \,\mathrm{d}p_t)\,\mathrm{d}x_t\right], \tag{2}$$

where $p_{t+\mathrm{d}t} = p_t + \mathrm{d}p_t$ is the price at which the order $\mathrm{d}x_t$ is executed.[10]

The aggregate position of the noise traders at $t$ is denoted by $u_t$, which is an exogenous Gaussian process given by

$$u_t \; = \; u_0 + \int_0^t \mathrm{d}u_\tau, \quad \text{with} \quad \mathrm{d}u_t \; = \; \sigma_v \,\mathrm{d}B_t^u, \tag{3}$$

where $B_t^u$ is a Brownian motion independent from $B_t^v$.

The market maker also learns about the asset value. At $t + \mathrm{d}t$, she receives a noisy signal of the innovation in asset value:

$$\mathrm{d}z_t \; = \; \mathrm{d}v_t + \,\mathrm{d}e_t, \quad \text{with} \quad \mathrm{d}e_t \; = \; \sigma_e \,\mathrm{d}B_t^e, \tag{4}$$

where $B_t^e$ is a Brownian motion independent from all the others. She does not observe the individual orders, but only the aggregate order flow

$$\mathrm{d}y_t \; = \; \mathrm{d}u_t + \,\mathrm{d}x_t. \tag{5}$$

Because the market maker is competitive and risk-neutral, at any time the price equals the conditional expectation of $v_1$ given the information available to her until that point. In the following, we will refer to the conditional expectation of $v_1$ just before trading takes place at time $t+\mathrm{d}t$ as the quote, and we denote it by $q_t$. One possible interpretation for the quote $q_t$ is that it is the bid-ask midpoint in a limit order book with zero tick size and zero bid-ask spread.[11] The conditional expectation of $v_1$ just after trading takes

[10]Because the optimal trading strategy of the informed trader might have a stochastic component, we cannot set $\mathsf{E}(\mathrm{d}p_t\mathrm{d}x_t) = 0$ as, e.g., in the Kyle (1985) model.

[11]This interpretation is correct if the price impact is increasing in the signed order flow and a zero order flow has zero price impact. These conditions are satisfied in the linear equilibrium we consider in

place at time $t + \mathrm{d}t$ is the execution price and is denoted by $p_{t+\mathrm{d}t}$.

We consider two different models: the *benchmark model* and the *fast model*, which differ according to the timing of information arrival and trading. A simplified timing of each model is presented in Table 1. Figure 2 shows the exact sequence of quotes and prices in each model.

**Table 1: Timing of events during $[t, t + \mathrm{d}t]$ in the benchmark model and in the fast model**

| Benchmark Model | Fast Model |
|---|---|
| 1. Informed trader observes $\mathrm{d}v_t$ | 1. Informed trader observes $\mathrm{d}v_t$ |
| 2. Market maker observes $\mathrm{d}z_t = \mathrm{d}v_t + \mathrm{d}e_t$ | 2. Trading |
| 3. Trading | 3. Market maker observes $\mathrm{d}z_t = \mathrm{d}v_t + \mathrm{d}e_t$ |

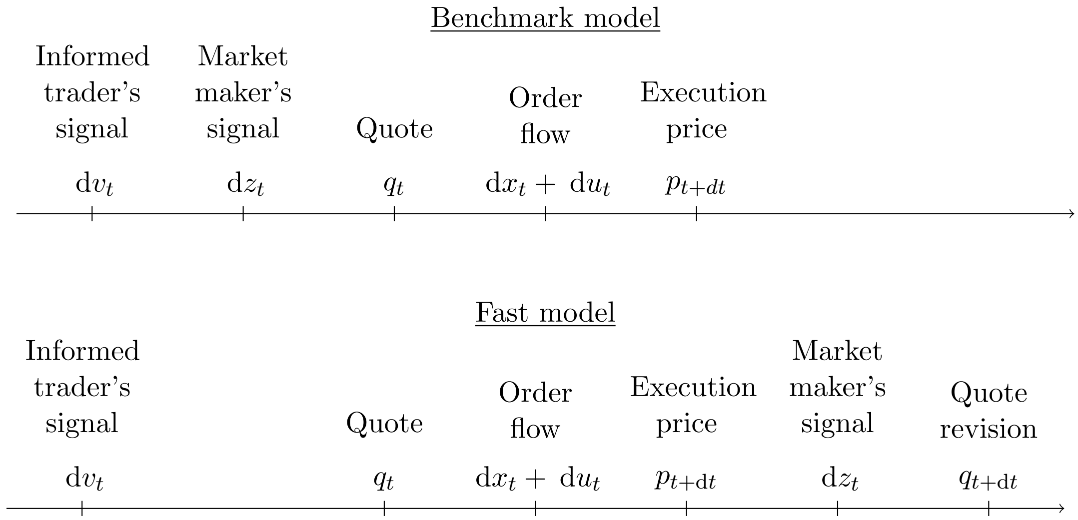
In the benchmark model, the order of events during the time interval $[t, t + \mathrm{d}t]$ is as follows. First, the informed trader observes $\mathrm{d}v_t$ and the market maker receives the signal $\mathrm{d}z_t$. The market maker sets the quote $q_t$ based on the information set $\mathcal{I}_t \cup \mathrm{d}z_t$, where $\mathcal{I}_t \equiv \{z_\tau\}_{\tau \le t} \cup \{y_\tau\}_{\tau \le t}$. The information set includes the order flow and the market maker's signal until time $t$, as well as the new signal $\mathrm{d}z_t$. Then, the informed trader submits a market order $\mathrm{d}x_t$ and noise traders also submit their order $\mathrm{d}u_t$. The information set of the market maker when she sets the execution price $p_{t+\mathrm{d}t}$ is $\mathcal{I}_t \cup \mathrm{d}z_t \cup \mathrm{d}y_t$ as it includes the new order flow.

In the fast model, the informed trader can move faster than the market maker. First, the informed trader observes $\mathrm{d}v_t$. Then, the market maker posts quotes before she observes her own signal. Therefore, the quote $q_t$ is based on the information set $\mathcal{I}_t$. The informed trader submits the market order $\mathrm{d}x_t$ along with the noise traders' orders $\mathrm{d}u_t$. The execution price $p_{t+\mathrm{d}t}$ is conditional on the information set $\mathcal{I}_t \cup \mathrm{d}y_t$. After trading has taken place, the market maker receives the signal $\mathrm{d}z_t$ and updates the quotes based on the information set $\mathcal{I}_t \cup \mathrm{d}z_t \cup \mathrm{d}y_t$. The new quote $q_{t+\mathrm{d}t}$ will be the prevalent quote in the next trading round.

The benchmark model is similar to models of the Kyle (1985) type. Formally, the

Section 3.

**Figure 2: Timing of events**




benchmark model is an extension of Back and Pedersen (1998) with the additional assumption that the market maker also learns about $\mathrm{d}v_t$. In all these versions of the Kyle model, the informed trader has more precise information than the market maker, but no speed advantage. By contrast, in the fast model, the informed trader has a speed advantage in addition to more precise information.

# 3 Equilibrium

The equilibrium concept is similar to that of Kyle (1985) or Back and Pedersen (1998). We look for linear equilibria defined as follows.

In the benchmark model, we look for an equilibrium in which the quote revision is linear in the market maker's signal

$$q_t \;=\; p_t + \mu_t \,\mathrm{d}z_t, \tag{6}$$

and the price impact is linear in the order flow

$$p_{t+\mathrm{d}t} \;=\; q_t + \lambda_t \,\mathrm{d}y_t. \tag{7}$$

In the fast model, we look for an equilibrium in which the price impact is linear in the order flow as in equation (7), and the subsequent quote revision is linear in the unexpected part of the market maker's signal[12]

$$q_{t+\mathrm{d}t} = p_{t+\mathrm{d}t} + \mu_t(\,\mathrm{d}z_t - \rho_t\,\mathrm{d}y_t). \tag{8}$$

In both models, we look for a strategy of the informed trader of the form

$$\mathrm{d}x_t = \beta_t(v_t - p_t)\,\mathrm{d}t + \gamma_t\,\mathrm{d}v_t, \tag{9}$$

i.e., we solve for $\beta_t$ and $\gamma_t$ so that the strategy defined in equation (9) maximizes the informed trader's expected profit (2). In Appendix A, we use the discrete time version of both models to show that, as long as the equilibrium has a linear pricing rule, the optimal strategy of the informed trader has the same form as in (9).[13]

In what follows, we refer to the first term of trading strategy, $\beta_t(v_t - p_t)\,\mathrm{d}t$, as *level trading*, as it consists in trading on the difference between the level of the asset value and the price level. This term appears in essentially all models of trading of the Kyle (1985) type, such as Back and Pedersen (1998), Back, Cao, and Willard (2000), etc. The second term of the trading strategy, $\gamma_t\,\mathrm{d}v_t$, consists in trading on the innovation of the asset value, and we call it *flow trading*. The next result shows that flow trading is zero in the benchmark model, but nonzero in the fast model.

**Theorem 1.** *In the benchmark model there is a unique linear equilibrium:*

$$\mathrm{d}x_t = \beta_t^B(v_t - p_t)\,\mathrm{d}t + \gamma_t^B\,\mathrm{d}v_t, \tag{10}$$

$$\mathrm{d}p_t = \mu_t^B\,\mathrm{d}z_t + \lambda_t^B\,\mathrm{d}y_t, \tag{11}$$

[12] In the fast model, the market maker's signal $\mathrm{d}z_t$ is predictable from the order flow $\mathrm{d}y_t$, thus the quote update is done only using the unexpected part of $\mathrm{d}z_t$.

[13] In fact, in discrete time the optimal strategy has $q_t$ instead of $p_t$. But because the difference between $p_t$ and $q_t$ is infinitesimal, the difference vanishes in continuous time when multiplying by $\mathrm{d}t$. In the proof of Theorem 1, we use $p_t$ for the benchmark model, and $q_t$ for the fast model, since these are well defined Itô processes with the same type of increment, $\lambda_t\,\mathrm{d}y_t + \mu_t(\,\mathrm{d}z_t - \rho_t\,\mathrm{d}y_t)$.

*with coefficients given by*

$$\beta_t^B = \frac{1}{1-t}\,\frac{\sigma_u}{\Sigma_0^{1/2}}\left(1+\frac{\sigma_v^2\sigma_e^2}{\Sigma_0(\sigma_v^2+\sigma_e^2)}\right)^{1/2}, \tag{12}$$

$$\gamma_t^B = 0, \tag{13}$$

$$\lambda_t^B = \frac{\Sigma_0^{1/2}}{\sigma_u}\left(1+\frac{\sigma_v^2\sigma_e^2}{\Sigma_0(\sigma_v^2+\sigma_e^2)}\right)^{1/2}, \tag{14}$$

$$\mu_t^B = \frac{\sigma_v^2}{\sigma_v^2+\sigma_e^2}. \tag{15}$$

*In the fast model there is a unique linear equilibrium:*[14]

$$\mathrm{d}x_t = \beta_t^F(v_t-q_t)\,\mathrm{d}t+\gamma_t^F\,\mathrm{d}v_t, \tag{16}$$

$$\mathrm{d}q_t = \lambda_t^F\,\mathrm{d}y_t+\mu_t^F(\,\mathrm{d}z_t-\rho_t^F\,\mathrm{d}y_t), \tag{17}$$

*with coefficients given by*

$$\beta_t^F = \frac{1}{1-t}\,\frac{\sigma_u}{(\Sigma_0+\sigma_v^2)^{1/2}}\,\frac{1}{\big(1+\frac{\sigma_e^2}{\sigma_v^2}f\big)^{1/2}}\left(1+\frac{(1-f)\sigma_v^2}{\Sigma_0}\,\frac{1+\frac{\sigma_e^2}{\sigma_v^2}+\frac{\sigma_e^2}{\sigma_v^2}f}{2+\frac{\sigma_e^2}{\sigma_v^2}+\frac{\sigma_e^2}{\sigma_v^2}f}\right), \tag{18}$$

$$\gamma_t^F = \frac{\sigma_u}{\sigma_v}\,f^{1/2} = \frac{\sigma_u}{(\Sigma_0+\sigma_v^2)^{1/2}}\,\frac{\big(1+\frac{\sigma_e^2}{\sigma_v^2}f\big)^{1/2}(1+f)}{2+\frac{\sigma_e^2}{\sigma_v^2}+\frac{\sigma_e^2}{\sigma_v^2}f}, \tag{19}$$

$$\lambda_t^F = \frac{(\Sigma_0+\sigma_v^2)^{1/2}}{\sigma_u}\,\frac{1}{\big(1+\frac{\sigma_e^2}{\sigma_v^2}f\big)^{1/2}(1+f)}, \tag{20}$$

$$\mu_t^F = \frac{1+f}{2+\frac{\sigma_e^2}{\sigma_v^2}+\frac{\sigma_e^2}{\sigma_v^2}f}, \tag{21}$$

$$\rho_t^F = \frac{\sigma_v^2}{\sigma_u(\Sigma_0+\sigma_v^2)^{1/2}}\,\frac{(1+\frac{\sigma_e^2}{\sigma_v^2}f)^{1/2}}{2+\frac{\sigma_e^2}{\sigma_v^2}+\frac{\sigma_e^2}{\sigma_v^2}f}, \tag{22}$$

*and $f$ is the unique root in $(0,1)$ of the cubic equation*

$$f = \frac{\big(1+\frac{\sigma_e^2}{\sigma_v^2}f\big)(1+f)^2}{\big(2+\frac{\sigma_e^2}{\sigma_v^2}+\frac{\sigma_e^2}{\sigma_v^2}f\big)^2}\,\frac{\sigma_v^2}{\sigma_v^2+\Sigma_0}. \tag{23}$$

[14]Note that the level trading component in (16) has $q_t$ instead of $p_t$. This is the same formula, since (8) implies $(p_t-q_t)\,\mathrm{d}t=0$. We use $q_t$ as a state variable, because it is a well defined Itô process.

*In both models, when $\sigma_v \to 0$, the equilibrium converges to the unique linear equilibrium in the continuous time version of Kyle (1985).*

*Proof.* See Appendix B. □

To give some intuition for the theorem, note that in both models the optimal strategy of the informed trader has a non-zero level trading component. This is because in both models the informed trader receives more precise signals than the market maker: the informed trader knows $v_t$ exactly, while the market maker's best forecast is $p_t$. Therefore, it is optimal for the informed trader to trade on the forecast error of the market maker $v_t - p_t$. This forecast error is slowly moving, because its change is of the order of $(\mathrm{d}v_t - \mathrm{d}p_t)\,\mathrm{d}t$, which at high frequencies is negligible. The informed trader trades smoothly on the forecast error, in the sense that the level trading component is of the order $\mathrm{d}t$.

The key difference between the two models is that only in the fast model the informed trader's optimal strategy has a flow trading component. The reason is that in the benchmark model, when the trader submits the order $\mathrm{d}x_t$, all the signals $\{\,\mathrm{d}v_\tau\}_{\tau \le t}$ that he has received are given the same weight in the optimal strategy. By contrast, in the fast model the marker maker has not incorporated the signal $\mathrm{d}z_t = \mathrm{d}v_t + \mathrm{d}e_t$ in the price yet. Therefore, it is optimal for the informed trader to trade aggressively on $\mathrm{d}v_t$ before the market maker receives information $\mathrm{d}z_t$.

The flow trading component is volatile since it is an innovation in a random walk process. It also generates a much larger order flow than the level trading component, because it is of order $\mathrm{d}t^{1/2}$.

We give some comparative statics for the coefficients from Theorem 1.

**Proposition 1.** *In the context of Theorem 1, for all values of the parameters we have the following inequalities:*

$$\beta_0^F < \beta_0^B \tag{24}$$

$$\lambda^F > \lambda^B \tag{25}$$

$$\mu^F < \mu^B. \tag{26}$$

*In both the benchmark equilibrium and the fast equilibrium,*

$\beta_0$ *increases in* $\sigma_v$, $\sigma_u$, $\sigma_e$*; and decreases in* $\Sigma_0$*;*

$\lambda$ *increases in* $\sigma_v$, $\sigma_e$, $\Sigma_0$*; and decreases in* $\sigma_u$*;*

$\mu$ *increases in* $\sigma_v$*; decreases in* $\sigma_e$*; and is constant in* $\sigma_u$*;*

$\mu$ *is constant in* $\Sigma_0$ *in the benchmark, but decreases in* $\Sigma_0$ *in the fast equilibrium.*

*Additionally, in the fast equilibrium,*

$\gamma$ *increases in* $\sigma_v$, $\sigma_u$*; and decreases in* $\sigma_e$, $\Sigma_0$*;*

$\rho$ *increases in* $\sigma_v$*; and decreases in* $\sigma_u$, $\sigma_e$, $\Sigma_0$.

*Proof.* See Appendix B. □

The intuition for some of these comparative statics is discussed in the next section.

# 4 Empirical Implications

## 4.1 High Frequency News Trading

In this section we argue that the informed trader of the fast model shares some of the characteristics that are attributed to the broad category of High Frequency Traders (HFTs). Specifically, we show that the informed trader (i) is responsible for a large fraction of the order flow; (ii) his order flow exhibits low serial correlation; and (iii) he engages in anticipatory trading. This is not to say that our model can be applied to study all types of HFTs. Indeed, the spectrum of strategies which can be classified under the umbrella of high frequency trading is quite large.[15] Our paper focuses on one of these strategies, namely, high frequency news trading (HFNT). Therefore, the empirical predictions and policy implications of our model apply to HFNT, but not necessarily to other categories of HFT.

[15] For instance, SEC (2010) places high frequency trading under four categories: (a) Passive Market Making, which generates large volumes by submitting and canceling many limit orders; (b) Arbitrage, which is based on correlation strategies (statistical arbitrage, pairs trading, index arbitrage, etc.); (c) Structural, which involves identifying and exploiting other market participants that are slow; and (d) Directional, which implies taking significant, unhedged positions based on anticipation of intraday price movements.

First, we define the *Informed Participation Rate (IPR)* as the contribution of the informed trader to total order flow

$$IPR_t \;=\; \frac{\mathsf{Var}(\,\mathrm{d}x_t)}{\mathsf{Var}(\,\mathrm{d}y_t)} \;=\; \frac{\mathsf{Var}(\,\mathrm{d}x_t)}{\mathsf{Var}(\,\mathrm{d}u_t) + \mathsf{Var}(\,\mathrm{d}x_t)}. \tag{27}$$

**Proposition 2.** *The informed participation rate is zero in the benchmark while it is positive in the fast model,*

$$IPR_t^B \;=\; 0, \qquad IPR_t^F \;=\; \frac{f}{1+f}, \tag{28}$$

*where $f$ is defined in Theorem 1.*

*Proof.* See Appendix B. □

In the benchmark model, the informed trader's optimal strategy has only a level trading component. The level trading component consists in a drift in the asset holding $x_t$. This generates a trading volume that is an order of magnitude smaller than the trading volume generated by the noise traders. Formally, informed trading volume is of the order $\mathrm{d}t$ while noise trading volume is of the order $\mathrm{d}t^{1/2}$. By contrast, in the fast model, the informed trader's optimal strategy includes a flow trading component. The flow trading component is volatile (i.e., stochastic), which generates a trading volume that is of the same order of magnitude as the noise trading volume.

In the discrete time version of the model, informed trading volume is non zero but it converges quickly to zero as the trading frequency increases. In Figure 1 in the Introduction, we have already seen that in the benchmark model the trading volume is already small when trading takes place at the daily frequency. In the fast model, informed trading volume converges to a limit between zero and one when the trading frequency becomes very large.

Next, we consider the serial correlation of the informed order flow.

**Proposition 3.** *The autocorrelation of the informed order flow is positive in the bench-*

*mark while it is zero in the fast model: for $\tau > 0$,*

$$\mathsf{Corr}(\,\mathrm{d}x_t^B,\,\mathrm{d}x_{t+\tau}^B) = \left(\frac{1-t-\tau}{1-t}\right)^{\frac{1}{2}+\lambda^B\beta_0^B} > 0, \tag{29}$$

$$\mathsf{Corr}(\,\mathrm{d}x_t^F,\,\mathrm{d}x_{t+\tau}^F) = 0. \tag{30}$$

*Proof.* See Appendix B. □

The intuition behind Proposition 3 is that the level trading component is slowly moving, i.e., it is serially correlated. This explains why the informed order flow is positively autocorrelated in the benchmark model. By contrast, the flow trading component is not serial correlated as it only depends on the current innovation in the asset value. Since the flow trading component generates a much larger order flow than the level component, the autocorrelation of the informed order flow is zero in the fast model. Note that the fact that the autocorrelation is exactly zero may depend on the specific assumptions of the model, e.g., the informed trader has no inventory cost. The more robust result related to Proposition 3 is that the serial correlation of the informed order flow is lower in the fast model than in the benchmark.

The empirical evidence about HFT order flow autocorrelation is mixed. For instance, using US stock trading data aggregated across all HFTs, Brogaard (2011) and Hendershott and Riordan (2011) find a positive autocorrelation of the aggregate HFT order flow. By contrast, Menkveld (2011) using data on a single HFT on the European stock market, and Kirilenko, Kyle, Samadi, and Tuzun (2011) using data on the Flash Crash of May 2010, find clear evidence of mean reverting inventories. These opposite results reflect the fact that HFT strategies are diverse and may come in a wide variety of autocorrelation patterns.[16] Empirical studies which consider HFTs as a whole measures the average autocorrelation across all types of HFT strategies, and HFNT is only one of those.

Finally, we measure *Anticipatory Trading (AT)* by the correlation between the in-

[16]In addition, the definition of HFTs can introduce a bias. For instance, in Brogaard (2011), Hendershott and Riordan (2011), and Kirilenko et al. (2011), one of the criteria to classify a trader as HFT is that it keeps its inventories close to zero.

formed order flow and the next instant return,

$$AT_t \;=\; \mathsf{Corr}(\,\mathrm{d}x_t, q_{t+\,\mathrm{d}t} - p_{t+\,\mathrm{d}t}), \tag{31}$$

where we recall that $p_{t+\,\mathrm{d}t}$ is the price at which the order flow $\mathrm{d}x_t$ is executed, and $q_{t+\,\mathrm{d}t}$ is the next quote revision that takes place when the market maker receives her next signal.

**Proposition 4.** *Anticipatory trading is zero in the benchmark while it is positive in the fast model*

$$AT_t^B \;=\; 0, \qquad AT_t^F \;=\; \frac{(1-\rho^F\gamma^F)\sigma_v}{\sqrt{(1-\rho^F\gamma^F)^2\sigma_v^2+\sigma_e^2+(\rho^F)^2\sigma_u^2}} > 0. \tag{32}$$

There is anticipatory trading in the fast model because the flow trading component of the strategy anticipates the very next quote revision. This is consistent with Kirilenko et al. (2011) and Hendershott and Riordan (2011) who find that, on average over all categories of high frequency trading strategies, HFTs' aggressive orders are correlated with future price changes at a short horizon.

## 4.2 The Effect of High Frequency News Trading

In this section we study the effect of HFNT on several market outcomes: liquidity, price discovery, price volatility, and price impact. To do that, we compare the equilibrium of the market when one moves from the benchmark to the fast model. Indeed, in the fast model, the informed trader is able to access information and trade based on it quickly, that is, before the information is incorporated into quotes. In practice, this can occur because the exchange increases automation, offers co-location services, or implements any other change that lowers the execution time for market orders. Alternatively, one can view a shift from the fast model to the benchmark model as the result of a move by the regulator or the trading platform to dampen HFNT.

We already proved the following result in Proposition 1:

**Corollary 1.** *Liquidity is lower in the fast model than in the benchmark:* $\lambda^F > \lambda^B$.

The market is less liquid in the fast model since there is more adverse selection than in the benchmark model. Indeed, the informed trader has more precise information in both models, and, in the fast model only, the informed is also faster. This generates a second source of adverse selection, coming from speed.

Previous empirical work has investigated the effect of high frequency trading in general on liquidity. Some find evidence of a positive (e.g., Hendershott, Jones, and Menkveld (2011), Hasbrouck and Saar (2010)) while others find the opposite (e.g., Hendershott and Moulton (2011)). These papers have considered high frequency traders as a group, and have therefore measured their average impact across the entire spectrum of HFT strategies. We predict that HFNT reduces liquidity, but it may be the case that high frequency market making improves the liquidity, and that the overall effect on liquidity is positive.

Another measure of the price impact of trades is the *Cumulative Price Impact (CPI)* defined as the covariance between the informed order flow trade per unit of time at $t$ and the subsequent price change over the time interval $[t, t+\tau]$ for $\tau > 0$:[17]

$$CPI_t(\tau) \;=\; \mathsf{Cov}\left(\frac{\mathrm{d}x_t}{\mathrm{d}t}, p_{t+\tau} - p_t\right). \tag{33}$$

Because the optimal strategy of the informed trader is of the type $\mathrm{d}x_t = \beta_t(v_t - p_t)\,\mathrm{d}t + \gamma_t\,\mathrm{d}v_t$, the cumulative price impact can be decomposed into two terms:

$$CPI_t(\tau) \;=\; \beta_t\,\mathsf{Cov}(v_t - p_t, p_{t+\tau} - p_t) + \frac{1}{\mathrm{d}t}\gamma_t\,\mathsf{Cov}(\,\mathrm{d}v_t, p_{t+\tau} - p_t), \tag{34}$$

and note that the second term is well defined, because $\mathsf{Cov}(\,\mathrm{d}v_t, p_{t+\tau} - p_t)$ is of the order of $\mathrm{d}t$, since the asset value, $v_t$, is a Gaussian process.

**Proposition 5.** *In the benchmark model, the cumulative price impact is*

$$CPI_t^B(\tau) \;=\; k_1^B\left[1 - \left(1 - \frac{\tau}{1-t}\right)^{\lambda^B\beta_0^B}\right], \tag{35}$$

[17]Using $p_t$ or $q_t$ in the definition of $CPI_t(\tau)$ is equivalent because the difference between the two is smaller than $p_{t+\tau} - p_t$ by an order of magnitude.

*while in the fast model it is*

$$CPI_t^F(\tau) \;=\; k_0^F + k_1^F \left[ 1 - \left( 1 - \frac{\tau}{1-t} \right)^{(\lambda^F - \mu^F \rho^F)\beta_0^F} \right], \tag{36}$$

*where*

$$k_1^B \;=\; \beta_0^B \Sigma_0, \tag{37}$$

$$k_0^F \;=\; \gamma^F ((\lambda^F - \mu^F \rho^F)\gamma^F + \mu^F)\sigma_v^2, \tag{38}$$

$$k_1^F \;=\; \beta_0^F \Sigma_0 + \gamma^F (1 - (\lambda^F - \mu^F \rho^F)\gamma^F - \mu^F)\sigma_v^2. \tag{39}$$

*Proof.* See Appendix B. □

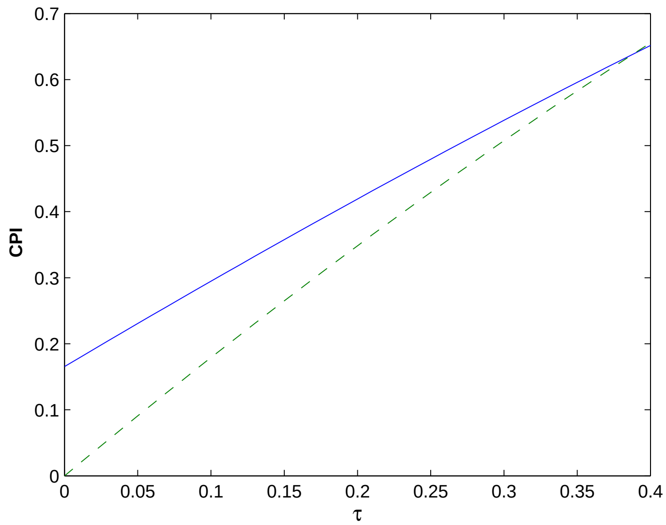
One can see from the formulas, or from Figure 3, that in the benchmark model the cumulative price impact starts from near-zero values when $\tau$ is very small, while in the fast model it starts from a positive value, $k_0^F$. Then, the cumulative price builds up over time in both models, because the level trading component is correlated with all prices changes in the future. To sum up, the intercept in Figure 3 is evidence of flow trading, while the positive slope is evidence of level trading. Note that the cumulative price impact is a univariate covariance. If we want to obtain a causal impact of trades, we need to control for the future order flow. This can be done using a VAR model, as will be shown in Section 4.4.

Next, we consider the effect of HFNT on the price discovery process. We define price informativeness at any given point in time $t$ as the (squared) pricing error

$$\Sigma_t \;=\; \mathsf{E}\big( (v_t - p_t)^2 \big). \tag{40}$$

More insight can be gained by decomposing this pricing error into errors about the last change in asset value and errors about the level of the asset value. First, we note that (40) can rewritten as follows:

**Figure 3: Cumulative Price Impact at Different Horizons.** The figure plots the cumulative price impact at $t = 0$, $\mathsf{Cov}\left(\frac{\mathrm{d}x_0}{\mathrm{d}t}, p_\tau - p_0\right)$ against the horizon $\tau \in (0, 1]$ in (a) the benchmark model, with a dotted line; and (b) the fast model, with a solid line. The parameters used are $\sigma_u = \sigma_v = \sigma_e = \Sigma_0 = 1$ (see Theorem 1). The liquidation date $t = 1$ corresponds to 10 calendar years.




**Lemma 1.** *In both the benchmark and the fast models,*

$$\Sigma_t \;=\; (1+t)\Sigma_0 + 2t\sigma_v^2 - 2\int_0^t \mathsf{Cov}(\,\mathrm{d}p_\tau, v_{\tau+\,\mathrm{d}\tau}). \tag{41}$$

*Proof.* See Appendix B. □

Intuitively, if price changes are more correlated with the asset value ($\mathsf{Cov}(\mathrm{d}p_\tau, v_{\tau+\,\mathrm{d}\tau})$ is larger), the price ends up being closer on average to the asset value ($\Sigma_t$ is smaller). Moreover, we have the following decomposition:[18]

$$\mathsf{Cov}(\,\mathrm{d}p_t, v_{t+\mathrm{d}t}) \;=\; \mathsf{Cov}(\,\mathrm{d}p_t, v_t) + \mathsf{Cov}(\,\mathrm{d}p_t,\,\mathrm{d}v_t). \tag{42}$$

**Proposition 6.** $\mathsf{Cov}(\mathrm{d}p_t, \mathrm{d}v_t)$ *is higher in the fast model than in the benchmark; while* $\mathsf{Cov}(\mathrm{d}p_t, v_t)$ *is higher in the benchmark than in the fast model.* $\Sigma_t$ *is the same in both the benchmark and the fast models.*

Returns are more informative about the level of the asset value in the benchmark

[18]In this equation, $\mathrm{d}p_t$ denotes $p_{t+\mathrm{d}t} - p_t$ in the benchmark model, and $q_{t+\mathrm{d}t} - q_t$ in the fast model.

model, while they are more informative about changes in the asset value in the fast model. The reason for the latter comes from the flow trading component. In the benchmark, the contemporaneous correlation between changes in the price and in the asset value comes from quote revisions only:

$$\mathsf{Cov}(\,\mathrm{d}p_t^B,\,\mathrm{d}v_t) \;=\; \mathsf{Cov}(\mu^B\,\mathrm{d}z_t,\,\mathrm{d}v_t) \;=\; \mu^B\sigma_v^2\,\mathrm{d}t. \tag{43}$$

In the fast model, flow trading adds to this covariance:

$$\mathsf{Cov}(\mathrm{d}p_t^F,\mathrm{d}v_t) \;=\; \mathsf{Cov}(\lambda^F\mathrm{d}x_t^F+\mu^F(\mathrm{d}z_t-\rho^F\,\mathrm{d}x_t),\mathrm{d}v_t) \;=\; \big(\mu^B+(\lambda^F-\mu^F\rho^F)\big)\sigma_v^2\mathrm{d}t. \tag{44}$$

It implies that returns are more correlated with the innovations of the asset value in the fast model.

By contrast, the covariance of returns with the level of the asset value is higher in the benchmark model. The reason is that the level component of informed trading is less intense in the fast model than in the benchmark model. Indeed, there is a substitution between flow trading and level trading. The intuition for this *substitution effect* is that the informed trader competes with himself when using his information advantage. Trading more on news now consumes the profit from trading on the level in the future. Therefore, when flow trading increases in the fast model, level trading has to decrease.

In terms of total pricing error, these two effects—higher correlation of returns with changes and lower correlation with levels—exactly cancel out, and the pricing error is the same in both models. In the fast model, new information is incorporated more quickly into the price while older information is incorporated less quickly, leaving the total pricing error equal in both models. The more formal reason why these two effects exactly offset each other is that, in both the benchmark and the fast models, the informed trader finds it optimal to release information at a constant rate to minimize price impact. Therefore, $\Sigma_t$ decreases linearly over time in both models. Moreover, the transversality condition for optimization requires that no money is left on the table at $t = 1$, i.e., $\Sigma_1 = 0$. Since the initial value $\Sigma_0$ is exogenously given, the evolution of $\Sigma_t$ is the same in both models.

We now consider the effect of HFNT on price volatility. Following Hasbrouck (1991a, 1991b) we decompose price volatility into the volatility coming from trades and the volatility coming from quotes:

$$\mathsf{Var}(\mathrm{d}p_t) \;=\; \mathsf{Var}(p_{t+\,\mathrm{d}t} - q_t) + \mathsf{Var}(q_t - p_t). \tag{45}$$

The first term of the decomposition if the variance of the price impact of trades $(p_{t+\mathrm{d}t} - q_t)$. The second term of the decomposition is the variance of quote revisions unrelated to trading $(q_t - p_t)$.

**Proposition 7.** $\mathsf{Var}(p_{t+\mathrm{d}t} - q_t)$ *is higher in the fast model than in the benchmark; while* $\mathsf{Var}(q_t - p_t)$ *is higher in the benchmark than in the fast model.* $Var(\mathrm{d}p_t)$ *is the same in benchmark and in the fast models and it equals*

$$\mathsf{Var}(\,\mathrm{d}p_t) \;=\; \sigma_v^2 + \Sigma_0. \tag{46}$$

More information is incorporated through trading in the fast model. This is because the informed trader acts on the news before the market marker revises the quotes. Therefore, trading is more intense and price volatility coming from trades is higher in the fast model. The flip side is that the quote revision is less intense, and the price volatility coming from quotes is lower in the fast model.

In terms of total price volatility, these two effects cancel each other and price volatility is the same in both models. The reason why the two effects exactly offset each other is that in an efficient market price changes are not autocorrelated. Therefore, the short-term price variance per unit of time is always equal to the long-term price variance per unit of time, which is itself equal to the variance per unit of time of the (exoegenous) asset value.

## 4.3 The Determinants of High Frequency News Trading

Because we identify HFNT with the activity of the informed trader in the fast model, in this section we study the determinants of HFNT by doing comparative statics on various parameters in the fast model. We measure HFNT activity by the informed participation rate defined in Equation (27).

Consider first the effect of the precision of public news. Holding constant the variance of the innovation of the asset value $\sigma_v^2$, more precise public news about the changes in asset value amounts to a lower $\sigma_z^2 = \sigma_v^2 + \sigma_e^2$, or, equivalently, a lower $\sigma_e^2$.

**Proposition 8.** *An increase in the precision of public news, i.e., a decrease in $\sigma_e$, increases HFNT activity (increases $IPR_t^F$) and improves liquidity (decreases $\lambda^F$).*

*Proof.* By Propositon 2, $IPR^F = \frac{f}{1+f}$, thus the informed participation rate in the fast model has the same dependence on $\sigma_e$ as $f$. From (19), $\gamma^F = \frac{\sigma_u}{\sigma_v} f^{1/2}$, thus $f$ has the same dependence on $\sigma_e$ as $\gamma^F$. Therefore, $IPR^F$ has the same depdendence on $\sigma_e$ as $\gamma^F$. But Proposition 1 shows that $\gamma^F$ is decreasing in $\sigma_e$. Finally, we use again Proposition 1 to show that $\lambda^F$ is increasing in $\sigma_e$. □

The fast trader needs a precise news environment in order to take advantage of anticipatory trading. Otherwise, if the public signal is imprecise, i.e. $\sigma_e$ is large, the market maker does not adjust quotes by much ($\mu^F$ is small), the informed trader cannot benefit much from his speed advantage and does not trade intensely on the news component. This prediction can be tested in the cross-section of securities, if one has a proxy for the amount of public news that is released over time. It can also be tested in the time-series of a specific security, if there is time-variation in the amount of public news.

As stated in Proposition 8, more public news also improves liquidity because it reduces adverse selection. Interestingly, it implies that if the amount of public news changes (over time or across securities) then HFNT and liquidity move in the same direction. This is not because HFNT improves liquidity; instead, this is because more public news increases both HFNT and liquidity.

Next, we consider the effect of price volatility. From Equation (46), $\mathsf{Var}(\mathrm{d}p_t) = \sigma_v^2 + \Sigma_0$, thus we model an increase in price volatility as an increase in the variance of the innovation of the asset value, $\sigma_v^2$, while holding costant the relative precision of public news, i.e., the ratio $\sigma_e^2/\sigma_v^2$. We can prove the following result.

**Proposition 9.** *An increase in price volatility (higher $\sigma_v$ while holding $\sigma_e/\sigma_v$ constant) increases HFNT activity (increases $IPR_t^F$) and reduces liquidity (increases $\lambda^F$).*

Because the informed trader acts in anticipation of price changes, more volatility increases the intensity of flow trading, and therefore the informed participation rate ($IPR$), or HFNT activity. As a result, there is more adverse selection, and liquidity is thus negatively affected.

## 4.4 Methodological Issues in Empirical Analysis of HFNT

Our framework can be used to shed light on some methodological issues in the empirical analysis of HFNT. In order to make our model more comparable to econometric models, we consider the discrete time version of our continuous time model, as in Appendix A. It works very similarly to the continuous time model, the main difference being that the infinitesimal time interval $\mathrm{d}t$ is replaced by a real number $\Delta t > 0$. We also consider that $\Delta t$ is small and we approximate the equilibrium variables ($\beta_t$, $\gamma_t$, $\lambda_t$, $\mu_t$, $\rho_t$) in the discrete time model by their continuous time counterpart. Letting $T = \frac{1}{\Delta t}$ be the number of trading periods, time is indexed by $t = 0, 1, \ldots, T-1$. The informed order flow at time $t$ is equal to

$$\Delta x_t \;=\; \beta_t(v_t - q_t)\Delta t + \gamma_t \Delta v_t, \tag{47}$$

where $q_t$ is the quote just before the order flow arrives, and $p_{t+1}$ is the execution price.

### 4.4.1 Timing Issues in Defining Returns

There are several issues when one measures returns empirically. For instance, when returns are computed from trade to trade, the econometrician can either use the transaction price, or the mid-quote just after the trade, or the bid or the ask depending on the direction of the order flow, or the mid-quote after the next quote revision, etc. Lags in trade reporting and time aggregation of data can also impose constraints on how trade-to-trade returns are defined. To emphasize the consequence of these timing assumptions, we contrast two different definitions of returns in the context of our model.

A first option is to compute returns using the quotes just after the order is filled ("post-trade quotes"). With this assumption, the return contemporaneous to the order flow $\Delta x_t + \Delta u_t$ is $r_t = p_{t+1} - p_t$. A second possibility is to compute returns using the quotes just before the next trade takes place ("pre-trade quotes"). In this case, the return contemporaneous to the order flow $\Delta x_t + \Delta u_t$ is $r_t = q_{t+1} - q_t$.

To illustrate the implications of these two assumptions for the empirical analysis, we consider the following VAR model with $K \geq 1$ lags in the spirit of Hasbrouck (1996):[19]

$$
\begin{aligned}
r_t &= \sum_{k=1}^{K} a_k r_{t-k} + \sum_{k=0}^{K} b_k \, \mathrm{d}x_{t-k} + \sum_{k=0}^{K} c_k \, \mathrm{d}u_{t-k} + \varepsilon_t, && (48) \\
dx_t &= \sum_{k=1}^{K} d_k r_{t-k} + \sum_{k=1}^{K} e_k \, \mathrm{d}x_{t-k} + \sum_{k=1}^{K} f_k \, \mathrm{d}u_{t-k} + \eta_t, && (49) \\
du_t &= \sum_{k=1}^{K} g_k r_{t-k} + \sum_{k=1}^{K} h_k \, \mathrm{d}x_{t-k} + \sum_{k=1}^{K} i_k \, \mathrm{d}u_{t-k} + \zeta_t. && (50)
\end{aligned}
$$

We compute the coefficients of the VAR model under the two timing assumptions. To allege the notations, we now omit the superscript $^F$ when we refer to the equilibrium variables in the fast model.

**Proposition 10.** *When post-trade quotes are used:* $b_0 = c_0 = \lambda$, $b_1 = \mu(1 - \rho\gamma)/\gamma$, $c_1 = -\mu\rho$, *and all other coefficients are zero. When pre-trade quotes are used:* $b_0 = \lambda - \mu\rho + \mu/\gamma$, $c_0 = \lambda - \mu\rho$, *and all other coefficients are zero.*

*Proof.* See Appendix B. □

Depending on how returns are measured, the estimated $b_1$ may be positive or equal to zero. When returns are computed using post-trade quotes, the informed order flow is positively related to the next period return ($b_1 > 0$). The economic interpretation is that the informed trader engages in anticipatory trading. By contrast, when returns are measured using pre-trade quotes, $b_1 = 0$ because the time $t$ order flow is incorrectly considered as being contemporaneous to the subsequent quote revision $q_{t+1} - p_{t+1}$. In this case, we fail to reject the incorrect null hypothesis of no anticipatory trading. This

[19] This specification is used, e.g., by Brogaard (2010).

suggests that using the quotes immediately after trading takes place, or the price at which the last unit of the order flow is executed, may be necessary to detect anticipatory trading in the data.

### 4.4.2 Sampling Issues

It is customary to aggregate data over time. This can be due to limited data availability, or it may be a deliberate choice of the econometrician to make data analysis more manageable. In this section we look at the consequence of time aggregation and we show that the time interval at which data are aggregated affects the results of the empirical analysis. In particular, when the sampling frequency is low relative to the trading frequency, the empirical moments are biased in the sense that they differ from the theoretical moments of the model.

Assume that each observation in the data spans $n \geq 1$ trading rounds. In this case, the data are a time-series of length $T/n$. For $j = 1, \ldots, \frac{T}{n}$, the $j$th observation corresponds to trading during the $n$ trading rounds starting at time $t = (j-1)n$. The order flow of the informed trader is $\Delta x_j(n) \equiv \Delta x_t + \cdots + \Delta x_{t+n-1}$, and, assuming that prices are defined as post-trade quotes, the return is $r_j(n) \equiv p_{t+n} - p_t$.

First, we consider the measure of anticipatory trading defined in equation (31). Its empirical counterpart when data are sampled every $n$ trading rounds is

$$AT_j(n) \;=\; \mathsf{Corr}(\Delta x_j(n), r_{j+1}(n)). \tag{51}$$

**Proposition 11.** *The empirical measure of anticipatory trading $AT_j(n)$ decreases with $n$ and converges to zero when $n \to +\infty$.*

*Proof.* See Appendix B. □

The aggregated order flow spans $n$ trading periods. Moreover, each trade anticipates news that is incorporated in the quotes in the next trading round. Therefore, only the last trade of the aggregated order flow $\Delta x_j(n)$ is correlated with the next aggregated return $r_{j+1}(n)$. As a result, when $n$ increases, the correlation between $\Delta x_j(n)$ and $r_{j+1}(n)$ decreases. When $n$ becomes too large, the correlation becomes almost zero.

This result suggests that sampling data at a sufficiently high frequency is important for detecting anticipatory trading.

We now turn to the informed participation rate (27). Its empirical counterpart when data are sampled each $n$ trading periods can be defined as

$$IPR_j(n) \;=\; \frac{\mathsf{Var}(\Delta x_j(n))}{\mathsf{Var}(\Delta x_j(n)) + \mathsf{Var}(\Delta u_j(n))}. \tag{52}$$

In order to obtain closed-form formula solutions, we consider the limit case where the trading frequency is large, holding fixed the time interval $\tau = n\Delta t$ at which data are aggregated. In this case, the informed participation can therefore be written as a function of $\tau$: $IPR_j(\tau) = \lim_{\Delta t\to 0} IPR_j(\tau/\Delta t)$.

**Proposition 12.** *The empirical informed participation rate $IPR_j(\tau)$ increases with the sampling interval $\tau$.*

*Proof.* See Appendix B. □

The level trading component is positively autocorrelated over time. Therefore, the variance of the informed order flow increases faster than the time horizon $\tau$ at which the variance is computed. Since the noise trading order flow is serially uncorrelated, the fraction of the order flow variance due to the informed trader increases with the sampling interval $\tau$.

Finally, consider $\mathsf{Corr}(\Delta x_j(n), \Delta x_{j+1}(n))$, the empirical autocorrelation of the informed order flow. Again, in order to obtain closed-form formulas, we hold constant the sampling interval $\tau = n\Delta t$ and we let $\Delta t \to 0$. As a result, the autocorrelation of the informed order flow is now a function of $\tau$.

**Proposition 13.** *The informed order flow autocorrelation $\mathsf{Corr}(\Delta x_j(\tau), \Delta x_{j+1}(\tau))$ increases with $\tau$.*

*Proof.* See Appendix B. □

The level trading component of the informed order flow is positively correlated over time, while the flow trading component is not. When data are sampled at a very high frequency, the flow trading component represents a large fraction of the informed order flow variance. In this case, the autocorrelation of the informed order is therefore close to zero. By contrast, at a lower frequency, the level trading component becomes a larger part of the the variance of the informed order flow, and the autocorrelation of the informed order flow increases.

# 5 Conclusions

We have argued that adverse selection has two components: a precision component, and a speed component. To analyze the effect of speed on market quality, we have proposed two models of trading with an informed trader who continuously observes a stream of news. In the benchmark model, the informed trader learns about the asset value at the same time as the market maker. In the fast model, the informed trader has an infinitesimal speed advantage. We have shown that the difference in equilibrium outcomes between the two models is large. In particular, we have shown that in the fast model the optimal strategy of the informed trader has a flow trading component, which is an order of magnitude larger and more volatile than the level trading component.

As a consequence, in the fast model the fraction of trading volume due to the informed investor is large, while in the benchmark model this fraction is essentially zero at high frequencies. As a result of an extra component of adverse selection, liquidity is lower in the fast model, compared to the benchmark. Nevertheless, price volatility and price informativeness are the same, due to a substitution effect. In the fast model, there is more flow trading, but less level trading.

Our results are consistent with stylized facts about high frequency trading, and we generate additional predictions about (i) the causal effect of high frequency trading on various market performance measures; (ii) the effect of various determinants of high frequency trading, both in the cross section and in the time series. For example, we find that an increase in the precision of public news increases the amount of high frequency trading, yet, surprisingly, liquidity is improved.

# A Models in Discrete Time

## A.1 Discrete Time Fast Model

We divide the interval $[0, 1]$ into $T$ equally spaced intervals of length $\Delta t = \frac{1}{T}$. Trading takes place at equally spaced times, $t = 1, 2, \ldots, T-1$. The sequence of events is as follows. At $t = 0$, the informed trader observes $v_0$. At each $t = 1, \ldots, T-1$, the informed trader observes $\Delta v_t = v_t - v_{t-1}$; and the market maker observes $\Delta z_{t-1} = \Delta v_{t-1} + \Delta e_{t-1}$, except at $t = 1$. The error in the market maker's signal is normally distributed, $\Delta e_{t-1} \sim \mathcal{N}(0, \sigma_e^2 \Delta t)$. The market maker quotes the bid price = the ask price = $q_t$. The informed trader then submits $\Delta x_t$, and the liquidity traders submit in aggregate $\Delta u_t \sim \mathcal{N}(0, \sigma_u^2 \Delta t)$. The market maker observes only the aggregate order flow, $\Delta y_t = \Delta x_t + \Delta u_t$, and sets the price at which the trading takes place, $p_t$. The market maker is competitive, i.e., makes zero profit. This translates into the following formulas:

$$p_t = \mathsf{E}(v_t \mid \mathcal{I}_t^p), \qquad \mathcal{I}_t^p = \{\Delta y_1, \ldots, \Delta y_t, \Delta z_1, \ldots, \Delta z_{t-1}\}, \tag{53}$$

$$q_{t+1} = \mathsf{E}(v_t \mid \mathcal{I}_t^q), \qquad \mathcal{I}_t^q = \{\Delta y_1, \ldots, \Delta y_t, \Delta z_1, \ldots, \Delta z_t\}. \tag{54}$$

We also denote

$$\Omega_t = \mathsf{Var}(v_t \mid \mathcal{I}_t^p), \tag{55}$$

$$\Sigma_t = \mathsf{Var}(v_t \mid \mathcal{I}_t^q). \tag{56}$$

**Definition 1.** *A pricing rule $p_t$ is called linear if it is of the form $p_t = q_t + \lambda_t \Delta y_t$, for all $t = 1, \ldots, T-1$.*[20] *An equilibrium is called linear if the pricing rule is linear, and the informed trader's strategy $\Delta x_t$ is linear in $\{v_\tau\}_{\tau \le t}$ and $\{q_\tau\}_{\tau \le t}$.*

The next result shows that if the pricing rule is linear, the informed trader's strategy is also linear, and furthermore it can be decomposed into a level trading component, $\beta_t(v_{t-1} - q_t)$, and a flow trading component, $\gamma_t \Delta v_t$.

[20] We could defined more generally, a pricing rule to be linear in the whole history $\{\Delta y_\tau\}_{\tau \le t}$, but as Kyle (1985) shows, this is equivalent to the pricing rule being linear only in $\Delta y_t$.

**Theorem 2.** *Any linear equilibrium must be of the form*

$$\begin{aligned} \Delta x_t &= \beta_t(v_{t-1} - q_t)\Delta t + \gamma_t \Delta v_t, && (57) \\ p_t &= q_t + \lambda_t \Delta y_t, && (58) \\ q_{t+1} &= p_t + \mu_t(\Delta z_t - \rho_t \Delta y_t), && (59) \end{aligned}$$

*for $t = 1, \dots, T-1$, where $\beta_t$, $\gamma_t$, $\lambda_t$, $\mu_t$, $\rho_t$, $\Omega_t$, and $\Sigma_t$ are constants that satisfy*

$$\lambda_t = \frac{\beta_t \Sigma_{t-1} + \gamma_t \sigma_v^2}{\beta_t^2 \Sigma_{t-1}\Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2}, \tag{60}$$

$$\mu_t = \frac{\left(\sigma_u^2 + \beta_t^2 \Sigma_{t-1}\Delta t - \beta_t \gamma_t \Sigma_{t-1}\right)\sigma_v^2}{\left(\beta_t^2 \Sigma_{t-1}\Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2\right)\sigma_e^2 + \left(\beta_t^2 \Sigma_{t-1}\Delta t + \sigma_u^2\right)\sigma_v^2}, \tag{61}$$

$$m_t = \lambda_t - \rho_t \mu_t = \frac{\beta_t \Sigma_{t-1}(\sigma_v^2 + \sigma_e^2) + \gamma_t \sigma_v^2 \sigma_e^2}{\left(\beta_t^2 \Sigma_{t-1}\Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2\right)\sigma_e^2 + \left(\beta_t^2 \Sigma_{t-1}\Delta t + \sigma_u^2\right)\sigma_v^2}, \tag{62}$$

$$\rho_t = \frac{\gamma_t \sigma_v^2}{\beta_t^2 \Sigma_{t-1}\Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2}, \tag{63}$$

$$\Omega_t = \Sigma_{t-1} + \sigma_v^2 \Delta t - \frac{\beta_t^2 \Sigma_{t-1}^2 + 2\beta_t \gamma_t \Sigma_{t-1}\sigma_v^2 + \gamma_t^2 \sigma_v^4}{\beta_t^2 \Sigma_{t-1}\Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2}\, \Delta t, \tag{64}$$

$$\Sigma_t = \Sigma_{t-1} + \sigma_v^2 \Delta t - \frac{\beta_t^2 \Sigma_{t-1}^2(\sigma_v^2 + \sigma_e^2) + \beta_t^2 \Sigma_{t-1}\Delta t \sigma_v^4 + \sigma_v^4 \sigma_u^2 + \gamma_t^2 \sigma_v^4 \sigma_e^2 + 2\beta_t \gamma_t \Sigma_{t-1}\sigma_v^2 \sigma_e^2}{\left(\beta_t^2 \Sigma_{t-1}\Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2\right)\sigma_e^2 + \left(\beta_t^2 \Sigma_{t-1}\Delta t + \sigma_u^2\right)\sigma_v^2}\, \Delta t. \tag{65}$$

*The value function of the informed trader is quadratic for all $t = 1, \dots, T-1$:*

$$\pi_t = \alpha_{t-1}(v_{t-1} - q_t)^2 + \alpha'_{t-1}(\Delta v_t)^2 + \alpha''_{t-1}(v_{t-1} - q_t)\Delta v_t + \delta_{t-1}. \tag{66}$$

*The coefficients of the optimal trading strategy and the value function satisfy*

$$\beta_t \Delta t = \frac{1 - 2\alpha_t m_t}{2(\lambda_t - \alpha_t m_t^2)}, \tag{67}$$

$$\gamma_t = \frac{1 - 2\alpha_t m_t (1 - \mu_t)}{2(\lambda_t - \alpha_t m_t^2)}, \tag{68}$$

$$\alpha_{t-1} = \beta_t \Delta t (1 - \lambda_t \beta_t \Delta t) + \alpha_t (1 - m_t \beta_t \Delta t)^2, \tag{69}$$

$$\alpha'_{t-1} = \alpha_t (1 - \mu_t - m_t \gamma_t)^2 + \gamma_t (1 - \lambda_t \gamma_t), \tag{70}$$

$$\alpha''_{t-1} = \beta_t \Delta t + \gamma_t (1 - 2\lambda_t \beta_t \Delta t) + 2\alpha_t (1 - m_t \beta_t \Delta t)(1 - \mu_t - m_t \gamma_t), \tag{71}$$

$$\delta_{t-1} = \alpha_t \big( m_t^2 \sigma_u^2 + \mu_t^2 \sigma_e^2 \big) \Delta t + \alpha'_t \sigma_v^2 \Delta t + \delta_t. \tag{72}$$

*The terminal conditions are*

$$\alpha_T = \alpha'_T = \alpha''_T = \delta_T = 0. \tag{73}$$

*The second order condition is*

$$\lambda_t - \alpha_t m_t^2 > 0. \tag{74}$$

*Given $\Sigma_0$, conditions (60)–(74) are necessary and sufficient for the existence of a linear equilibrium.*

*Proof.* First, we show that Equations (60)–(65) are equivalent to the zero profit conditions of the market maker. Second, we show that Equations (67)–(74) are equivalent to the informed trader's strategy (57) being optimal.

**Zero Profit of Market Maker:** Let us start with with the market maker's update due to the order flow at $t = 1, \ldots, T-1$. Conditional on $\mathcal{I}^q_{t-1}$, the variables $v_{t-1} - q_t$ and $\Delta v_t$ have a bivariate normal distribution:

$$\begin{bmatrix} v_{t-1} - q_t \\ \Delta v_t \end{bmatrix} \mid \mathcal{I}^q_{t-1} \sim \mathcal{N}\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_{t-1} & 0 \\ 0 & \sigma_v^2 \end{bmatrix} \right). \tag{75}$$

The aggregate order flow at $t$ is of the form

$$\Delta y_t \;=\; \beta_t(v_{t-1} - q_t)\Delta t + \gamma_t \Delta v_t + \Delta u_t. \tag{76}$$

Denote by

$$\Phi_t \;=\; \mathsf{Cov}\left(\begin{bmatrix} v_{t-1} - q_t \\ \Delta v_t \end{bmatrix}, \Delta y_t\right) \;=\; \begin{bmatrix} \beta_t \Sigma_{t-1} \\ \gamma_t \sigma_v^2 \end{bmatrix} \Delta t. \tag{77}$$

Then, conditional on $\mathcal{I}_t = \mathcal{I}_{t-1}^q \cup \{\Delta y_t\}$, the distribution of $v_{t-1} - q_t$ and $\Delta v_t$ is bivariate normal:

$$\begin{bmatrix} v_{t-1} - q_t \\ \Delta v_t \end{bmatrix} \mid \mathcal{I}_t \;\sim\; \mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}\right), \tag{78}$$

where

$$\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \;=\; \Phi_t \,\mathsf{Var}(\Delta y_t)^{-1} \Delta y_t \;=\; \begin{bmatrix} \beta_t \Sigma_{t-1} \\ \gamma_t \sigma_v^2 \end{bmatrix} \frac{1}{\beta_t^2 \Sigma_{t-1} \Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2} \,\Delta y_t, \tag{79}$$

and

$$\begin{aligned} \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \;&=\; \mathsf{Var}\left(\begin{bmatrix} v_{t-1} - q_t \\ \Delta v_t \end{bmatrix}\right) - \Phi_t \,\mathsf{Var}(\Delta y_t)^{-1} \Phi_t' \qquad (80) \\ =\; \begin{bmatrix} \Sigma_{t-1} & 0 \\ 0 & \sigma_v^2 \Delta t \end{bmatrix} &- \frac{1}{\beta_t^2 \Sigma_{t-1} \Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2} \begin{bmatrix} \beta_t^2 \Sigma_{t-1}^2 & \beta_t \gamma_t \Sigma_{t-1} \sigma_v^2 \\ \beta_t \gamma_t \Sigma_{t-1} \sigma_v^2 & \gamma_t^2 \sigma_v^4 \end{bmatrix} \Delta t. \end{aligned}$$

We compute

$$p_t - q_t \;=\; \mathsf{E}(v_t - q_t \mid \mathcal{I}_t) \;=\; \mu_1 + \mu_2 \;=\; \frac{\beta_t \Sigma_{t-1} + \gamma_t \sigma_v^2}{\beta_t^2 \Sigma_{t-1} \Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2} \,\Delta y_t, \tag{81}$$

which proves Equation (60) for $\lambda_t$. Also,

$$\begin{aligned} \Omega_t \;&=\; \mathsf{Var}(v_t - q_t \mid \mathcal{I}_t) \;=\; \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2 \\ &=\; \Sigma_{t-1} + \sigma_v^2 \Delta t - \frac{\beta_t^2 \Sigma_{t-1}^2 + 2\beta_t \gamma_t \Sigma_{t-1} \sigma_v^2 + \gamma_t^2 \sigma_v^4}{\beta_t^2 \Sigma_{t-1} \Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2} \,\Delta t, \end{aligned} \tag{82}$$

which proves (64).

Next, to compute $q_{t+1} = \mathsf{E}(v_t \mid \mathcal{I}_t^q)$, we start from the same prior as in (75), but we consider the impact of both the order flow at $t$ and the market maker's signal at $t+1$:

$$\Delta y_t = \beta_t(v_{t-1} - q_t)\Delta t + \gamma_t \Delta v_t + \Delta u_t, \tag{83}$$

$$\Delta z_t = \Delta v_t + \Delta e_t. \tag{84}$$

Denote by

$$\Psi_t = \mathsf{Cov}\left( \begin{bmatrix} v_{t-1} - q_t \\ \Delta v_t \end{bmatrix}, \begin{bmatrix} \Delta y_t \\ \Delta z_t \end{bmatrix} \right) = \begin{bmatrix} \beta_t \Sigma_{t-1} & 0 \\ \gamma_t \sigma_v^2 & \sigma_v^2 \end{bmatrix} \Delta t, \tag{85}$$

$$V_t^{yz} = \mathsf{Var}\left( \begin{bmatrix} \Delta y_t \\ \Delta z_t \end{bmatrix} \right) = \begin{bmatrix} \beta_t^2 \Sigma_{t-1} \Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2 & \gamma_t \sigma_v^2 \\ \gamma_t \sigma_v^2 & \sigma_v^2 + \sigma_e^2 \end{bmatrix} \Delta t. \tag{86}$$

Conditional on $\mathcal{I}_t^q = \mathcal{I}_{t-1}^q \cup \{\Delta y_t, \Delta z_t\}$, the distribution of $v_{t-1} - q_t$ and $\Delta v_t$ is bivariate normal:

$$\begin{bmatrix} v_{t-1} - q_t \\ \Delta v_t \end{bmatrix} \mid \mathcal{I}_t^q \sim \mathcal{N}\left( \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \right), \tag{87}$$

where

$$\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \Psi_t \, (V_t^{yz})^{-1} \begin{bmatrix} \Delta y_t \\ \Delta z_t \end{bmatrix} = \frac{\begin{bmatrix} \beta_t \Sigma_{t-1} (\sigma_v^2 + \sigma_e^2) \Delta y_t \; - \; \beta_t \gamma_t \Sigma_{t-1} \sigma_v^2 \Delta z_t \\ \gamma_t \sigma_v^2 \sigma_e^2 \Delta y_t \; + \; (\beta_t^2 \Sigma_{t-1} \Delta t + \sigma_u^2) \sigma_v^2 \Delta z_t \end{bmatrix}}{(\beta_t^2 \Sigma_{t-1} \Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2)\sigma_e^2 + (\beta_t^2 \Sigma_{t-1} \Delta t + \sigma_u^2)\sigma_v^2}, \tag{88}$$

and

$$\begin{aligned} \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} &= \mathsf{Var}\left( \begin{bmatrix} v_{t-1} - q_t \\ \Delta v_t \end{bmatrix} \right) - \Psi_t \, (V_t^{yz})^{-1} \, \Psi_t' \\ &= \begin{bmatrix} \Sigma_{t-1} & 0 \\ 0 & \sigma_v^2 \Delta t \end{bmatrix} - \frac{\begin{bmatrix} \beta_t^2 \Sigma_{t-1}^2 (\sigma_v^2 + \sigma_e^2) & \beta_t \gamma_t \Sigma_{t-1} \sigma_v^2 \sigma_e^2 \\ \beta_t \gamma_t \Sigma_{t-1} \sigma_v^2 \sigma_e^2 & (\beta_t^2 \Sigma_{t-1} \Delta t + \gamma_t^2 \sigma_e^2 + \sigma_u^2) \sigma_v^4 \end{bmatrix}}{(\beta_t^2 \Sigma_{t-1} \Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2)\sigma_e^2 + (\beta_t^2 \Sigma_{t-1} \Delta t + \sigma_u^2)\sigma_v^2} \Delta t. \end{aligned} \tag{89}$$

Therefore,

$$
\begin{aligned}
q_{t+1} - q_t &= \mu_1 + \mu_2 \\
&= \frac{\big(\beta_t \Sigma_{t-1}(\sigma_v^2 + \sigma_e^2) + \gamma_t \sigma_v^2 \sigma_e^2\big)\Delta y_t \; + \; \big(\sigma_u^2 + \beta_t^2 \Sigma_{t-1}\Delta t - \beta_t \gamma_t \Sigma_{t-1}\big)\sigma_v^2 \Delta z_t}{(\beta_t^2 \Sigma_{t-1}\Delta t + \gamma_t^2 \sigma_v^2 + \sigma_u^2)\, \sigma_e^2 + (\beta_t^2 \Sigma_{t-1}\Delta t + \sigma_u^2)\, \sigma_v^2} \qquad (90) \\
&= m_t \Delta y_t \; + \; \mu_t \Delta z_t \; = \; (\lambda_t - \rho_t \mu_t)\Delta y_t \; + \; \mu_t \Delta z_t, \qquad (91)
\end{aligned}
$$

which proves Equations (61), (62), and (63). Also,

$$
\begin{aligned}
\Sigma_t &= \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2 \\
&= \Sigma_{t-1} + \sigma_v^2 \Delta t - \frac{\beta_t^2 \Sigma_{t-1}^2 (\sigma_v^2 + \sigma_e^2) + \beta_t^2 \Sigma_{t-1}\Delta t \sigma_v^4 + \sigma_v^4 \sigma_u^2 + \gamma_t^2 \sigma_v^4 \sigma_e^2 + 2\beta_t \gamma_t \Sigma_{t-1} \sigma_v^2 \sigma_e^2}{(\beta_t^2 \Sigma_{t-1} + (\beta_t + \gamma_t)^2 \sigma_v^2 + \sigma_u^2)\, \sigma_e^2 + (\beta_t^2 \Sigma_{t-1} + \sigma_u^2)\, \sigma_v^2} \, \Delta t, \qquad (92)
\end{aligned}
$$

which proves (65).

**Optimal Strategy of Informed Trader:** At each $t = 1, \ldots, T-1$, the informed trader maximizes the expected profit: $\pi_t = \max \sum_{\tau=t}^{T-1} \mathsf{E}\big((v_T - p_\tau)\Delta x_\tau\big)$. We prove by backward induction that the value function is quadratic and of the form given in (66): $\pi_t = \alpha_{t-1}(v_{t-1} - q_t)^2 + \alpha'_{t-1}(\Delta v_t)^2 + \alpha''_{t-1}(v_{t-1} - q_t)\Delta v_t + \delta_{t-1}$. At the last decision point $(t = T-1)$ the next value function is zero, i.e., $\alpha_T = \alpha'_T = \alpha''_T = \delta_T = 0$, which are the terminal conditions (73). This is the transversality condition: no money is left on the table. In the induction step, if $t = 1, \ldots, T-1$, we assume that $\pi_{t+1}$ is of the desired form. The Bellman principle of intertemporal optimization implies

$$
\pi_t \; = \; \max_{\Delta x} \; \mathsf{E}\Big((v_t - p_t)\Delta x + \pi_{t+1} \mid \mathcal{I}_t^q, v_t, \Delta v_t\Big). \qquad (93)
$$

Equations (58) and (59) show that the quote $q_t$ evolves by $q_{t+1} = q_t + m_t \Delta y_t + \mu_t \Delta z_t$, where $m_t = \lambda_t - \rho_t \mu_t$. This implies that the informed trader's choice of $\Delta x$ affects the trading price and the next quote by

$$
\begin{aligned}
p_t &= q_t + \lambda_t(\Delta x + \Delta u_t), && (94)\\
q_{t+1} &= q_t + m_t(\Delta x + \Delta u_t) + \mu_t \Delta z_t. && (95)
\end{aligned}
$$

Substituting these into the Bellman equation, we get

$$
\begin{aligned}
\pi_t \;=\; & \max_{\Delta x}\; \mathsf{E}\Big(\Delta x(v_{t-1} + \Delta v_t - q_t - \lambda_t \Delta x - \lambda_t \Delta u_t) \\
& \;+\; \alpha_t(v_{t-1} + \Delta v_t - q_t - m_t \Delta x - m_t \Delta u_t - \mu_t \Delta v_t - \mu_t \Delta e_t)^2 \;+\; \alpha'_t \Delta v_{t+1}^2 \qquad (96)\\
& \;+\; \alpha''_t(v_{t-1} + \Delta v_t - q_t - m_t \Delta x - m_t \Delta u_t - \mu_t \Delta v_t - \mu_t \Delta e_t)\Delta v_{t+1} \;+\; \delta_t\Big) \\
=\; & \max_{\Delta x}\; \Delta x(v_{t-1} - q_t + \Delta v_t - \lambda_t \Delta x) \\
& \;+\; \alpha_t\Big((v_{t-1} - q_t - m_t \Delta x + (1-\mu_t)\Delta v_t)^2 + (m_t^2 \sigma_u^2 + \mu_t^2 \sigma_e^2)\Delta t\Big) \;+\; \alpha'_t \sigma_v^2 \Delta t \qquad (97)\\
& \;+\; 0 \;+\; \delta_t.
\end{aligned}
$$

The first order condition with respect to $\Delta x$ is

$$
\Delta x \;=\; \frac{1 - 2\alpha_t m_t}{2(\lambda_t - \alpha_t m_t^2)}(v_{t-1} - q_t) \;+\; \frac{1 - 2\alpha_t m_t(1-\mu_t)}{2(\lambda_t - \alpha_t m_t^2)}\Delta v_t, \qquad (98)
$$

and the second order condition for a maximum is $\lambda_t - \alpha_t m_t^2 > 0$, which is (74). Thus, the optimal $\Delta x$ is indeed of the form $\Delta x_t = \beta_t(v_{t-1} - q_t)\Delta t + \gamma_t \Delta v_t$, where $\beta_t \Delta t$ and $\gamma_t$ are as in Equations (67) and (68). We substitute $\Delta x_t$ in the formula for $\pi_t$ to obtain

$$
\begin{aligned}
\pi_t \;=\; & \Big(\beta_t \Delta t(1 - \lambda_t \beta_t \Delta t) + \alpha_t(1 - m_t \beta_t \Delta t)^2\Big)(v_{t-1} - q_t)^2 \\
& + \Big(\alpha_t(1 - \mu_t - m_t \gamma_t)^2 + \gamma_t(1 - \lambda_t \gamma_t)\Big)\Delta v_t^2 \qquad (99)\\
& + \Big(\beta_t \Delta t + \gamma_t(1 - 2\lambda_t \beta_t \Delta t) + 2\alpha_t(1 - m_t \beta_t \Delta t)(1 - \mu_t - m_t \gamma_t)\Big)(v_{t-1} - q_t)\Delta v_t \\
& + \alpha_t\big(m_t^2 \sigma_u^2 + \mu_t^2 \sigma_e^2\big)\Delta t \;+\; \alpha'_t \sigma_v^2 \Delta t \;+\; \delta_t.
\end{aligned}
$$

This proves that indeed $\pi_t$ is of the form $\pi_t = \alpha_{t-1}(v_{t-1} - q_t)^2 + \alpha'_{t-1}(\Delta v_t)^2 + \alpha''_{t-1}(v_{t-1} - q_t)\Delta v_t + \delta_{t-1}$, with $\alpha_{t-1}$, $\alpha'_{t-1}$, $\alpha''_{t-1}$ and $\delta_{t-1}$ as in Equations (69)–(72). □

We now briefly discuss the existence of a solution for the recursive system given in

Theorem 2. The system of equations (60)–(72) can be numerically solved backwards, starting from the boundary conditions (73). We also start with an arbitrary value of $\Sigma_T > 0$.[21] By backward induction, suppose $\alpha_t$ and $\Sigma_t$ are given. One verifies that Equation (65) implies

$$\Sigma_{t-1} = \frac{\Sigma_t\big(\sigma_v^2\sigma_u^2 + \sigma_v^2(\sigma_u^2+\gamma_t^2\sigma_e^2)\big) - \sigma_v^2\sigma_u^2\sigma_e^2\Delta t}{\big(\sigma_u^2\sigma_e^2 + \sigma_v^2(\sigma_u^2+\gamma_t^2\sigma_e^2) + \beta_t^2\Delta t^2\sigma_v^2\sigma_e^2 - 2\gamma_t\beta_t\Delta t\sigma_v^2\sigma_e^2\big) - \Sigma_t\beta_t^2\Delta t\big(\sigma_v^2+\sigma_e^2\big)}. \tag{100}$$

Then, Equations (60)–(62) can be rewritten to express $\lambda_t, \mu_t, m_t$ as functions of $(\Sigma_t, \beta_t, \gamma_t)$ instead of $(\Sigma_{t-1}, \beta_t, \gamma_t)$. Next, we use (67) and (68) to express $\lambda_t, \mu_t, m_t$ as functions of $(\lambda_t, \mu_t, m_t, \alpha_t, \Sigma_t)$. This gives a system of polynomial equations, whose solution $\lambda_t, \mu_t, m_t$ depends only on $(\alpha_t, \Sigma_t)$. Numerical simulations show that the solution is unique under the second order condition (74), but the authors have not been able to prove theoretically that this is true in all cases. Once the recursive system is computed for all $t = 1, \ldots, T-1$, the only condition left to do is to verify that the value obtained for $\Sigma_0$ is the correct one. However, unlike in Kyle (1985), the recursive equation for $\Sigma_t$ is not linear, and therefore the parameters cannot be simply rescaled. Instead, one must numerically modify the initial choice of $\Sigma_T$ until the correct value of $\Sigma_0$ is reached.

## A.2 Discrete Time Benchmark Model

The setup is the same as for the fast model, except that the market maker gets the signal $\Delta z$ at the same time as the informed trader observes $\Delta v$. The sequence of events is as follows. At $t = 0$, the informed trader observes $v_0$. At each $t = 1, \ldots, T-1$, the informed trader observes $\Delta v_t = v_t - v_{t-1}$; and the market maker observes $\Delta z_t = \Delta v_t + \Delta e_t$, with $\Delta e_t \sim \mathcal{N}(0, \sigma_e^2\Delta t)$. The market maker quotes the bid price = the ask price = $q_t$. The informed trader then submits $\Delta x_t$, and the liquidity traders submit in aggregate $\Delta u_t \sim \mathcal{N}(0, \sigma_u^2\Delta t)$. The market maker observes only the aggregate order flow, $\Delta y_t = \Delta x_t + \Delta u_t$, and sets the price at which the trading takes place, $p_t$. The

[21] Numerically, it should be of the order of $\Delta t$.

market maker is competitive, i.e., makes zero profit. This implies

$$
\begin{aligned}
p_t &= \mathsf{E}(v_t \mid \mathcal{I}_t^p), \qquad \mathcal{I}_t^p = \{\Delta y_1, \ldots, \Delta y_t, \Delta z_1, \ldots, \Delta z_t\}, && (101)\\
q_t &= \mathsf{E}(v_t \mid \mathcal{I}_t^q), \qquad \mathcal{I}_t^q = \{\Delta y_1, \ldots, \Delta y_{t-1}, \Delta z_1, \ldots, \Delta z_t\}. && (102)
\end{aligned}
$$

We also denote

$$
\begin{aligned}
\Sigma_t &= \mathsf{Var}(v_t \mid \mathcal{I}_t^p), && (103)\\
\Omega_t &= \mathsf{Var}(v_t \mid \mathcal{I}_t^q). && (104)
\end{aligned}
$$

The next result shows that if the pricing rule is linear, the informed trader's strategy is also linear, and furthermore it only has a level trading component, $\beta_t(v_t - q_t)$.

**Theorem 3.** *Any linear equilibrium must be of the form*

$$
\begin{aligned}
\Delta x_t &= \beta_t(v_t - q_t)\Delta t, && (105)\\
p_t &= q_t + \lambda_t \Delta y_t, && (106)\\
q_t &= p_{t-1} + \mu_{t-1}\Delta z_t, && (107)
\end{aligned}
$$

*for* $t = 1, \ldots, T-1$, *where by convention* $p_0 = 0$, *and* $\beta_t$, $\gamma_t$, $\lambda_t$, $\mu_t$, $\Omega_t$, *and* $\Sigma_t$ *are constants that satisfy*

$$
\begin{aligned}
\lambda_t &= \frac{\beta_t \Sigma_t}{\sigma_u^2}, && (108)\\
\mu_t &= \frac{\sigma_v^2}{\sigma_v^2 + \sigma_e^2}, && (109)\\
\Omega_t &= \frac{\Sigma_t \sigma_u^2}{\sigma_u^2 - \beta_t^2 \Sigma_t \Delta t}, && (110)\\
\Sigma_{t-1} &= \Sigma_t + \frac{\beta_t^2 \Sigma_t^2}{\sigma_u^2 - \beta_t^2 \Sigma_t \Delta t}\,\Delta t - \frac{\sigma_v^2 \sigma_e^2}{\sigma_v^2 + \sigma_e^2}\,\Delta t. && (111)
\end{aligned}
$$

*The value function of the informed trader is quadratic for all* $t = 1, \ldots, T-1$*:*

$$
\pi_t = \alpha_{t-1}(v_t - q_t)^2 + \delta_{t-1}. \qquad (112)
$$

*The coefficients of the optimal trading strategy and the value function satisfy*

$$\beta_t \Delta t = \frac{1 - 2\alpha_t\lambda_t}{2\lambda_t(1-\alpha_t\lambda_t)}, \tag{113}$$

$$\alpha_{t-1} = \beta_t\Delta t(1-\lambda_t\beta_t\Delta t) + \alpha_t(1-\lambda_t\beta_t\Delta t)^2, \tag{114}$$

$$\delta_{t-1} = \alpha_t\big(\lambda_t^2\sigma_u^2 + \mu_t^2(\sigma_v^2+\sigma_e^2)\big)\Delta t + \delta_t. \tag{115}$$

*The terminal conditions are*

$$\alpha_T = \delta_T = 0. \tag{116}$$

*The second order condition is*

$$\lambda_t(1-\alpha_t\lambda_t) > 0. \tag{117}$$

*Given $\Sigma_0$, conditions (108)–(117) are necessary and sufficient for the existence of a linear equilibrium.*

*Proof.* First, we show that Equations (108)–(111) are equivalent to the zero profit conditions of the market maker. Second, we show that Equations (113)–(117) are equivalent to the informed trader's strategy (105) being optimal.

**Zero Profit of Market Maker:** Let us start with with the market maker's update due to the order flow at $t = 1, \ldots, T-1$. Conditional on $\mathcal{I}_t^q$, $v_t$ has a normal distribution, $v_t|\mathcal{I}_t^q \sim \mathcal{N}(q_t, \Omega_t)$. The aggregate order flow at $t$ is of the form $\Delta y_t = \beta_t(v_t - q_t)\Delta t + \Delta u_t$. Denote by

$$\Phi_t = \mathsf{Cov}(v_t - q_t, \Delta y_t) = \beta_t\Omega_t\Delta t. \tag{118}$$

Then, conditional on $\mathcal{I}_t^p = \mathcal{I}_t^q \cup \{\Delta y_t\}$, $v_t \sim \mathcal{N}(p_t, \Sigma_t)$, with

$$p_t = q_t + \lambda_t\Delta y_t, \tag{119}$$

$$\lambda_t = \Phi_t \,\mathsf{Var}(\Delta y_t)^{-1} = \frac{\beta_t\Omega_t}{\beta_t^2\Omega_t\Delta t + \sigma_u^2}, \tag{120}$$

$$\begin{aligned}\Sigma_t &= \mathsf{Var}(v_t - q_t) - \Phi_t\,\mathsf{Var}(\Delta y_t)^{-1}\Phi_t' \\ &= \Omega_t - \frac{\beta_t^2\Omega_t^2}{\beta_t^2\Omega_t\Delta t + \sigma_u^2}\,\Delta t = \frac{\Omega_t\sigma_u^2}{\beta_t^2\Omega_t\Delta t + \sigma_u^2}.\end{aligned} \tag{121}$$

To obtain Equation (108) for $\lambda_t$, note that the above equations for $\lambda_t$ and $\Sigma_t$ imply $\frac{\lambda_t}{\Sigma_t} = \frac{\beta_t}{\sigma_u^2}$. Equation (110) is obtained by solving for $\Sigma_t$ in Equation (121).

Next, consider the market maker's update at $t = 1, \dots, T-1$ due to the signal $\Delta z_t = \Delta v_t + \Delta e_t$. From $v_{t-1} | \mathcal{I}_{t-1}^p \sim \mathcal{N}(p_{t-1}, \Sigma_{t-1})$, we have $v_t | \mathcal{I}_{t-1}^p \sim \mathcal{N}(p_{t-1}, \Sigma_{t-1} + \sigma_v^2 \Delta t)$ . Denote by

$$\Psi_t \ = \ \mathsf{Cov}(v_t - p_{t-1}, \Delta z_t) \ = \ \sigma_v^2 \Delta t. \tag{122}$$

Then, conditional on $\mathcal{I}_t^q = \mathcal{I}_{t-1}^p \cup \{\Delta z_t\}$, $v_t | \mathcal{I}_t^q \sim \mathcal{N}(q_t, \Omega_t)$, with

$$q_t \ = \ p_{t-1} + \mu_t \Delta z_t, \tag{123}$$

$$\mu_t \ = \ \Psi_t \, \mathsf{Var}(\Delta z_t)^{-1} \ = \ \frac{\sigma_v^2}{\sigma_v^2 + \sigma_e^2}, \tag{124}$$

$$\begin{aligned} \Omega_t \ &= \ \mathsf{Var}(v_t - p_{t-1}) - \Psi_t \, \mathsf{Var}(\Delta z_t)^{-1} \Psi_t' \\ &= \ \Sigma_{t-1} + \sigma_v^2 \Delta t - \frac{\sigma_v^4}{\sigma_v^2 + \sigma_e^2} \, \Delta t \ = \ \Sigma_{t-1} + \frac{\sigma_v^2 \sigma_e^2}{\sigma_v^2 + \sigma_e^2} \, \Delta t. \end{aligned} \tag{125}$$

which proves Equation (109) for $\mu_t$. Note that Equation (125) gives a formula for $\Sigma_{t-1}$ as a function of $\Omega_t$, and we already proved (110), which expresses $\Omega_t$ as a function of $\Sigma_t$. We therefore get $\Sigma_{t-1}$ as a function of $\Sigma_t$, which is (111).

**Optimal Strategy of Informed Trader:** At each $t = 1, \dots, T-1$, the informed trader maximizes the expected profit: $\pi_t = \max \sum_{\tau=t}^{T-1} \mathsf{E}\big((v_T - p_\tau)\Delta x_\tau\big)$. We prove by backward induction that the value function is quadratic and of the form given in (112): $\pi_t = \alpha_{t-1}(v_t - q_t)^2 + \delta_{t-1}$. At the last decision point ($t = T-1$) the next value function is zero, i.e., $\alpha_T = \delta_T = 0$, which are the terminal conditions (116). In the induction step, if $t = 1, \dots, T-1$, we assume that $\pi_{t+1}$ is of the desired form. The Bellman principle of intertemporal optimization implies

$$\pi_t \ = \ \max_{\Delta x} \ \mathsf{E}\Big((v_t - p_t)\Delta x + \pi_{t+1} \mid \mathcal{I}_t^q, v_t, \Delta v_t\Big). \tag{126}$$

Equations (106) and (107) show that the quote $q_t$ evolves by $q_{t+1} = q_t + m_t \Delta y_t + \mu_t \Delta z_{t+1}$. This implies that the informed trader's choice of $\Delta x$ affects the trading price and the next quote by

$$
\begin{aligned}
p_t &= q_t + \lambda_t(\Delta x + \Delta u_t), && (127)\\
q_{t+1} &= q_t + \lambda_t(\Delta x + \Delta u_t) + \mu_t \Delta z_{t+1}. && (128)
\end{aligned}
$$

Substituting these into the Bellman equation, we get

$$
\begin{aligned}
\pi_t &= \max_{\Delta x} \mathsf{E}\Big(\Delta x(v_t - q_t - \lambda_t \Delta x - \lambda_t \Delta u_t) \\
&\quad + \alpha_t(v_t + \Delta v_{t+1} - q_t - \lambda_t \Delta x - \lambda_t \Delta u_t - \mu_t \Delta z_{t+1})^2 + \delta_t\Big) && (129)\\
&= \max_{\Delta x} \Delta x(v_t - q_t - \lambda_t \Delta x) \\
&\quad + \alpha_t\Big((v_t - q_t - \lambda_t \Delta x)^2 + (\lambda_t^2 \sigma_u^2 + \mu_t^2(\sigma_v^2 + \sigma_e^2))\Delta t\Big) + \delta_t. && (130)
\end{aligned}
$$

The first order condition with respect to $\Delta x$ is

$$
\Delta x = \frac{1 - 2\alpha_t \lambda_t}{2\lambda_t(1 - \alpha_t \lambda_t)}(v_t - q_t), \tag{131}
$$

and the second order condition for a maximum is $\lambda_t(1 - \alpha_t \lambda_t) > 0$, which is (117). Thus, the optimal $\Delta x$ is indeed of the form $\Delta x_t = \beta_t(v_t - q_t)\Delta t$, where $\beta_t \Delta t$ satisfies Equation (113). We substitute $\Delta x_t$ in the formula for $\pi_t$ to obtain

$$
\pi_t = \Big(\beta_t \Delta t(1 - \lambda_t \beta_t \Delta t) + \alpha_t(1 - \lambda_t \beta_t \Delta t)^2\Big)(v_t - q_t)^2 + \alpha_t\big(\lambda_t^2 \sigma_u^2 + \mu_t^2(\sigma_v^2 + \sigma_e^2)\big)\Delta t + \delta_t. \tag{132}
$$

This proves that indeed $\pi_t$ is of the form $\pi_t = \alpha_{t-1}(v_t - q_t)^2 + \delta_{t-1}$, with $\alpha_{t-1}$ and $\delta_{t-1}$ as in Equations (114) and (115). □

Equations (108)–(111) and (113)–(115) form a system of equations. As before, it is solved backwards, starting from the boundary conditions (116), and so that $\Sigma_t = \Sigma_0$ at $t = 0$.

# B Proofs

## B.1 Proof of Theorem 1

**Benchmark model:** We compute the optimal strategy of the informed trader at $t + \mathrm{d}t$. As we have seen in the discrete version of the model, in Appendix A, we need to consider only strategies $\mathrm{d}x_\tau$ of the type $\mathrm{d}x_\tau = \beta_\tau(v_\tau - p_\tau)\,\mathrm{d}\tau + \gamma_\tau\,\mathrm{d}v_\tau$. Recall that $\mathcal{I}_t^p$ is the market maker's information set immediately after trading at $t$. If we denote by $\mathcal{J}_t^p = \mathcal{I}_t^p \cup \{v_\tau\}_{\tau \le t+\mathrm{d}t}$ the trader's information set before trading at $t + \mathrm{d}t$, the expected profit from trading after $t$ is

$$\pi_t \;=\; \mathsf{E}\left( \int_t^1 (v_1 - p_{\tau+\mathrm{d}\tau})\,\mathrm{d}x_\tau \mid \mathcal{J}_t^p \right). \tag{133}$$

From (11), $p_{\tau+\mathrm{d}\tau} = p_\tau + \mu_\tau(\mathrm{d}v_\tau + \mathrm{d}e_\tau) + \lambda_\tau(\mathrm{d}x_\tau + \mathrm{d}u_\tau)$. For $\tau \ge t$, denote by

$$V_\tau \;=\; \mathsf{E}\big( (v_\tau - p_\tau)^2 \mid \mathcal{J}_t^p \big). \tag{134}$$

Then the expected profit is

$$\begin{aligned} \pi_t \;&=\; \mathsf{E}\left( \int_t^1 (v_\tau + \,\mathrm{d}v_\tau - p_\tau - \mu_\tau\,\mathrm{d}v_\tau - \lambda_\tau\,\mathrm{d}x_\tau)\,\mathrm{d}x_\tau \mid \mathcal{J}_t^p \right) &(135)\\ &=\; \int_t^1 \Big( \beta_\tau V_\tau + (1 - \mu_\tau - \lambda_\tau\gamma_\tau)\gamma_\tau\sigma_v^2 \Big)\,\mathrm{d}\tau. &(136) \end{aligned}$$

$V_\tau$ can be computed recursively:

$$\begin{aligned} V_{\tau+\mathrm{d}\tau} \;&=\; \mathsf{E}\big( (v_{\tau+\mathrm{d}\tau} - p_{\tau+\mathrm{d}\tau})^2 \mid \mathcal{J}_t^p \big) \\ &=\; \mathsf{E}\big( (v_\tau + \,\mathrm{d}v_\tau - p_\tau - \mu_\tau\,\mathrm{d}v_\tau - \mu_\tau\,\mathrm{d}e_\tau - \lambda_\tau\,\mathrm{d}x_\tau - \lambda_\tau\,\mathrm{d}u_\tau)^2 \mid \mathcal{J}_t^p \big) \qquad (137)\\ &=\; V_\tau + (1 - \mu_\tau - \lambda_\tau\gamma_\tau)^2\sigma_v^2\,\mathrm{d}\tau + \mu_\tau^2\sigma_e^2\,\mathrm{d}\tau + \lambda_\tau^2\sigma_u^2\,\mathrm{d}\tau - 2\lambda_t\beta_t V_\tau\,\mathrm{d}\tau. \end{aligned}$$

therefore the law of motion of $V_\tau$ is a first order differential equation

$$V_\tau' \;=\; -2\lambda_t\beta_t V_\tau + (1 - \mu_\tau - \lambda_\tau\gamma_\tau)^2\sigma_v^2 + \mu_\tau^2\sigma_e^2 + \lambda_\tau^2\sigma_u^2, \tag{138}$$

or equivalently $\beta_\tau V_\tau = \frac{-V_\tau' + (1-\mu_\tau-\lambda_\tau\gamma_\tau)^2\sigma_v^2 + \mu_\tau^2\sigma_e^2 + \lambda_\tau^2\sigma_u^2}{2\lambda_\tau}$. Substitute this into (133) and integrate by parts. Since $V_t = 0$, we get

$$\begin{aligned}\pi_t \;=\; & -\frac{V_1}{2\lambda_1} + \int_t^1 V_\tau \Big(\frac{1}{2\lambda_\tau}\Big)' \,\mathrm{d}\tau \\ & + \int_t^1 \bigg( \frac{(1-\mu_\tau-\lambda_\tau\gamma_\tau)^2\sigma_v^2 + \mu_\tau^2\sigma_e^2 + \lambda_\tau^2\sigma_u^2}{2\lambda_\tau} + (1-\mu_\tau-\lambda_\tau\gamma_\tau)\gamma_\tau\sigma_v^2 \bigg) \,\mathrm{d}\tau.\end{aligned} \tag{139}$$

This is essentially the argument of Kyle (1985): we have eliminated the choice variable $\beta_\tau$ and replaced it by $V_\tau$. Since $V_\tau > 0$ can be arbitrarily chosen, in order to get an optimum we must have $\big(\frac{1}{2\lambda_\tau}\big)' = 0$, which is equivalent to

$$\lambda_\tau \;=\; \text{constant}. \tag{140}$$

For a maximum, the transversality condition $V_1 = 0$ must be also satisfied.

We next turn to the choice of $\gamma_\tau$. The first order condition is

$$-(1-\mu_\tau-\lambda_\tau\gamma_\tau) + (1-\mu_\tau-\lambda_\tau\gamma_\tau) - \lambda_\tau\gamma_\tau \;=\; 0 \quad\Longrightarrow\quad \gamma_\tau \;=\; 0. \tag{141}$$

Thus, there is no flow trading in the benchmark model. Note also that the second order condition is $\lambda_\tau > 0$.[22]

Next, we derive the pricing rules from the market maker's zero profit conditions. The equations $p_t = \mathsf{E}(v_1 | \mathcal{I}_t^p)$ and $q_t = \mathsf{E}(v_1 | \mathcal{I}_t^p, \mathrm{d}z_t)$ imply that $q_t = p_t + \mu_t \,\mathrm{d}z_t$, where

$$\mu_t \;=\; \frac{\mathsf{Cov}(v_1,\, \mathrm{d}z_t \mid \mathcal{I}_t^p)}{\mathsf{Var}(\,\mathrm{d}z_t \mid \mathcal{I}_t^p)} \;=\; \frac{\mathsf{Cov}(v_0 + \int_0^1 \mathrm{d}v_\tau,\, \mathrm{d}v_t + \,\mathrm{d}e_t \mid \mathcal{I}_t^p)}{\mathsf{Var}(\,\mathrm{d}v_t + \,\mathrm{d}e_t \mid \mathcal{I}_t^p)} \;=\; \frac{\sigma_v^2}{\sigma_v^2 + \sigma_e^2}. \tag{142}$$

The equations $q_t = \mathsf{E}(v_1 | \mathcal{I}_{t+\mathrm{d}t}^q)$ and $p_{t+\mathrm{d}t} = \mathsf{E}(v_1 | \mathcal{I}_{t+\mathrm{d}t}^q, \mathrm{d}y_t)$ imply that $p_{t+\mathrm{d}t} = q_t + \lambda_t \mathrm{d}y_t$, where

$$\lambda_t \;=\; \frac{\mathsf{Cov}(v_1,\, \mathrm{d}y_t \mid \mathcal{I}_{t+\mathrm{d}t}^q)}{\mathsf{Var}(\,\mathrm{d}y_t \mid \mathcal{I}_{t+\mathrm{d}t}^q)} \;=\; \frac{\mathsf{Cov}(v_1, \beta_t(v_t - p_t)\,\mathrm{d}t + \,\mathrm{d}u_t \mid \mathcal{I}_{t+\mathrm{d}t}^q)}{\mathsf{Var}(\beta_t(v_t - p_t)\,\mathrm{d}t + \,\mathrm{d}u_t \mid \mathcal{I}_{t+\mathrm{d}t}^q)} \;=\; \frac{\beta_t \Sigma_t}{\sigma_u^2}, \tag{143}$$

[22] The condition $\lambda_\tau > 0$ is also a second order condition with respect to the choice of $\beta_\tau$. To see this, suppose $\lambda_\tau < 0$. Then if $\beta_\tau > 0$ is chosen very large, Equation (138) shows that $V_\tau$ is very large as well, and thus $\beta_\tau V_\tau$ can be made arbitrarily large. Thus, there would be no maximum.

where $\Sigma_t = \mathsf{E}\big((v_t - p_t)^2 | \mathcal{I}_t^p\big)$.[23] The information set of the informed trader, $\mathcal{J}_t^p$, is a refinement of the market maker's information set, $\mathcal{I}_t^p$. Therefore, by the law of iterated expectations, $\Sigma_t$ satisfies the same equation as $V_t$:

$$\Sigma_t' \;=\; -2\lambda_t\beta_t\Sigma_t + (1-\mu_t-\lambda_t\gamma_t)^2\sigma_v^2 + \mu_t^2\sigma_e^2 + \lambda_t^2\sigma_u^2, \tag{144}$$

except that it has a different initial condition. One can solve this differential equation explicitly and show that the transversality condition $V_1 = 0$ is equivalent to $\int_0^1 \beta_t \,\mathrm{d}t = +\infty$, and in turn this is equivalent to $\Sigma_1 = 0$. Since $\lambda_t$, $\mu_t$ and $\gamma_t = 0$ are constant, by (143) $\beta_t\Sigma_t$ is also constant. Equation (144) then implies that $\Sigma_t'$ is constant. Since $\Sigma_1 = 0$, $\Sigma_t = (1-t)\Sigma_0$, and $\beta_t = \frac{\beta_0}{1-t}$. Finally, we integrate (144) between 0 and 1, and substituting for $\lambda = \frac{\beta_0\Sigma_0}{\sigma_u^2}$, $\mu = \frac{\sigma_v^2}{\sigma_v^2+\sigma_e^2}$, and $\gamma = 0$, we obtain $\beta_0$ and $\lambda$ as stated in the Theorem.

**Fast model:** The informed trader has the same objective function as in (133):

$$\pi_t \;=\; \mathsf{E}\left(\int_t^1 (v_1 - p_{\tau+\,\mathrm{d}\tau})\,\mathrm{d}x_\tau \mid \mathcal{J}_t^p\right). \tag{145}$$

but here we use $q_t$ instead of $p_t$ as a state variable. From (7), $p_{t+\mathrm{d}t} = q_t + \lambda_t \mathrm{d}y_t$. Also, from (17), $q_{\tau+\,\mathrm{d}\tau} = q_\tau + \mu_\tau(\mathrm{d}z_\tau - \rho_\tau \mathrm{d}y_\tau) + \lambda_\tau(\mathrm{d}y_\tau)$, and we obtain

$$q_{\tau+\,\mathrm{d}\tau} \;=\; \mu_\tau\,\mathrm{d}z_\tau + m_\tau\,\mathrm{d}y_\tau, \quad \text{with} \tag{146}$$

$$m_\tau \;=\; \lambda_\tau - \mu_\tau\rho_\tau. \tag{147}$$

As we have seen in the discrete version of the model, in Appendix A, we need to consider only strategies $\mathrm{d}x_\tau$ of the type (16), $\mathrm{d}x_\tau = \beta_\tau(v_\tau - q_\tau)\,\mathrm{d}\tau + \gamma_\tau\,\mathrm{d}v_\tau$. For $\tau \geq t$, denote by

$$V_\tau \;=\; \mathsf{E}\big((v_\tau - q_\tau)^2 \mid \mathcal{J}_t^p\big). \tag{148}$$

[23]Because $\mathcal{I}_{t+\mathrm{d}t}^q = \mathcal{I}_t^p \cup \{\mathrm{d}z_t\}$, the two information sets differ only by the infinitesimal quantity $\mathrm{d}z_t$, and thus we can also write $\Sigma_t = \mathsf{E}\big((v_t - p_t)^2 | \mathcal{I}_{t+\mathrm{d}t}^q\big) = \mathsf{E}\big((v_t - p_t)^2 | \mathcal{I}_t^p\big)$.

The expected profit is

$$\pi_t = \mathsf{E}\Big(\int_t^1 (v_\tau + \mathrm{d}v_\tau - q_\tau - \lambda_\tau \,\mathrm{d}x_\tau)\,\mathrm{d}x_\tau \mid \mathcal{J}_t^p\Big) \tag{149}$$

$$= \int_t^1 \Big(\beta_\tau V_\tau + (1-\lambda_\tau\gamma_\tau)\gamma_\tau\sigma_v^2\Big)\,\mathrm{d}\tau. \tag{150}$$

$V_\tau$ is computed as in the benchmark model, except that $\lambda_\tau$ is replaced by $m_\tau$:

$$\begin{aligned} V_{\tau+\mathrm{d}\tau} &= \mathsf{E}\big((v_{\tau+\mathrm{d}\tau} - q_{\tau+\mathrm{d}\tau})^2 \mid \mathcal{J}_t^p\big) \\ &= V_\tau + (1-\mu_\tau - m_\tau\gamma_\tau)^2\sigma_v^2\,\mathrm{d}\tau + \mu_\tau^2\sigma_e^2\,\mathrm{d}\tau + m_\tau^2\sigma_u^2\,\mathrm{d}\tau - 2m_t\beta_t V_\tau\,\mathrm{d}\tau. \end{aligned} \tag{151}$$

therefore the law of motion of $V_\tau$ is a first order differential equation

$$V_\tau' = -2m_t\beta_t V_\tau + (1-\mu_\tau - m_\tau\gamma_\tau)^2\sigma_v^2 + \mu_\tau^2\sigma_e^2 + m_\tau^2\sigma_u^2, \tag{152}$$

or equivalently $\beta_\tau V_\tau = \frac{-V_\tau' + (1-\mu_\tau - m_\tau\gamma_\tau)^2\sigma_v^2 + \mu_\tau^2\sigma_e^2 + m_\tau^2\sigma_u^2}{2m_\tau}$. Substitute this into (133) and integrate by parts. Since $V_t = 0$, we get

$$\begin{aligned} \pi_t &= -\frac{V_1}{2m_1} + \int_t^1 V_\tau\Big(\frac{1}{2m_\tau}\Big)'\,\mathrm{d}\tau \\ &\quad + \int_t^1 \left(\frac{(1-\mu_\tau - m_\tau\gamma_\tau)^2\sigma_v^2 + \mu_\tau^2\sigma_e^2 + m_\tau^2\sigma_u^2}{2m_\tau} + (1-\lambda_\tau\gamma_\tau)\gamma_\tau\sigma_v^2\Big)\right)\,\mathrm{d}\tau. \end{aligned} \tag{153}$$

Since $V_\tau > 0$ can be arbitrarily chosen, in order to get an optimum we must have $\big(\frac{1}{2m_\tau}\big)' = 0$, which is equivalent to $m_\tau =$ constant. For a maximum, the transversality condition $V_1 = 0$ must be also satisfied.

We next turn to the choice of $\gamma_\tau$. The first order condition is

$$-(1-\mu_\tau - m_\tau\gamma_\tau) + (1-\lambda_\tau\gamma_\tau) - \lambda_\tau\gamma_\tau = 0 \quad\Longrightarrow\quad \gamma_\tau = \frac{\mu_\tau}{2\lambda_\tau - m_\tau} = \frac{\mu_\tau}{\lambda_\tau + \mu_\tau\rho_\tau}. \tag{154}$$

Thus, we obtain a nonzero flow trading component. The second order condition is $\lambda_\tau + \mu_\tau\rho_\tau > 0$. There is also a second order condition with respect to $\beta$: $m_\tau > 0$: see Footnote 22.

Next, we derive the pricing rules from the market maker's zero profit conditions. As in the benchmark model, we compute

$$\lambda_t \;=\; \frac{\mathsf{Cov}_t(v_1,\, \mathrm{d}y_t)}{\mathsf{Var}_t(\,\mathrm{d}y_t)} \;=\; \frac{\mathsf{Cov}_t(v_1, \beta_t(v_t - p_t)\,\mathrm{d}t + \gamma_t\,\mathrm{d}v_t +\, \mathrm{d}u_t)}{\mathsf{Var}(\beta_t(v_t - p_t)\,\mathrm{d}t + \gamma_t\,\mathrm{d}v_t +\, \mathrm{d}u_t)} \;=\; \frac{\beta_t\Sigma_t + \gamma_t\sigma_v^2}{\gamma_t^2\sigma_v^2 + \sigma_u^2}, \quad (155)$$

$$\rho_t \;=\; \frac{\mathsf{Cov}_t(\,\mathrm{d}z_t,\, \mathrm{d}y_t)}{\mathsf{Var}_t(\,\mathrm{d}y_t)} \;=\; \frac{\gamma_t\sigma_v^2}{\gamma_t^2\sigma_v^2 + \sigma_u^2}, \quad (156)$$

$$\mu_t \;=\; \frac{\mathsf{Cov}_t(v_1,\, \mathrm{d}z_t - \rho_t\,\mathrm{d}y_t)}{\mathsf{Var}_t(\,\mathrm{d}z_t - \rho_t\,\mathrm{d}y_t)} \;=\; \frac{-\rho_t\beta_t\Sigma_t + (1-\rho_t\gamma_t)\sigma_v^2}{(1-\rho_t\gamma_t)^2\sigma_v^2 + \rho_t^2\sigma_u^2 + \sigma_e^2}. \quad (157)$$

By the same arguments as for the benchmark model, $\Sigma_t = (1-t)\Sigma_0$, $\beta_t = \frac{\beta_0}{1-t}$, and $\beta_t\Sigma_t$, $\lambda_t$, $\rho_t$, $\mu_t$ are constant. Since $\Sigma_t$ satisfies the same Equation (152) as $V_t$, and $\Sigma_t' = -\Sigma_0$, we obtain

$$-\Sigma_0 \;=\; -2m_t\beta_t\Sigma_t + (1-\mu_\tau - m_\tau\gamma_\tau)^2\sigma_v^2 + \mu_\tau^2\sigma_e^2 + m_\tau^2\sigma_u^2. \quad (158)$$

We now define the following constants:

$$a \;=\; \frac{\sigma_u^2}{\sigma_v^2}, \quad b \;=\; \frac{\sigma_e^2}{\sigma_v^2}, \quad c \;=\; \frac{\Sigma_0}{\sigma_v^2}, \quad (159)$$

$$f \;=\; \frac{\gamma^2}{a}, \quad \tilde{\lambda} \;=\; \lambda\gamma, \quad \tilde{\rho} \;=\; \rho\gamma, \quad \nu = \frac{\beta_0\Sigma_0}{\sigma_u^2}\gamma, \quad \tilde{m} = m\gamma. \quad (160)$$

With these notations, Equations (154)–(158) become

$$\begin{aligned} \tilde{\lambda} \;&=\; \mu(1-\tilde{\rho}), \quad \tilde{\lambda} \;=\; \frac{\nu + f}{1+f}, \quad \tilde{\rho} \;=\; \frac{f}{1+f}, \quad \mu \;=\; \frac{1-\nu}{1+b(1+f)} \\ c \;&=\; \frac{2\nu}{f} - (1-\mu-\tilde{m})^2 - \mu^2 b - \frac{\tilde{m}^2}{f}. \end{aligned} \quad (161)$$

Substitute $\tilde{\lambda}$, $\tilde{\rho}$, $\mu$ in $\tilde{\lambda} = \mu(1-\tilde{\rho})$ and solve for $\nu$:

$$\nu \;=\; \frac{1-(1+b)f - bf^2}{2+b+bf} \;=\; \frac{1+f}{2+b+bf} - f. \quad (162)$$

The other equations, together with $\tilde{m} = \tilde{\lambda} - \mu\tilde{\rho}$, imply

$$\begin{aligned} \tilde{\lambda} \;&=\; \frac{1}{2+b+bf}, \quad \tilde{\rho} \;=\; \frac{f}{1+f}, \quad \mu \;=\; \frac{1+f}{2+b+bf}, \quad \tilde{m} \;=\; \frac{1-f}{2+b+bf}, \\ 1+c \;&=\; \frac{(1+bf)(1+f)^2}{f(2+b+bf)^2}. \end{aligned} \quad (163)$$

Putting together (162) and the last equation in (163), we compute

$$\beta_0 \;=\; \frac{a}{c\gamma}\,\nu \;=\; \frac{a^{1/2}}{cf^{1/2}}\,\nu \;=\; \frac{a^{1/2}}{cf^{1/2}(1+c)}\,\frac{1}{(1+f)^{1/2}}\left(c+(1-f)\frac{1+b+bf}{2+b+bf}\right). \tag{164}$$

Now substitute $a$, $b$, $c$ from (159) in Equations (163)–(164) and use $\gamma = a^{1/2}f^{1/2}$ to obtain Equations (18)–(23). One can also check that the second order conditions $\lambda+\mu\rho > 0$ and $m > 0$ are equivalent to $f \in (-1,1)$. Next, we show that the equation $1+c = \frac{(1+bf)(1+f)^2}{f(2+b+bf)^2}$ has a unique solution $f \in (-1,1)$, which in fact lies in $(0,1)$. This can be shown by noting that

$$F_b(f) \;=\; 1+c, \quad \text{with} \quad F_b(x) \;=\; \frac{(1+bx)(1+x)^2}{x(2+b+bx)^2}. \tag{165}$$

One verifies $F_b'(x) = \frac{(x+1)(x-1)(2+b+3bx)}{x^2(2+b+bx)^2}$, so $F_b(x)$ decreases on $(0,1)$. Since $F_b(0) = +\infty$ and $F_b(1) = \frac{1}{1+b} < 1$, there is a unique $f \in (0,1)$ so that $F_b(f) = 1+c$.[24]

## B.2 Proof of Proposition 1

We use the notations from the proof of Theorem 1. We start by showing that $\mu^F < \mu^B$; by computation, $\frac{1+f}{2+b+bf} < \frac{1}{1+b}$ is equivalent to $f < 1$, which is true since $f \in (0,1)$.

We show that $\lambda^F > \lambda^B$, i.e., $\frac{(c+1)^{1/2}}{a^{1/2}}\frac{1}{(1+bf)^{1/2}(1+f)} > \frac{c^{1/2}}{a^{1/2}}\left(1+\frac{b}{c(1+b)}\right)^{1/2}$. After squaring the two sides, and using $1+c = \frac{(1+bf)(1+f)^2}{f(2+b+bf)^2}$, we need to prove that $\frac{1}{f(2+b+bf)^2} > c + 1 - \frac{1}{1+b}$, or equivalently $\frac{1}{1+b} > \frac{(1+bf)(1+f)^2}{f(2+b+bf)^2} - \frac{1}{f(2+b+bf)^2}$. This can be reduced to proving $1+b+(1-f)(1+bf) > 0$, which is true.

The same type of calculations can be used to show that $\beta_0^F < \beta_0^B$, or to prove the other comparative statics.

## B.3 Proof of Proposition 2

In the benchmark model, $Var(dx_t) = (\beta_t^B)^2\Sigma_t \mathrm{d}t^2$ and $Var(du_t) = \sigma_u^2\mathrm{d}t$. Therefore, $IPR_t^B = 0$.

In the fast model, $Var(dx_t) = (\gamma_t^B)^2\sigma_v^2\mathrm{d}t$. Therefore, $IPR_t^F = (\gamma_t^B)^2\sigma_v^2/((\gamma_t^B)^2\sigma_v^2 +$

[24] One can check that $F_b(x) = 1+c$ has no solution on $(-1,0)$: When $b \le 1$, $F_b(x) < 0$ on $(-1,0)$. When $b > 1$, $F_b(x)$ attains its maximum on $(-1,0)$ at $x^* = -\frac{2+b}{3b}$, for which $F_b(x^*) = \frac{(b-1)^3}{b(b+2)^3} < 1$.

$\sigma_u^2) = f/(f+1)$, using the equation for $\gamma_t^F$ in Theorem 1.

## B.4 Proof of Proposition 3

We start with a useful preliminary result:

**Lemma 2.** *In the benchmark model and in the fast model, for all $s < u$, we have*

$$Cov(v_s - p_s, v_u - p_u) = \Sigma_s \left(\frac{1-u}{1-s}\right)^{m\beta_0}, \tag{166}$$

$$Cov(dv_s, v_u - p_u) = (1 - m\gamma - \mu)\sigma_v^2 \left(\frac{1-u}{1-s}\right)^{m\beta_0} ds, \tag{167}$$

*where $m \equiv \lambda - \mu\rho$.*

*Proof.* We start from

$$\begin{aligned} Cov(v_s - p_s, v_u - p_u) &= Cov(v_s - p_s, v_s - p_s) - \int_s^u Cov(v_s - p_s, dp_h)\, dh \\ &= \Sigma_s - \int_s^u Cov(v_s - p_s, m\beta_h(v_h - p_h))\, dh \end{aligned}$$

Differentiating with respect to $u$ we obtain

$$\frac{\partial}{\partial u} Cov(v_s - p_s, v_u - p_u) = -m\beta_u Cov(v_s - p_s, v_u - p_u),$$

which rewrites as

$$\frac{\partial}{\partial \tau} \log Cov(v_s - p_s, v_u - p_u) = -m\beta_u = -m\beta_0 \frac{1}{1-u} = m\beta_0 \frac{\partial}{\partial u} \log(1-u).$$

Integrating between $s$ and $u$ and using $Cov(v_s - p_s, v_s - p_s) = \Sigma_s$, we obtain equation (166).

Similarly, we have

$$\begin{aligned} Cov(dv_s, v_u - p_u) &= Cov(dv_s, dv_s - dp_s) - \int_s^u Cov(dv_s, dp_h)\, dh \\ &= (1 - m\gamma - \mu)\sigma_v^2 ds - \int_s^u m\beta_h Cov(dv_s, v_h - p_h)\, dh. \end{aligned}$$

Proceeding as above we obtain (167). $\square$

We can now prove Proposition 3. The formula in the benchmark model follows immediately from Equation (166). In the fast model, the auto-covariance of the order flow is of order $\mathrm{d}t^2$ while the variance is of order $\mathrm{d}t$, therefore the autocorrelation is of order $\mathrm{d}t$, which is zero in continuous time.

## B.5 Proof of Proposition 5

Follows immediately from Lemma 2.

## B.6 Proof of Lemma 1

We start from

$$\Sigma_t = Var(v_t - p_t) = Var(v_t) + Var(p_t) - 2Cov(v_t, \int_0^t dp_\tau).$$

We have $Var(v_t) = \Sigma_0 + t\sigma_v^2$. Since the price is a martingale and given that we prove in the proof of Proposition 7 that the volatility of the price is equal to the volatility of the asset value, we have $Var(p_t) = t(\Sigma_0 + \sigma_v^2)$. Finally, using that price change cannot be correlated with future innovation in asset value, we obtain equation (41).

## B.7 Proof of Proposition 7

In the benchmark model, $Var(p_{t+\mathrm{d}t}-q_t) = (\lambda^B)^2\sigma_u^2\mathrm{d}t$ and $Var(q_{t+\mathrm{d}t}-p_{t+\mathrm{d}t}) = (\mu^B)^2\sigma_v^2\mathrm{d}t$. Using the equilibrium parameter values of Theorem 1 we obtain $Var(dp_t) = \Sigma_0 + \sigma_v^2 t$.

Similarly, in the fast model, $Var(p_{t+\mathrm{d}t} - q_t) = (\lambda^F)^2((\gamma^F)^2\sigma_v^2 + \sigma_u^2)\mathrm{d}t$ and $Var(q_{t+\mathrm{d}t} - p_{t+\mathrm{d}t}) = (\mu^B)^2((1 - \rho^F\gamma^F)^2\sigma_v^2 + \sigma_e^2 + (\rho^F)^2\sigma_u^2)\mathrm{d}t$. Using the equilibrium parameter values of Theorem 1, we obtain that $Var(p_{t+\mathrm{d}t} - q_t)$ is higher than in the benchmark, $Var(q_{t+\mathrm{d}t} - p_{t+\mathrm{d}t})$ is lower than in the benchmark, and $Var(dp_t) = \Sigma_0 + \sigma_v^2 t$ is the same as in the benchmark.

## B.8 Proof of Proposition 10

When returns are computed using post-trade quotes, we have

$$r_t = \mu_{t-\Delta t}(\Delta z_{t-\Delta t} - \rho_{t-\Delta t}\Delta y_{t-\Delta t}) + \lambda_t \Delta y_t.$$

Using that $\Delta x_{t-\Delta t} \approx \gamma_{t-\Delta t}\Delta v_{t-\Delta t}$ when $\Delta t \to 0$, and that $\gamma_t$, $\lambda_t$, $\mu_t$, and $\rho_t$ are constant over time, the above equation rewrites as

$$r_t \approx \lambda \Delta x_t + \lambda \Delta u_t + \mu(1/\gamma - \rho)\Delta x_{t-\Delta t} - \mu\rho\Delta u_{t-\Delta t} + \mu\Delta e_{t-\Delta t}.$$

Similarly, when pre-trade quotes are used, we have

$$\begin{aligned} r_t &= \lambda_t \Delta y_t + \mu_t(\Delta z_t - \rho_t \Delta y_t) \\ &= (\lambda - \mu\rho + \mu/\gamma)\Delta x_t + (\lambda - \mu\rho)\Delta u_t + \mu \Delta e_t. \end{aligned}$$

## B.9 Proof of Proposition 11

In the limit $\Delta t \to 0$, we have

$$\begin{aligned} &Cov(\Delta x_j(n), r_{j+1}(n)) = \mu\gamma(1 - \rho\gamma)\sigma_v^2 \Delta t, \\ &Var(\Delta x_j) = n\gamma^2\sigma_v^2\Delta t, \\ &Var(r_{j+1}) = (\sigma_v^2 + \Sigma_0)n\Delta t. \end{aligned}$$

Therefore

$$Corr(\Delta x_j, r_{j+1}) = \frac{\mu(1-\rho\gamma)\sigma_v}{n\sqrt{\sigma_v^2 + \Sigma_0}}$$

is decreasing in $n$ and goes to 0 when $n$ goes to infinity.

## B.10 Proof of Proposition 12

We consider the limit $\Delta t \to 0$ and $n \to +\infty$ such that $n\Delta t = \tau$ is fixed. In this case, we have $Var(\Delta x_j) = Var(x_{t+\tau} - x_t)$, where $t = (j-1)\tau$. Then, we can write

$$\begin{aligned} Var(x_j) &= Var(\int_{s=t}^{t+\tau} \beta_s(v_s - p_s)ds + \gamma_s dv_s) \\ &= \int_{s=t}^{t+\tau} \gamma_s^2 Var(dv_s) \\ &\quad +2\int_{s=t}^{t+\tau}\int_{u=s}^{t+\tau} \beta_s\beta_u Cov\left(v_s - p_s, v_u - p_u\right) ds\, du \\ &\quad +2\int_{s=t}^{t+\tau}\int_{u=s}^{t+\tau} \gamma_s\beta_u Cov\left(dv_s, v_u - p_u\right) du. \end{aligned}$$

It then follows from Lemma 2 that

$$Var(x_j) = \gamma^2\sigma_v^2\tau + \beta_t(\beta_0\Sigma_0 + \gamma(1 - m\gamma - \mu)\sigma_v^2)\tau^2 + o(\tau^2)$$

when $\tau$ is small and where $m \equiv \lambda - \mu\rho$. Using that $Var(u_j) = \sigma_u^2\tau$, we obtain

$$IPR_j = \frac{\gamma^2\sigma_v^2}{\gamma^2\sigma_v^2 + \sigma_u^2} + \frac{\sigma_u^2}{(\gamma^2\sigma_v^2 + \sigma_u^2)^2}\beta_t(\beta_0\Sigma_0 + \gamma(1 - m\gamma - \mu)\sigma_v^2)\tau + o(\tau).$$

## B.11 Proof of Proposition 13

We consider the limit $\Delta t \to 0$ and $n \to +\infty$ such that $n\Delta t = \tau$ is fixed. In this case, we have

$$\begin{aligned} &Cov(\Delta x_j, \Delta x_{j+1}) = \beta_t(\beta_0\Sigma_0 + \gamma(1 - m\gamma - \mu)\sigma_v^2)\tau^2 + o(\tau^2), \\ &Var(\Delta x_j) = Var(\Delta x_{j+1}) = \gamma^2\sigma_v^2\tau + \beta_t(\beta_0\Sigma_0 + \gamma(1 - m\gamma - \mu)\sigma_v^2)\tau^2 + o(\tau^2). \end{aligned}$$

Therefore

$$Corr(\Delta x_j, \Delta x_{j+1}) = \frac{\beta_t^2\Sigma_t + \beta_t\gamma_t(1 - m\gamma - \mu)\sigma_v^2}{\gamma^2\sigma_v^2}\tau + o(\tau).$$

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