Information and Trading Targets in a Dynamic Market Equilibrium[1]

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May 3, 2015

[1]The authors thank Steve Shreve, Mihai Sirbu, and Gordan Žitković for useful discussions. The second author has been supported by the National Science Foundation under Grant No. DMS-1411809 (2014 - 2017). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation (NSF).

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ABSTRACT: This paper investigates the equilibrium interactions between trading targets and private information in a multi-period Kyle (1985) market. There are two investors who each follow dynamic trading strategies: A strategic portfolio rebalancer who engages in order splitting to reach a cumulative trading target and an unconstrained strategic insider who trades on long-lived information. We consider a baseline case in which the rebalancer is initially uninformed and also cases in which the rebalancer is initially partially informed. We derive a linear Bayesian Nash equilibrium, describe an algorithm for computing such equilibria, and present numerical results on properties of these equilibria.

KEYWORDS: Market microstructure, optimal order execution, price discovery, asymetric information, liquidity, portfolio rebalancing

AMS SUBJECT CLASSIFICATIONS: 93E20

JEL-CLASSIFICATION: G12, G11, D53

1 Introduction

Price discovery and liquidity in financial markets arise from the interactions of different investors with different information and trading motives using a variety of order execution strategies.[2] An important insight from Akerlof (1970), Grossman and Stiglitz (1980), Kyle (1985), and Glosten and Milgrom (1985) is that trading noise plays a critical role in markets subject to adverse selection when some investors trade on superior private information. However, orders from investors with non-informational reasons to trade also presumably reflect optimizing behavior such as minimizing trading costs, optimizing hedging objectives, and other portfolio structuring objectives. Moreover, while informed and uninformed investors trade differently, the opportunities available to them for how to trade are presumably similar.

Our paper is the first to model optimal dynamic trading by both informed and rebalancing investors without exogenous restrictions on information life and trading strategies. We specifically investigate a multi-period Kyle (1985) market in which there are two strategic investors with different trading motives who each follow optimal but different dynamic trading strategies. One investor is the standard strategic informed investor with long-lived information. The other investor is a strategic portfolio rebalancer who can trade over multiple rounds to minimize the cost of hitting a terminal trading target. In addition, the model has noise traders and competitive market makers. In our model, the informed investor's orders are masked by two types of trading noise over time: Independently and identically distributed noise trader orders and correlated randomness in the optimally chosen orders submitted by the rebalancer with the trading target.

Our main results are:

- Sufficient conditions for a linear Bayesian Nash equilibrium are characterized for this market.
- An algorithm for computing such equilibria numerically is provided.
- The presence of the rebalancer introduces several new features: i) the aggregate order flow is autocorrelated, ii) expected trading for the insider and rebalancer is U-shaped over time, and iii) the price impact of the order flow is S-shaped

[2] The heterogeneity of the investing public is an important fact underlying current debates about high frequency trading (SEC 2010).

with initial price impacts above those in Kyle and later price impacts below Kyle's.

- The rebalancer's trading is driven by the rebalancing target, minimizing trading costs to reach the trading target, and profiting from any private information he acquires endogenously over time through the trading process. As a result, the rebalancer sometimes buys/sells more than his ultimate target and then partially unwinds his position at the end to achieve his trading target.

Our analysis integrates two literatures on pricing and trading. The first is research on price discovery. Kyle (1985) described equilibrium pricing and dynamic trading in a market with noise traders and a single investor who has long-lived private information. Subsequent work by Holden and Subrahmanyam (1992), Foster and Viswanathan (1996), and Back, Cao, and Willard (2000) extended the model to allow for multiple informed investors with long-lived information.

A second literature studies optimal dynamic order execution for uninformed investors with trading targets. This work includes Bertsimas and Low (1998), Almgren and Chriss (1999, 2000), Gatheral and Scheid (2011), Engel, Ferstenberg, and Russell (2012) and Predoiu, Shaikhet, and Shreve (2011) on optimal dynamic order execution with trading targets and Bunnermeier and Pedersen (2005) on predatory trading in response to predictable uninformed trading. This research all takes the price impact function for orders as exogenous. In contrast, we model optimal order execution in an equilibrium setting that endogenizes the price impact of orders and that reflects, in particular, the impact of strategic uninformed trading on price impacts.[3]

Models combining both informed trading and optimized rebalancing have largely been restricted to static settings or to multi-period settings with short-lived information and/or exogenous restrictions on the rebalancer's trading strategies. Admati and Pfleider (1988) study a dynamic market consisting of a series of repeating one-period trading rounds with short-lived information and uninformed liquidity traders who only trade once but decide when to time their trading. An exception is Seppi (1990) who models an informed investor and an uninformed strategic investor with a trading target in a market in which both can trade dynamically. His model is solved

[3]In our model, order flow has a price impact due to adverse selection because of the insider's private information. Alternatively, one could model price impacts due to inventory costs and imperfect competition in liquidity provision.

for separating and partial pooling equilibria with upstairs block trading, but only for a restricted set of particular model parameterizations.

Our paper is related to Degryse, de Jong, and van Kervel (DJK 2014). Both their paper and our analysis model dynamic order splitting by an uninformed investor in a multi-period market. However, the informed investors in DJK have short-lived private information (i.e., they only have one chance to trade on high-frequency value innovations before they become public) whereas our insider can trade on long-lived information over multiple intra-day time periods. Both papers have autocorrelated (predictable) order flows because of the dynamic rebalancing. Order flow autocorrelation is empirically significant but absent in previous Kyle models.[4] However, there are several notable differences between our work and DJK. First, we show that the zero price impact of predictable orders is robust to dynamic informed trading. Thus, our rebalancer engages in "sunshine trading," using early trading to signal later trading. However, the numerical magnitude of "sunshine trading" is smaller in our setting than in DJK. This is because our informed insider can trade dynamically whereas DJK's series of informed traders are, by construction, unable to trade predictably over time. Second, our analysis is possible because we use the approach of Foster and Vishwanathan (1996) to circumvent the large state space problem mentioned in DJK. This means that our rebalancer's orders depend dynamically on the realized path of aggregate orders as well as on their rebalancing target. In contrast, the rebalancer in DJK trades deterministically over time. Third, the insider's and the rebalancer's orders interact in our model. In particular, the rebalancer can learn about the insider's information, and the insider can identify and benefit from mechanical price pressure from the rebalancer's orders. Fourth, we derive intertemporal price impacts and order flow patterns that differ from those in both Kyle and in DJK.

2 Model

We model a multi-period discrete-time market for a risky stock. A trading day is normalized to the interval $[0, 1]$ during which there are $N \in \mathbb{N}$ time points at which trade can occur where $\Delta := \frac{1}{N} > 0$ is the time step. As in Kyle (1985), the stock's true value $\tilde{v}$ becomes publicly known at time $N + 1$ after the market closes at the

[4] For empirical evidence on order flow autocorrelation, see Hasbrouck (1991a,b) and also the related empirical references in Degryse, de Jong, and van Kervel (2014).

end of the day. The value $\tilde{v}$ is normally distributed with mean zero and variance $\sigma_{\tilde{v}}^2$. Additionally, there is a money market account that pays a zero interest rate.

Four types of investors trade in our model:

1. An informed trader (who we will call the insider) knows the true stock value $\tilde{v}$ at the beginning of trading and has zero initial positions in both the stock and the money market account. The insider is risk-neutral and maximizes the expected value of her final wealth. The insider's order for the stock at time n, $n = 1, ..., N$, is denoted by $\Delta\theta_n^I$ where θ_n^I is her accumulated total stock position at time n.

2. A constrained investor needs to rebalance his portfolio by buying or selling stock to reach a terminal trading target constraint $\tilde{a}$ on his ending stock position θ_N^R by the close of the trading day. He starts the day with zero initial positions in both the stock and the money market account.[5] The target $\tilde{a}$ is jointly normally distributed with $\tilde{v}$. The variable $\tilde{a}$ has zero-mean and variance $\sigma_{\tilde{a}}^2$ and a correlation $\rho \in [0, 1]$ with the stock value $\tilde{v}$. When ρ is 0, the rebalancer is initially uninformed. However, if $\rho > 0$, then we can think of the rebalancer as being initially informed about $\tilde{v}$ but subject to random binding risk limits.[6] The rebalancer is risk-neutral and maximizes the expected value of his final wealth subject to the terminal stock position constraint. The rebalancer's order for the stock at time n, $n = 1, ..., N$, is $\Delta\theta_n^R$, and the terminal constraint requires $\Delta\theta_N^R = \tilde{a} - \theta_{N-1}^R$ at time N.

3. Noise traders submit net orders for stock at times n, $n = 1, ..., N$, that are exogenously given by Brownian motion increments Δw_n. These increments are normally distributed with zero-mean and variance $\mathbb{V}[\Delta w_n] = \sigma_w^2 \Delta$ for a constant $\sigma_w > 0$. We assume that w is independent of $\tilde{v}$ and $\tilde{a}$.

4. Competitive risk-neutral market makers observe the aggregate net order flow y_n at times n, $n = 1, ..., N$, where

$$y_n := \Delta\theta_n^I + \Delta\theta_n^R + \Delta w_n, \quad y_0 := 0. \tag{2.1}$$

[5] Both the insider and the rebalancer finance their stock trading by borrowing/lending. This assumption simplifies the notation for their objective functions but is without loss of generality.

[6] The fact that the terminal value $\tilde{v}$ is measured in dollars while the trading target $\tilde{a}$ is measured in shares is not problematic for $\tilde{v}$ and $\tilde{a}$ being correlated random variables.

Given competition and risk-neutrality, market makers clear the market (i.e., trade $-y_n$) at a stock price p_n set to be

$$p_n := \mathbb{E}[\tilde{v}|\sigma(y_1, ..., y_n)], \quad n = 1, 2, ..., N, \quad p_0 := 0, \tag{2.2}$$

where $\sigma(y_1, ..., y_n)$ is the sigma-algebra generated by the order flow history.

The constrained rebalancer's presence is the main difference between our setting and Kyle (1985) as well as the multi-agent settings in Holden and Subrahmanyam (1992) and Foster and Viswanathan (1996). As we shall see, the rebalancer's presence produces new stylized features, such as autocorrelated order flow, relative to the existing models.

Because all initial positions are assumed to be zero (i.e., $\theta_0^I = \theta_0^R = 0$), the insider chooses orders $\Delta\theta_n^I \in \sigma(\tilde{v}, y_1, ..., y_{n-1})$ at times n, $n = 1, 2, ..., N$, to maximize

$$\mathbb{E}\left[\theta_N^I(\tilde{v} - p_N) + \theta_{N-1}^I \Delta p_N + ... + \theta_1^I \Delta p_2 \Big| \sigma(\tilde{v})\right] = \mathbb{E}\left[\sum_{n=1}^N (\tilde{v} - p_n)\Delta\theta_n^I \Big| \sigma(\tilde{v})\right]. \tag{2.3}$$

On the other hand, the rebalancer faces the terminal constraint $\theta_N^R = \tilde{a}$. Therefore, he submits orders $\Delta\theta_n^R \in \sigma(\tilde{a}, y_1, ..., y_{n-1})$ at times n, $n = 1, 2, ..., N-1$, to maximize

$$\mathbb{E}\left[\tilde{a}(\tilde{v} - p_N) + \theta_{N-1}^R \Delta p_N + ... + \theta_1^R \Delta p_2 \Big| \sigma(\tilde{a})\right] = \frac{\rho\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a}^2 - \mathbb{E}\left[\sum_{n=1}^N (\tilde{a} - \theta_{n-1}^R)\Delta p_n \Big| \sigma(\tilde{a})\right], \tag{2.4}$$

given the trading target constraint $\theta_N^R = \tilde{a}$. Here the equality follows from $p_N = \sum_{n=1}^N \Delta p_n$, $p_0 = 0$, and $\mathbb{E}[\tilde{v}|\sigma(\tilde{a})] = \frac{\rho\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a}$. As proven in the appendix, the insider's problem (2.3) and the rebalancer's problem (2.4) are both quadratic optimization problems.

Definition 2.1. A *Baysian Nash* equilibrium is a collection of random variables $\{\theta_n^I, \theta_n^R, p_n\}$ such that

(i) given $\{\theta_n^R, p_n\}$, the strategy θ_n^I solves the insider's problem (2.3):

$$\max_{\substack{\Delta\theta_k^I \in \sigma(\tilde{v}, y_1, ..., y_{k-1}) \\ 1 \le k \le N}} \mathbb{E}\Big[\sum_{k=1}^N (\tilde{v} - p_k)\Delta\theta_k^I \Big| \sigma(\tilde{v})\Big], \tag{2.5}$$

(ii) given $\{\theta_n^I, p_n\}$, the strategy θ_n^R solves the rebalancer's problem (2.4):

$$\max_{\substack{\Delta\theta_k^R \in \sigma(\tilde{v}, y_1, \dots, y_{k-1}) \\ 1\le k\le N-1,\, \theta_N^R = \tilde{a}}} -\mathbb{E}\Big[\sum_{k=1}^N (\tilde{a} - \theta_{k-1}^R)\Delta p_k \Big| \sigma(\tilde{a})\Big], \tag{2.6}$$

(iii) given $\{\theta_n^I, \theta_n^R\}$, the pricing rule p_n satisfies (2.2).

To clarify this definition, we recall the Doob-Dynkin lemma: For any random variable B and any $\sigma(B)$-measurable random variable A we can find a deterministic function f such that $A = f(B)$. Therefore, we can write $\theta_n^R = f_n^R(\tilde{a}, y_1, \dots, y_{n-1})$, $\theta_n^I = f_n^I(\tilde{v}, y_1, \dots, y_n)$, and $p_n = f_n^p(y_1, \dots, y_n)$ for three deterministic functions f_n^R, f_n^I, and f_n^p. In (i), (ii), and (iii) we then mean that the functions f_n^R, f_n^I, and f_n^p are fixed whereas the random variables $y_1, ..., y_n$ vary with the controls θ^I and θ^R.

In what follows, our goal is to construct a linear Bayesian Nash equilibrium in which (i) the insider's and rebalancer's trading strategies take the forms:

$$\Delta\theta_n^R = \beta_n^R\Big(\tilde{a} - \theta_{n-1}^R\Big) + \alpha_n^R q_{n-1}, \quad \theta_0^R = 0, \tag{2.7}$$

$$\Delta\theta_n^I = \beta_n^I\Big(\tilde{v} - p_{n-1}\Big) + \alpha_n^I q_{n-1}, \quad \theta_0^I = 0, \tag{2.8}$$

where $\beta_n^R, \beta_n^I, \alpha_n^R, \alpha_n^I$, $n = 1, 2, ..., N$, are constants with $\beta_N^R = 1$ and $\alpha_N^R = 0$, and (ii) the pricing rule has the dynamics

$$\Delta p_n = \lambda_n y_n + \mu_n q_{n-1}, \quad p_0 := 0, \tag{2.9}$$

where λ_n, μ_n are constants, and (iii) where the process q_n has the dynamics

$$\Delta q_n = r_n y_n + s_n q_{n-1}, \quad q_0 := 0, \tag{2.10}$$

for constants r_n and s_n, $n = 1, 2, ..., N$. The rebalancer and insider are not restricted to use linear strategies like (2.7) and (2.8). However, we will prove that they optimally choose such strategies in the equilibrium we construct.

The rebalancer's trading target necessitates the introduction of the process q_n which is our model's main new feature. Much like p_n is a state variable giving the market maker beliefs about the stock valuation, q_n is a state variable indicating market maker beliefs about the rebalancer's remaining trading given the prior trading history.

There are two things to note about q_n. First, the rebalancer's trading is not limited to be a deterministic function of his target $\tilde{a}$. Rather, his trades can also depend on the realized prior order flow history as reflected in q_n. This is in contrast to the deterministic rebalancer trades in Degryse, de Jong, and van Kervel (2014). Second, if equations (2.7) through (2.10) define a linear Bayesian Nash equilibrium, then the same equilibrium (with the same prices and orders) is obtained if r_n and s_n are replaced with xr_n and xs_n and μ_n, α_n^L, and α_n^I are replaced with μ_n/x, α_n^R/x, and α_n^I/x for any scaler $x > 0$. Thus, in the equilibrium considered below, we normalize r_n and s_n so that q_n is the market makers' expectation of the rebalancer's remaining demand $\tilde{a} - \theta_n^R$ at time n based on the observed history of aggregate orders[7]

$$q_n = \mathbb{E}[\tilde{a} - \theta_n^R | \sigma(y_1, ..., y_n)], \quad n = 1, ..., N. \tag{2.11}$$

The term $\tilde{a} - \theta_{n-1}^R$ in (2.7) plays two roles in the rebalancer's strategy. It is the distance between the rebalancer's current position and his final trading target $\tilde{a}$, and, in equilibrium, it is also private information about possible misvaluation of the stock value $\tilde{v} - p_{n-1}$:

$$\begin{aligned} \mathbb{E}[\tilde{v} - p_{n-1} | \sigma(\tilde{a}, y_1, ..., y_{n-1})] &= \mathbb{E}[\tilde{v} - p_{n-1} | \sigma(\tilde{a} - \theta_{n-1}^R - q_{n-1}, y_1, ..., y_{n-1})] \\ &= \mathbb{E}[\tilde{v} - p_{n-1} | \sigma(\tilde{a} - \theta_{n-1}^R - q_{n-1})]. \end{aligned} \tag{2.12}$$

The first equality follows from $q_{n-1} \in \sigma(y_1, ..., y_{n-1})$ and $\theta_{n-1}^R \in \sigma(\tilde{a}, y_1, ..., y_{n-1})$. The second equality follows from the independence between $\tilde{v} - p_{n-1}$ and $y_1, ..., y_{n-1}$ as well as the independence between $\tilde{a} - \theta_{n-1}^R - q_{n-1}$ and $y_1, ..., y_{n-1}$. Thus, $\tilde{a}$ is, in general, incrementally informative about $\tilde{v}$ beyond the past order flow history. In particular, it is informative at $n > 1$ even if $\rho = 0$.

Similarly, the term $\tilde{v} - p_{n-1}$ in (2.8) plays two roles in the insider's strategy. It is both private information about the stock value and, in equilibrium, informative about the remaining demand $\tilde{a} - \theta_{n-1}^R$ for the rebalancer:

$$\begin{aligned} &\mathbb{E}[\tilde{a} - \theta_{n-1}^R | \sigma(\tilde{v}, y_1, ..., y_{n-1})] \\ &\quad = q_{n-1} + \mathbb{E}[\tilde{a} - \theta_{n-1}^R - q_{n-1} | \sigma(\tilde{v} - p_{n-1}, y_1, ..., y_{n-1})] \\ &\quad = q_{n-1} + \mathbb{E}[\tilde{a} - \theta_{n-1}^R - q_{n-1} | \sigma(\tilde{v} - p_{n-1})]. \end{aligned} \tag{2.13}$$

[7] An alternative scaling would be to set q_{n-1} equal to $\mathbb{E}[y_n | \sigma(y_1, ..., y_{n-1})]$.

The first equality follows from $q_{n-1}, p_{n-1} \in \sigma(y_1, ..., y_{n-1})$. The second equality follows from the independence between $\tilde{v} - p_{n-1}$ and $y_1, ..., y_{n-1}$ as well as the independence between $\tilde{a} - \theta^R_{n-1} - q_{n-1}$ and $y_1, ..., y_{n-1}$.

2.1 Equilibrium

In this section we characterize sufficient conditions for existence of a linear Bayesian Nash equilibrium of the form in (2.7) through (2.10). The characterization follows the logic of Foster and Viswanathan (1996) closely. Figure 1 graphically illustrates the steps we use to describe sufficient equilibrium conditions.

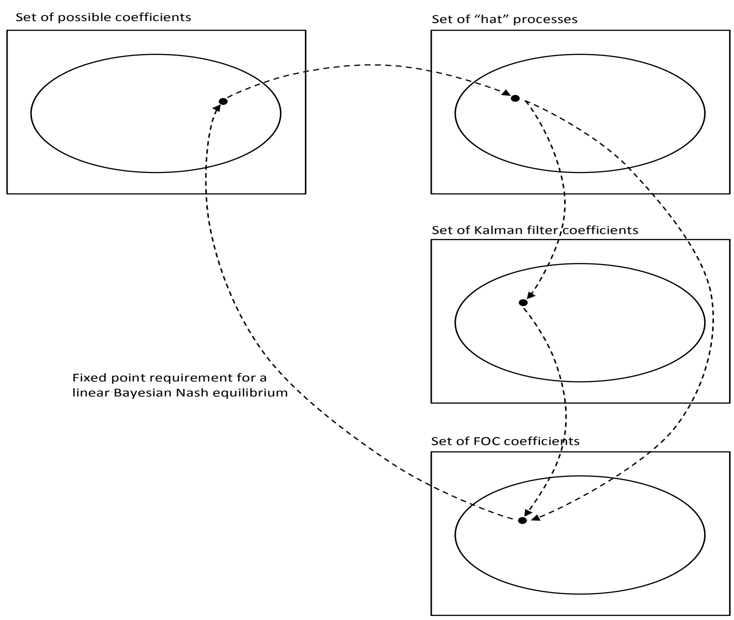


Figure 1: Venn diagrams illustrating the various constants entering the fixed point which describes the Baysian Nash equilibrium.

To begin, we consider the complete set of all possible candidate values for the equilibrium constants

$$\lambda_n, \mu_n, r_n, s_n, \beta^R_n, \alpha^R_n, \beta^I_n, \alpha^I_n, \quad n = 1, \ldots, N, \tag{2.14}$$

with

$$\beta_N^R = 1, \quad \alpha_N^R = 0. \tag{2.15}$$

The restrictions in (2.15) at date N reflect the fact that the rebalancer must achieve his target $\tilde{a}$ after his last round of trade. Our goal is to identify sufficient conditions for a candidate set of specific coefficient values to be an equilibrium. We do this in three steps.

The first step takes a set of candidate constants (2.14)-(2.15) and computes (using the terminology and notation of Foster and Viswanathan 1996) a corresponding system of "hat" price and order flow processes

$$\Delta\hat{\theta}_n^I = \beta_n^I(\tilde{v} - \hat{p}_{n-1}) + \alpha_n^I \hat{q}_{n-1}, \quad \hat{\theta}_0^I := 0, \tag{2.16}$$

$$\Delta\hat{\theta}_n^R = \beta_n^R(\tilde{a} - \hat{\theta}_{n-1}^R) + \alpha_n^R \hat{q}_{n-1}, \quad \hat{\theta}_0^R := 0, \tag{2.17}$$

$$\hat{y}_n = \Delta\hat{\theta}_n^I + \Delta\hat{\theta}_n^R + \Delta w_n, \tag{2.18}$$

$$\Delta\hat{p}_n = \lambda_n \hat{y}_n + \mu_n \hat{q}_{n-1}, \quad \hat{p}_0 := 0, \tag{2.19}$$

$$\Delta\hat{q}_n = r_n \hat{y}_n + s_n \hat{q}_{n-1}, \quad \hat{q}_0 := 0. \tag{2.20}$$

The system of processes $(\Delta\hat{p}_n, \Delta\hat{q}_n, \hat{y}_n, \Delta\hat{\theta}_n^I, \Delta\hat{\theta}_n^R)$ is fully specified (autonomous) by the coefficients (2.14)-(2.15). Furthermore, given the zero-mean and joint normality of $\tilde{v}$, $\tilde{a}$, and w, the "hat" system is also zero-mean and jointly normal. We define the variances and covariance for the "hat" dynamics, $n = 2, ..., N$, by

$$\Sigma_n^{(1)} := \mathbb{V}\big[\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}\big], \tag{2.21}$$

$$\Sigma_n^{(2)} := \mathbb{V}\big[\tilde{v} - \hat{p}_{n-1}\big], \tag{2.22}$$

$$\Sigma_n^{(3)} := \mathbb{E}\big[\big(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}\big)(\tilde{v} - \hat{p}_{n-1})\big], \tag{2.23}$$

where the initial variances and covariance at $n = 1$ are

$$\Sigma_1^{(1)} = \sigma_{\tilde{a}}^2, \quad \Sigma_1^{(2)} = \sigma_{\tilde{v}}^2, \quad \Sigma_1^{(3)} = \rho. \tag{2.24}$$

The "hat" processes will be used to make (2.3) and (2.4) analytically tractable in the sense that both the insider's problem and the rebalancer's problem can be described by a five-dimensional state process; see (2.34) and (2.37) below. In particular, the "hat" processes denote the processes that agents believe other agents believe describe the equilibrium. In equilibrium, these beliefs must be correct. This consistency requirement imposes two groups of conditions that a set of candidate constants must satisfy to be equilibrium constants. The next two steps explain these conditions.

The second step requires the coefficients, λ_n, μ_n, s_n, and r_n, of the price and order flow state variable processes to be consistent in equilibrium with Bayesian updating. In particular, if market makers believe that the insider and rebalancer are following the "hat" strategies, then we can re-write (2.2) as

$$\begin{aligned}\Delta p_n &= \lambda_n\big(y_n - \mathbb{E}[y_n|\sigma(y_1,...,y_{n-1})]\big)\\ &= \lambda_n\big(y_n - [\beta_n^R\,\mathbb{E}[\tilde{a} - \hat{\theta}_{n-1}^R|\sigma(y_1,...,y_{n-1})] + \alpha_n^R\hat{q}_{n-1} + \alpha_n^I\hat{q}_{n-1}]\big)\\ &= \lambda_n\big(y_n - (\alpha_n^I + \alpha_n^R + \beta_n^R)q_{n-1}\big),\end{aligned} \tag{2.25}$$

for $n = 1,...,N$. The first equality follows from the fact that, given the jointly Gaussian structure of the "hat" processes, conditional expectations are linear projections. The second equality follows from (i) the definition of the aggregate order flow, (ii) the fact that $\tilde{v} - \hat{p}_{n-1}$ is independent of past order flows, and (iii) the assumption that the noise trader orders are zero–mean and i.i.d. over time. The final equality follows from the assumption that in our conjectured equilibrium p_n is linear in q_{n-1} and the normalization that $q_{n-1} = \mathbb{E}[\tilde{a} - \hat{\theta}_{n-1}^R|\sigma(y_1,...,y_{n-1})]$. Comparing the last line of (2.25) with (2.9) and using the fact that λ_n equals the projection coefficient

$$\frac{\mathrm{Cov}(\tilde{v} - p_{n-1}, y_n - \mathbb{E}[y_n|\sigma(y_1,...,y_{n-1})])}{\mathbb{V}(y_n - \mathbb{E}[y_n|\sigma(y_1,...,y_{n-1})])} \tag{2.26}$$

gives restrictions on the coefficients of the price process in terms of the insider and rebalancer strategy coefficients. A similar logic can also be used to derive restrictions on the coefficients of the q_n process in terms of the investor strategy coefficients. Thus, these calculations lead to four restrictions on the state–variable and strategy constants in a linear Bayesian Nash equilibrium for $n = 1,...,N$:

Condition 2.2. In equilibrium, the price and order flow state variable coefficients depend on the insider and rebalancer order coefficients as follow:

$$\lambda_n = \frac{\beta_n^I \Sigma_n^{(2)} + \beta_n^R \Sigma_n^{(3)}}{(\beta_n^I)^2 \Sigma_n^{(2)} + (\beta_n^R)^2 \Sigma_n^{(1)} + 2\beta_n^I \beta_n^R \Sigma_n^{(3)} + \sigma_w^2 \Delta}, \tag{2.27}$$

$$r_n = \frac{(1-\beta_n^R)(\beta_n^I \Sigma_n^{(3)} + \beta_n^R \Sigma_n^{(1)})}{(\beta_n^I)^2 \Sigma_n^{(2)} + (\beta_n^R)^2 \Sigma_n^{(1)} + 2\beta_n^I \beta_n^R \Sigma_n^{(3)} + \sigma_w^2 \Delta}, \tag{2.28}$$

$$\mu_n = -(\alpha_n^I + \alpha_n^R + \beta_n^R)\lambda_n, \tag{2.29}$$

$$s_n = -(\alpha_n^I + \alpha_n^R + \beta_n^R) r_n - (\alpha_n^R + \beta_n^R), \tag{2.30}$$

where the conditional variances and covariance from (2.21)-(2.23) can now be explicitly specified as (see the proof of Lemma A.1 in Appendix A.1)

$$\Sigma_n^{(1)} = (1-\beta_{n-1}^R)\big((1-\beta_{n-1}^R - r_{n-1}\beta_{n-1}^R)\Sigma_{n-1}^{(1)} - r_{n-1}\beta_{n-1}^I \Sigma_{n-1}^{(3)}\big), \tag{2.31}$$

$$\Sigma_n^{(2)} = (1-\lambda_{n-1}\beta_{n-1}^I)\Sigma_{n-1}^{(2)} - \lambda_{n-1}\beta_{n-1}^R \Sigma_{n-1}^{(3)}, \tag{2.32}$$

$$\Sigma_n^{(3)} = (1-\beta_{n-1}^R)\big((1-\lambda_{n-1}\beta_{n-1}^I)\Sigma_{n-1}^{(3)} - \lambda_{n-1}\beta_{n-1}^R \Sigma_{n-1}^{(1)}\big). \tag{2.33}$$

$\diamondsuit$

Note that Condition 2.2 has a "block" structure. The variances and covariance $\Sigma_n^{(1)}$, $\Sigma_n^{(2)}$, and $\Sigma_n^{(3)}$ at time n just depend on prior coefficients and prior variances and covariance from time $n-1$. The values of λ_n and r_n just depend on the β_n^R and β_n^I strategy coefficients and the variances and covariance at time n (along with the exogenous noise trading variance σ_w^2). Lastly, μ_n and s_n depend on λ_n and μ_n and the full set of strategy coefficients at time n.

The third step begins by deriving value functions for the optimization problems for the two strategic investors. Consider first the insider at a generic time n. Given her trades $\Delta\theta_1^I, \dots, \Delta\theta_n^I$ — which need not be consistent with the candidate "hat" dynamics — the insider not only knows the final stock value $\tilde{v}$, but also the extent to which the actual prices and rebalancer's order flow at date n deviate from the values they would have had if she had instead traded as in the candidate "hat" processes. Thus, the state variables at date n for the insider's value function are

$$X_n^{(1)} := \tilde{v} - \hat{p}_n,\ X_n^{(2)} := \hat{q}_n,\ X_n^{(3)} := \hat{\theta}_n^R - \theta_n^R,\ X_n^{(4)} := \hat{q}_n - q_n,\ X_n^{(5)} := \hat{p}_n - p_n. \tag{2.34}$$

Here the "un-hatted" variables are the variable values given her actual (potentially arbitrary) orders, see (2.9) and (2.10), whereas the "hat" variables are not affected by actual orders. When the rebalancer's strategy is taken to be fixed by (2.7), it is characterized by the two sequences of candidate constants $\beta_1^R, \ldots, \beta_N^R$ and $\alpha_1^R, \ldots, \alpha_N^R$. However, even though the rebalancer's strategy is fixed, its realizations are subject to the insider's choice of control θ^I since the aggregate order flow affects the rebalancer's actual orders. Thus, the state variable $X_n^{(3)}$ measures the effect of the insider's actual orders on the rebalancer's actual orders. A similar interpretation applies to $X_n^{(4)}$ and $X_n^{(5)}$ and the order flow state variable q_n and prices p_n. In equilibrium, we will see that the three deviation state variables $X^{(3)}, X^{(4)}$, and $X^{(5)}$ are zero. However, in deriving the equilibrium, we need to allow for the possibility of past suboptimal play.

We show (see the appendix) that the insider's value function for $n = 0, 1, ..., N$ has a quadratic form

$$\max_{\substack{\Delta\theta_k^I \in \sigma(\tilde{v}, y_1, \ldots, y_{k-1}) \\ n+1 \leq k \leq N}} \mathbb{E}\Big[\sum_{k=n+1}^{N} (\tilde{v} - p_k)\Delta\theta_k^I \Big| \sigma(\tilde{v}, y_1, ..., y_n)\Big] = I_n^{(0)} + \sum_{1 \leq i \leq j \leq 5} I_n^{(i,j)} X_n^{(i)} X_n^{(j)}, \tag{2.35}$$

where $I_n^{(0)}$ and $I_n^{(i,j)}$ are coefficients computed recursively from the candidate coefficients (2.14)-(2.15). We use the Bellman principle to derive the value function coefficients at time n (i.e., $I_n^{(0)}$ and $I_n^{(i,j)}$) in terms of the value function coefficients at time $n + 1$, which, in turn, depend on the strategy and pricing coefficients at times $n + 1, \ldots, N$. The next section describes this recursion in detail.

Similarly, when the insider's strategy is given by (2.8), an analogous argument can be used to derive a quadratic value function for the rebalancer

$$\max_{\substack{\Delta\theta_k^R \in \sigma(\tilde{a}, y_1, \ldots, y_{k-1}) \\ n+1 \leq k \leq N-1}} -\mathbb{E}\Big[\sum_{k=n+1}^{N} (\tilde{a} - \theta_{k-1}^R)\Delta p_k \Big| \sigma(\tilde{a}, y_1, ..., y_n)\Big] = L_n^{(0)} + \sum_{1 \leq i \leq j \leq 5} L_n^{(i,j)} Y_n^{(i)} Y_n^{(j)}. \tag{2.36}$$

Here the state variables are

$$Y_n^{(1)} := \tilde{a} - \hat{\theta}_n^R,\ Y_n^{(2)} := \hat{q}_n,\ Y_n^{(3)} := \hat{\theta}_n^R - \theta_n^R,\ Y_n^{(4)} := \hat{q}_n - q_n,\ Y_n^{(5)} := \hat{p}_n - p_n, \tag{2.37}$$

given a prior sequence of (potentially off-equilibrium) trades $\Delta\theta_1^R, \ldots, \Delta\theta_n^R$. The coefficients $L_n^{(0)}$ and $L_n^{(i,j)}$ are again computed recursively from the candidate coefficients (2.14) and (2.15). In equilibrium, the deviation state variables $Y^{(3)}, Y^{(4)}$, and $Y^{(5)}$ are again zero.

The first– and second–order conditions for the insider's and rebalancer's maximization problems, given the insider's and rebalancer's value functions, lead to a group of equilibrium restrictions on the investor strategy coefficients β_n^I, β_n^R, α_n^I, and α_n^R in terms of the price and order flow state variable coefficients.

Condition 2.3. Given the conditional variances $\Sigma_N^{(1)}$ and $\Sigma_N^{(2)}$ and covariance $\Sigma_N^{(3)}$ at time N from (2.21)-(2.23), the candidate constants satisfy (recall that $\beta_N^R = 1$)

$$\beta_N^I = \Big(\frac{1}{2\lambda_N} - \frac{\Sigma_N^{(3)}}{2\Sigma_N^{(2)}}\Big), \quad \lambda_N > 0. \tag{2.38}$$

In addition, for $n = 1, \ldots, N-1$, given the variances and covariance $\Sigma_n^{(1)}, \Sigma_n^{(2)}, \Sigma_n^{(3)}$, the candidate coefficients in (2.14)-(2.15) solve the following four polynomial equations:

$$\begin{aligned} 2\lambda_n\beta_n^I &= -(r_n I_n^{(1,4)} + \lambda_n I_n^{(1,5)})\big(1 - \lambda_n(\beta_n^I + \frac{\beta_n^R\Sigma_n^{(3)}}{\Sigma_n^{(2)}})\big) \\ &\quad - (r_n I_n^{(2,4)} + \lambda_n I_n^{(2,5)}) r_n(\beta_n^I + \frac{\beta_n^R\Sigma_n^{(3)}}{\Sigma_n^{(2)}}) + 1 - \frac{\lambda_n\beta_n^R\Sigma_n^{(3)}}{\Sigma_n^{(2)}}, \end{aligned} \tag{2.39}$$

$$\begin{aligned} 0 &= \lambda_n + (1 - \beta_n^R)\big(L_n^{(1,3)} + r_n L_n^{(1,4)} + \lambda_n L_n^{(1,5)}\big) \\ &\quad + r_n\big(\beta_n^R + \frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}\big)\big(L_n^{(2,3)} + r_n L_n^{(2,4)} + \lambda_n L_n^{(2,5)}\big), \end{aligned} \tag{2.40}$$

$$\alpha_n^R = -\frac{\big(L_n^{(2,3)} + r_n L_n^{(2,4)} + \lambda_n L_n^{(2,5)}\big)\big(r_n\beta_n^R + r_n\frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}} + \beta_n^R - 1\big)}{L_n^{(1,3)} + r_n L_n^{(1,4)} + \lambda_n L_n^{(1,5)} + L_n^{(2,3)} + r_n L_n^{(2,4)} + \lambda_n L_n^{(2,5)}}, \tag{2.41}$$

$$\alpha_n^I = \frac{(r_n I_n^{(2,4)} + \lambda_n I_n^{(2,5)})(\alpha_n^R + \beta_n^R - 1)}{\lambda_n}, \tag{2.42}$$

where the I and L terms are from the two investor value functions (2.35) and (2.36). The candidate constants and the value function terms must also satisfy the two inequalities:

$$
\begin{aligned}
&-\lambda_n + I_n^{(4,5)}\lambda_n r_n + I_n^{(4,4)} r_n^2 + I_n^{(5,5)}\lambda_n^2 < 0, && (2.43)\\
&L_n^{(3,3)} + L_n^{(3,4)} r_n + L_n^{(3,5)}\lambda_n + L_n^{(4,4)} r_n^2 + L_n^{(5,5)}\lambda_n^2 + L_n^{(4,5)} r_n\lambda_n < 0, && (2.44)
\end{aligned}
$$

which (as we show in the appendix) ensure sufficiency of the first-order-conditions.

$\diamondsuit$

Here again, the equilibrium restrictions have a "block" structure in that α_n^I and α_n^R in (2.41) and (2.42) depend on β_n^I and β_n^R, whereas the linear equations (2.39) and (2.40) – which are the first-order conditions for the insider and rebalancer at date n – can be solved to express β_n^I and β_n^R in terms of the current updating coefficients and value function coefficients (which only depend on the updating and strategy coefficients at later dates) but not on α_n^I and α_n^R.

Our main theoretical result is the following:

Theorem 2.4. *If the constants* (2.14) *and* (2.15) *satisfy Conditions 2.2 and 2.3, then a linear Bayesian Nash equilibrium exists of the form given in equations* (2.7)-(2.10)*. Furthermore, we have*

$$
\begin{aligned}
&\alpha_n^I = I_n^{(1,2)} = I_n^{(2,2)} = I_n^{(2,3)} = I_n^{(2,4)} = I_n^{(2,5)} = 0 \quad \textit{for} \quad n = 1, 2, ..., N,\\
&r_N = 0, \quad \mu_N = -\lambda_N, \quad s_N = -1.
\end{aligned}
\tag{2.45}
$$

The new feature in our model, compared to Foster and Viswanathan (1996) and Kyle (1985), is the presence of the q_n process in the equilibrium price dynamics (2.9). This produces new stylized features including autocorrelation of the equilibrium aggregate order flow:

$$
\begin{aligned}
&\mathbb{E}[y_n|\sigma(y_1, ..., y_{n-1})]\\
&\quad = \mathbb{E}[\Delta\theta_n^I + \Delta\theta_n^R + \Delta w_n|\sigma(y_1, ..., y_{n-1})]\\
&\quad = \alpha_n^R\, q_{n-1} + \mathbb{E}[\beta_n^I(\tilde{v} - p_{n-1}) + \beta_n^R(\tilde{a} - \theta_{n-1}^R)|\sigma(y_1, ..., y_{n-1})]\\
&\quad = (\alpha_n^R + \beta_n^R)\, q_{n-1},
\end{aligned}
\tag{2.46}
$$

which, in general, is not zero. The second equality uses the fact from Theorem 2.4 that $\alpha_n^I = 0$. The last equality follows, in part, from the earlier observation that, in equilibrium q_{n-1} is the conditional expectation of $\tilde{a} - \theta_{n-1}^R$ given the prior trading history.

2.2 Algorithm

This section describes an algorithm for searching numerically for a linear Bayesian Nash equilibrium. The algorithm is similar in logic to the algorithm in Section V in Foster and Viswanathan (1996), except that our algorithm requires three constants as inputs (due to the presence of two strategic agents) whereas Foster and Viswanathan (1996) only requires one constant as an input.

To describe the algorithm we assume that the conclusions of Lemma A.2 and Lemma A.4 (see the appendix) are valid. The algorithm starts by taking as inputs three conjectured constants:

$$\Sigma_N^{(1)} > 0, \quad \Sigma_N^{(2)} > 0, \quad \Sigma_N^{(3)} \in \mathbb{R} \quad \text{such that} \quad \big(\Sigma_N^{(3)}\big)^2 \leq \Sigma_N^{(1)}\Sigma_N^{(2)}, \tag{2.47}$$

and proceeds through backward induction.[8]

Starting step for time N: We need λ_N and β_N^I to satisfy (2.27) for $n = N$ and (2.38). Given those two constants, we can define the constants

$$\beta_N^R := 1, \quad \alpha_N^R := \alpha_N^I := r_N := 0, \quad \mu_N := -\lambda_N, \quad s_N := -1. \tag{2.48}$$

Based on Lemma A.2 and Lemma A.4 we can then let $I_{N-1}^{(0)}, I_{N-1}^{(i,j)}, L_{N-1}^{(0)}$, and $L_{N-1}^{(i,j)}$, $1 \leq i \leq j \leq 5$, be the coefficients appearing in the two representations:

$$\max_{\Delta\theta_N^I} \mathbb{E}\Big[(\tilde{v} - p_N)\Delta\theta_N^I \Big| \sigma(\tilde{v}, y_1, ..., y_{N-1})\Big] = I_{N-1}^{(0)} + \sum_{1\leq i\leq j\leq 5} I_{N-1}^{(i,j)} X_{N-1}^{(i)} X_{N-1}^{(j)}, \tag{2.49}$$

$$\mathbb{E}\Big[- (\tilde{a} - \theta_{N-1}^R)\Delta p_N \Big| \sigma(\tilde{a}, y_1, ..., y_{N-1})\Big] = L_{N-1}^{(0)} + \sum_{1\leq i\leq j\leq 5} L_{N-1}^{(i,j)} Y_{N-1}^{(i)} Y_{N-1}^{(j)}. \tag{2.50}$$

Induction step: At each time n the algorithm takes the following constants as input:

$$\Sigma_{n+1}^{(1)}, \Sigma_{n+1}^{(2)}, \Sigma_{n+1}^{(3)}, I_n^{(0)}, (I_n^{(i,j)})_{1\leq i\leq j\leq 5}, L_n^{(0)}, (L_n^{(i,j)})_{1\leq i\leq j\leq 5}. \tag{2.51}$$

Given these constants, $(\lambda_n, r_n, \beta_n^I, \beta_n^R, \Sigma_n^{(1)}, \Sigma_n^{(2)}, \Sigma_n^{(3)})$ must satisfy (2.27)-(2.28), (2.39)-(2.40), and (2.31)-(2.33). This gives a system of seven polynomial equations in

[8] $\Sigma^{(2)}$ must be non-increasing over time (as in Kyle 1985) but $\Sigma^{(1)}$ might not be.

seven unknown constants. Given a solution to these seven equations, we obtain $(\mu_n, s_n, \alpha_n^I, \alpha_n^R)$ from (2.29), (2.30), (2.41), and (2.42).

Next, to compute the coefficients in the value functions at time $n-1$; that is,

$$I_{n-1}^{(0)}, (I_{n-1}^{(i,j)})_{1\le i\le j\le 5}, L_{n-1}^{(0)}, (L_{n-1}^{(i,j)})_{1\le i\le j\le 5}, \tag{2.52}$$

we consider the following two optimization problems:

$$\max_{\Delta\theta_n^I} \mathbb{E}\Big[(\tilde{v}-p_n)\Delta\theta_n^I + I_n^{(0)} + \sum_{1\le i\le j\le 5} I_n^{(i,j)} X_n^{(i)} X_n^{(j)} \Big| \sigma(\tilde{v}, y_1, ..., y_{n-1})\Big], \tag{2.53}$$

$$\max_{\Delta\theta_n^R} \mathbb{E}\Big[-(\tilde{a}-\theta_{n-1}^R)\Delta p_n + L_n^{(0)} + \sum_{1\le i\le j\le 5} L_n^{(i,j)} Y_n^{(i)} Y_n^{(j)} \Big| \sigma(\tilde{a}, y_1, ..., y_{n-1})\Big]. \tag{2.54}$$

According to Lemma A.2, the insider's problem (2.53) is quadratic in $\Delta\theta_n^I$ whereas Lemma A.4 ensures that the rebalancer's problem (2.54) is quadratic in $\Delta\theta_n^R$. The first-order-condition produces the candidate optimizer for the insider's order $\Delta\theta_n^I$

$$\sum_{i=1}^{5} \gamma_n^{(i)} X_{n-1}^{(i)}, \quad n = 1, ..., N, \tag{2.55}$$

where

$$\gamma_n^{(1)} := \frac{1}{2\Big(\lambda_n - I_n^{(4,5)}\lambda_n r_n - I_n^{(4,4)} r_n^2 - I_n^{(5,5)}\lambda_n^2\Big)}\Big(- (r_n I_n^{(1,4)} + \lambda_n I_n^{(1,5)})\big(1 - \lambda_n(\beta_n^I + \frac{\beta_n^R \Sigma_n^{(3)}}{\Sigma_n^{(2)}})\big)$$

$$- (r_n I_n^{(2,4)} + \lambda_n I_n^{(2,5)}) r_n (\beta_n^I + \frac{\beta_n^R \Sigma_n^{(3)}}{\Sigma_n^{(2)}}) \tag{2.56}$$

$$- 2\beta_n^I\big(\lambda_n r_n I_n^{(4,5)} + r_n^2 I_n^{(4,4)} + \lambda_n^2 I_n^{(5,5)}\big) + 1 - \frac{\lambda_n \beta_n^R \Sigma_n^{(3)}}{\Sigma_n^{(2)}}\Big),$$

$$\gamma_n^{(2)} := \frac{1}{2\Big(\lambda_n - I_n^{(4,5)}\lambda_n r_n - I_n^{(4,4)} r_n^2 - I_n^{(5,5)}\lambda_n^2\Big)}\Big((r_n I_n^{(1,4)} + \lambda_n I_n^{(1,5)})\big(\lambda_n(\alpha_n^I + \alpha_n^R + \beta_n^R) + \mu_n\big)$$

$$- (r_n I_n^{(2,4)} + \lambda_n I_n^{(2,5)})\big(1 + r_n(\alpha_n^I + \alpha_n^R + \beta_n^R) + s_n\big) \tag{2.57}$$

$$- 2\alpha_n^I\big(\lambda_n r_n I_n^{(4,5)} + r_n^2 I_n^{(4,4)} + \lambda_n^2 I_n^{(5,5)}\big) - \big(\lambda_n(\alpha_n^R + \beta_n^R) + \mu_n\big)\Big),$$

$$\gamma_n^{(3)} := \frac{-(r_n I_n^{(3,4)} + \lambda_n I_n^{(3,5)})(1 - \beta_n^R) + 2\beta_n^R\Big(\lambda_n r_n I_n^{(4,5)} + r_n^2 I_n^{(4,4)} + \lambda_n^2 I_n^{(5,5)}\Big) - \lambda_n \beta_n^R}{2\Big(\lambda_n - I_n^{(4,5)}\lambda_n r_n - I_n^{(4,4)} r_n^2 - I_n^{(5,5)}\lambda_n^2\Big)}, \tag{2.58}$$

$$\gamma_n^{(4)} := \frac{1}{2\Big(\lambda_n - I_n^{(4,5)}\lambda_n r_n - I_n^{(4,4)} r_n^2 - I_n^{(5,5)}\lambda_n^2\Big)}\Big(- (r_n I_n^{(3,4)} + \lambda_n I_n^{(3,5)})\alpha_n^R$$

$$- 2\big(\lambda_n r_n I_n^{(4,5)} + r_n^2 I_n^{(4,4)} + \lambda_n^2 I_n^{(5,5)}\big)\alpha_n^R - I_n^{(4,5)}(\lambda_n(s_n + 1) + r_n \mu_n) \tag{2.59}$$

$$- 2 r_n I_n^{(4,4)}(s_n + 1) - 2\lambda_n I_n^{(5,5)}\mu_n + (\lambda_n \alpha_n^R + \mu_n)\Big),$$

$$\gamma_n^{(5)} := \frac{1 - I_n^{(4,5)} r_n - 2\lambda_n I_n^{(5,5)}}{2\Big(\lambda_n - I_n^{(4,5)}\lambda_n r_n - I_n^{(4,4)} r_n^2 - I_n^{(5,5)}\lambda_n^2\Big)}. \tag{2.60}$$

Furthermore, the second-order condition (2.43) ensures that this candidate optimizer indeed maximizes the insider's objective. As an aside, we note that (2.39) and (2.42) come from (2.56) and (2.57) when the equilibrium conditions $\gamma_n^{(1)} = \beta_n^I$ and $\gamma_n^{(2)} = \alpha_n^I$ are imposed.

Similarly, the candidate optimizer for the rebalancer's order $\Delta\theta_n^R$ at time n is

$$\sum_{i=1}^{5} \delta_n^{(i)} Y_{n-1}^{(i)}, \quad n = 1, ..., N, \tag{2.61}$$

where

$$\delta_n^{(1)} := \frac{1}{2\Big(L_n^{(3,3)}+L_n^{(3,4)}r_n+L_n^{(3,5)}\lambda_n+L_n^{(4,4)}r_n^2+L_n^{(5,5)}\lambda_n^2+L_n^{(4,5)}r_n\lambda_n\Big)}\times \Big(\lambda_n + (1-\beta_n^R)\big(L_n^{(1,3)} + r_n L_n^{(1,4)} + \lambda_n L_n^{(1,5)}\big) + r_n\big(\beta_n^R + \frac{\beta_n^I \Sigma_n^{(3)}}{\Sigma_n^{(1)}}\big)\big(L_n^{(2,3)} + r_n L_n^{(2,4)} + \lambda_n L_n^{(2,5)}\big) + 2\beta_n^R\big(L_n^{(3,3)} + L_n^{(3,4)}r_n + L_n^{(3,5)}\lambda_n + L_n^{(4,4)}r_n^2 + L_n^{(5,5)}\lambda_n^2 + L_n^{(4,5)}r_n\lambda_n\big)\Big), \tag{2.62}$$

$$\delta_n^{(2)} := -\frac{1}{2\Big(L_n^{(3,3)}+L_n^{(3,4)}r_n+L_n^{(3,5)}\lambda_n+L_n^{(4,4)}r_n^2+L_n^{(5,5)}\lambda_n^2+L_n^{(4,5)}r_n\lambda_n\Big)}\times \Big(\alpha_n^R\big(L_n^{(1,3)} + r_n L_n^{(1,4)} + \lambda_n L_n^{(1,5)}\big) - \big(r_n(\alpha_n^I + \alpha_n^R - \frac{\beta_n^I \Sigma_n^{(3)}}{\Sigma_n^{(1)}}) + s_n + 1\big)\big(L_n^{(2,3)} + r_n L_n^{(2,4)} + \lambda_n L_n^{(2,5)}\big) - 2\alpha_n^R\big(L_n^{(3,3)} + L_n^{(3,4)}r_n + L_n^{(3,5)}\lambda_n + L_n^{(4,4)}r_n^2 + L_n^{(5,5)}\lambda_n^2 + L_n^{(4,5)}r_n\lambda_n\big)\Big), \tag{2.63}$$

$$\delta_n^{(3)} := \frac{\lambda_n + 2L_n^{(3,3)} + L_n^{(3,4)}r_n + L_n^{(3,5)}\lambda_n}{2\Big(L_n^{(3,3)}+L_n^{(3,4)}r_n+L_n^{(3,5)}\lambda_n+L_n^{(4,4)}r_n^2+L_n^{(5,5)}\lambda_n^2+L_n^{(4,5)}r_n\lambda_n\Big)}, \tag{2.64}$$

$$\delta_n^{(4)} := \frac{(L_n^{(3,4)}+2L_n^{(4,4)}r_n+L_n^{(4,5)}\lambda_n)(r_n\alpha_n^I+s_n+1)+(L_n^{(3,5)}+2L_n^{(5,5)}\lambda_n+L_n^{(4,5)}r_n)(\lambda_n\alpha_n^I+\mu_n)}{2\Big(L_n^{(3,3)}+L_n^{(3,4)}r_n+L_n^{(3,5)}\lambda_n+L_n^{(4,4)}r_n^2+L_n^{(5,5)}\lambda_n^2+L_n^{(4,5)}r_n\lambda_n\Big)}, \tag{2.65}$$

$$\delta_n^{(5)} := -\frac{\beta_n^I\Big(L_n^{(3,4)}r_n+L_n^{(3,5)}\lambda_n+2L_n^{(4,4)}r_n^2+2L_n^{(5,5)}\lambda_n^2+2L_n^{(4,5)}r_n\lambda_n\Big)-L_n^{(3,5)}-2\lambda_n L_n^{(5,5)}-r_n L_n^{(4,5)}}{2\Big(L_n^{(3,3)}+L_n^{(3,4)}r_n+L_n^{(3,5)}\lambda_n+L_n^{(4,4)}r_n^2+L_n^{(5,5)}\lambda_n^2+L_n^{(4,5)}r_n\lambda_n\Big)}. \tag{2.66}$$

Again, (2.44) ensures that this candidate optimizer indeed maximizes the rebalancer's objective. We also note that (2.40) and (2.41) come from (2.62) and (2.63) when the equilibrium conditions $\delta_n^{(1)} = \beta_n^R$ and $\delta_n^{(2)} = \alpha_n^R$ are imposed.

The value function constants for time $n-1$ are then found by inserting the optimal strategies (2.55) and (2.61) into the two problems (2.53) and (2.54) and then matching coefficients.

Termination: The iteration above is continued back to time $n = 1$. If the resulting values at $n = 1$ satisfy

$$\Sigma_1^{(1)} = \sigma_{\tilde{a}}^2, \quad \Sigma_1^{(2)} = \sigma_{\tilde{v}}^2, \quad \Sigma_1^{(3)} = \rho, \tag{2.67}$$

the algorithm terminates and the computed coefficients produce a linear Bayesian Nash equilibrium. Otherwise, we adjust the conjectured starting input values in (2.47) and start the algorithm all over.

3 Numerical results

As is common with multi-period Kyle-type models, we do not have analytic comparative results about the properties of our model. However, we have conducted a variety of numerical experiments to illustrate properties of the model. The baseline specification for our model has $N = 10$ rounds of trading, the variance of the terminal stock value $\tilde{v}$ is normalized to $\sigma_{\tilde{v}}^2 = 1$, the total variance of the Brownian motion noise trading order flow over the N periods is fixed at $\sigma_w^2 = 4$, the variance of the trading target $\tilde{a}$ is $\sigma_{\tilde{a}}^2 = 1$, and the correlation between the trading target $\tilde{a}$ and the terminal stock value $\tilde{v}$ is $\rho = 0$. In our analysis, we vary the correlation ρ and the variance of the trading target $\sigma_{\tilde{a}}^2$.

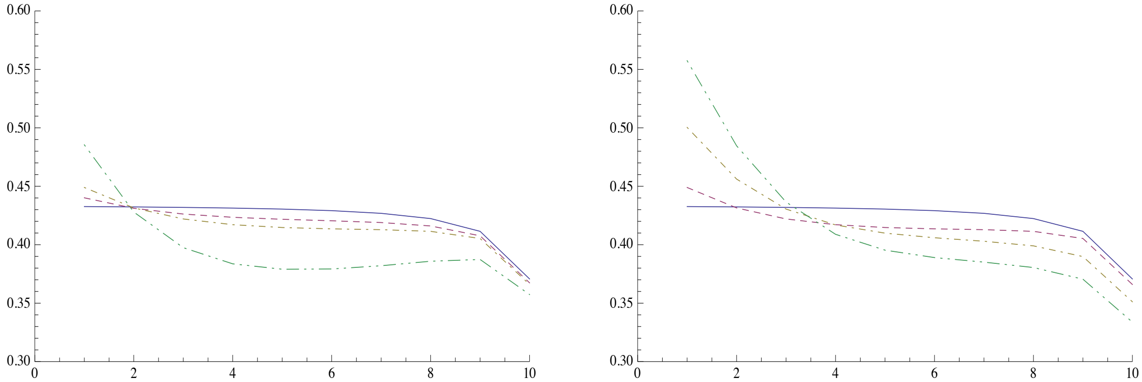
The two graphs in Figure 2 show the price impact of order flow parameter λ_n over time. The various dashed lines are for different parameterizations of our model. For comparison, the solid (blue) line is the corresponding price impact in Kyle (1985) in which the rebalancer is absent. In the first round of trading at time $n = 1$, rebalancing noise by itself would reduce the value of λ_1 relative to Kyle (1985). However, in equilibrium, the insider's trading strategy also changes. The net effect in this example is that λ_1 increases relative to Kyle (1985).[9] At later times $n > 2$, the price impacts are lower than in Kyle. The result is an S-shaped twist in λ_n over time. The price impact trajectory in our model also differs from Degryse, de Jong, and van Kervel (2014) in which price impacts have an inverted U-shape (see their Figure 1).

Figure 2A varies the variance of the trading target $\sigma_{\tilde{a}}^2$ while holding ρ fixed at 0. We see that the S-shaped twist in λ_n becomes stronger for larger values of $\sigma_{\tilde{a}}^2$. When $\sigma_{\tilde{a}}^2$ is high enough, the price impact of order flow can even be non-monotone over time (see the dashed line corresponding to $\sigma_{\tilde{a}}^2 = 3.7$, which is comparable to the total daily noise trader order variance $\sigma_w^2 = 4$). Figure 2B varies the correlation ρ between the terminal stock value $\tilde{v}$ and the trading target $\tilde{a}$ holding the variance $\sigma_{\tilde{a}}^2$ fixed at 1. Here again, there is an asymmetric impact of ρ over time relative to our baseline model with $\rho = 0$. At early times, λ_n is increasing in the correlation ρ, but at later times, λ_n is decreasing in ρ. This is because increasing ρ changes some rebalancing trades from noise into informative order flow.

Figure 3 shows the trajectory of the variance $\Sigma_n^{(2)}$ of $\tilde{v} - p_{n-1}$ over time where p_n are

[9] From equation (2.27) we see that λ_n is non-monotone in the aggressiveness of informed trading. Thus, there may also be parameterizations for which our model produces an inverted U-shape for λ_n.

Figure 2: Plot of $(\lambda_n)_{n=1}^N$ for the parameters $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 4$, $N = 10$, $\sigma_{\tilde{a}}^2 = 1$ (right only), and $\rho = 0$ (left only).

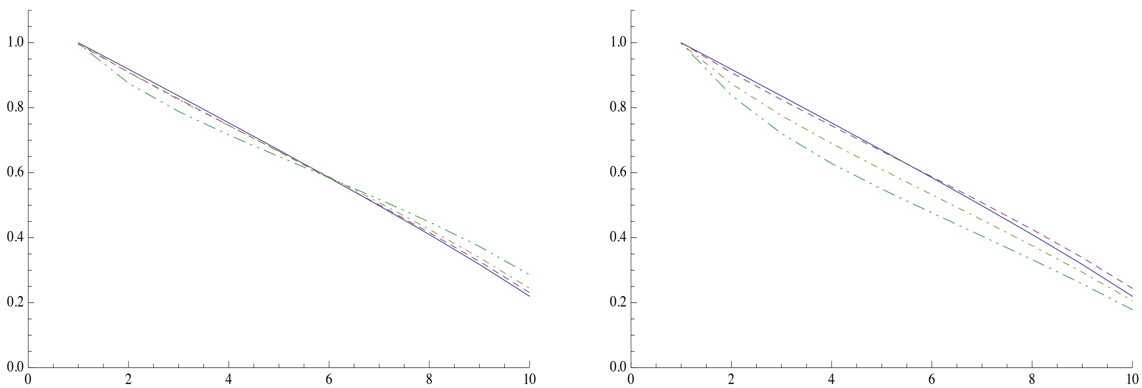


A: Kyle (———), $\sigma_{\tilde{a}}^2 = 0.48$ (− − −), $\sigma_{\tilde{a}}^2 = 1$ (− · − · −), $\sigma_{\tilde{a}}^2 = 3.7$ (− · · − · · −).

B: Kyle (———), $\rho = 0$ (− − −), $\rho = 0.25$ (− · − · −), $\rho = 0.47$ (− · · − · · −).

the equilibrium prices. In our baseline case where $\rho = 0$, there is faster information revelation at early times, but slower information revelation later towards the end. When $\rho > 0$, public uncertainty about $\tilde{v}$ falls faster in our model than in Kyle's model. This is because, with $\rho > 0$, the rebalancer also trades, from the beginning, on information about the stock value.

Figure 3: Plot of $(\Sigma_n^{(2)})_{n=1}^N$ for the parameters $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 4$, $N = 10$, $\sigma_{\tilde{a}}^2 = 1$ (right only), and $\rho = 0$ (left only).

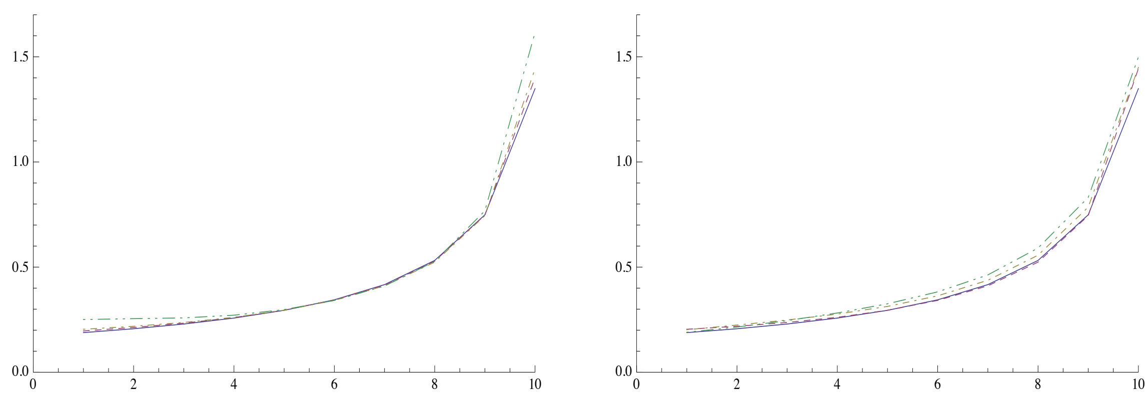


A: Kyle (———), $\sigma_{\tilde{a}}^2 = 0.48$ (− − −), $\sigma_{\tilde{a}}^2 = 1$ (− · − · −), $\sigma_{\tilde{a}}^2 = 3.7$ (− · · − · · −).

B: Kyle (———), $\rho = 0$ (− − −), $\rho = 0.25$ (− · − · −), $\rho = 0.47$ (− · · − · · −).

Figure 4 shows the insider's strategy coefficients β_n^I, which measures how aggressively she trades on her private information $\tilde{v} - p_{n-1}$ over time.[10] As in Kyle, the intensity of informed trading in our model also increases as time approaches the terminal time N. This is consistent with the fact that the price impact of order flow λ_n in Figure 2 shrinks as time passes. We also see that as the variance of the trading target $\sigma_{\tilde{a}}^2$ increases, the informed investor trades more aggressively at early dates, less so in the middle, and then slightly more aggressively again towards the end. The informed trader's increased initial aggressiveness reflects the fact that there is more noise, due to the rebalancer's trading target $\tilde{a}$, in which to hide the insider's orders. In addition, if $\rho > 0$, insider trading aggressiveness increases somewhat due to a Holden-Subrahmanyam race–to–trade competition effect. The apparent size of the changes in β_1^I – which are on the order of 10 percent – are visually understated in Figure 4 because of the vertical scaling (due to the size of β_{10}^I). In the next figure, we will see that the impact of these changes on order size is nontrivial.[11]

Figure 4: Plot of $(\beta_n^I)_{n=1}^N$ for the parameters $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 4$, $N = 10$, $\sigma_{\tilde{a}}^2 = 1$ (right only), and $\rho = 0$ (left only).



A: Kyle (———), $\sigma_{\tilde{a}}^2 = 0.48$ (− − −), $\sigma_{\tilde{a}}^2 = 1$ (− · − · −), $\sigma_{\tilde{a}}^2 = 3.7$ (− · · − · · −).

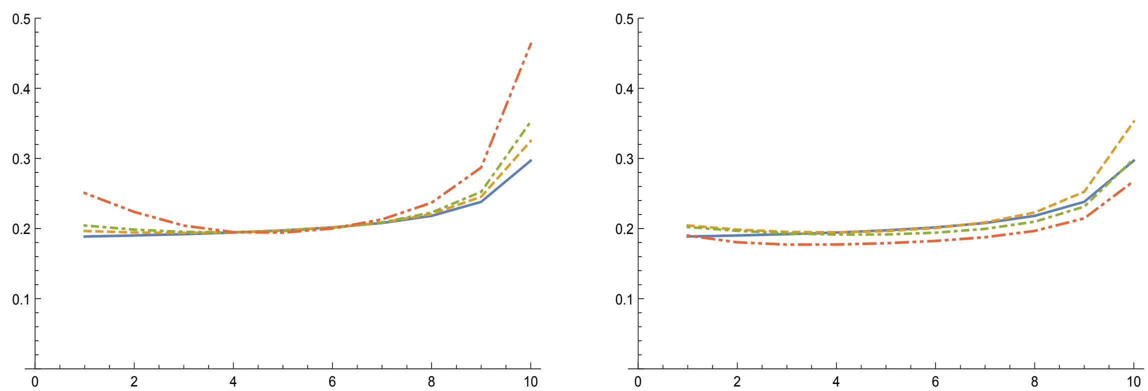
B: Kyle (———), $\rho = 0$ (− − −), $\rho = 0.25$ (− · − · −), $\rho = 0.47$ (− · · − · · −).

[10]Recall that $\alpha_n^I = 0$ so the state variable q_n has no direct impact on the insider's orders.

[11]With $\rho > 0$, there are two differences relative to Holden and Subrahmanyam (1992). First, the insider still has better information than the rebalancer if $\rho < 1$. Thus, our analysis with $\rho > 0$ is more comparable to Foster and Viswanathan (1994), which has two asymmetrically informed traders, one of which is better informed than the other. Second, trading by our rebalancer, when he is informed about $\tilde{v}$, is constrained by his terminal target $\tilde{a}$. This works against rat races with extremely aggressive rebalancer trading.

Figure 5 shows the insider's expected trades over the day for the specific value realization $\tilde{v} = 1$ and averaged over $\tilde{a}$ and noise trader paths w. Kyle's model is the solid (blue) line, whereas the dotted lines represent are various parameterizations of our model. Unlike Kyle's model, our model produces a slight U-shaped trading pattern; that is, our insider expects ex ante to trade somewhat more initially and again at the end of the day. However, the U-shape is not big. Since the trading expectations in Figure 5 are linear in the realization of $\tilde{v}$, the expected informed trades are also slightly U-shaped for other realizations of $\tilde{v}$.

Figure 5: Plot of $\mathbb{E}[\Delta\theta_n^I|\sigma(\tilde{v})]$ for $n = 1, 2, ..., 10$. The parameters are $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 4$, $N = 10$, $\sigma_{\tilde{a}}^2 = 1$ (right only), $\rho = 0$ (left only), and the realization of $\tilde{v}$ equals 1.



A: Kyle (———), $\sigma_{\tilde{a}}^2 = 0.48$ (− − −), $\sigma_{\tilde{a}}^2 = 1$ (− · − · −), $\sigma_{\tilde{a}}^2 = 3.7$ (− · · − · · −).

B: Kyle (———), $\rho = 0$ (− − −), $\rho = 0.25$ (− · − · −), $\rho = 0.47$ (− · · − · · −).

Next, we turn to the rebalancer. The rebalancer's trades reflect a variety of consideration: First, the rebalancer needs to reach his trading target $\tilde{a}$ at time N. Second, he wants to reach this target at the lowest cost possible. Thus, to the extent that his orders have a price impact, he splits up his orders to take into account the pattern of the price impact coefficients λ_n over time. Third, the rebalancer engages in "sunshine trading." In particular, early orders can be used to signal predictable future orders at later dates, which, from (2.25), will have no price impact. Fourth, the rebalancer understands that the mechanical impact of his trades on prices creates incentives for the insider to trade.[12] This can actually be beneficial for the rebalancer. For example, if early uninformed rebalancer orders raise prices, then, in expectation,

[12] The "mechanical impact" of an order refers the impact of an order on the aggregate order flow y_n, which, in turn, affects prices.

the insider should then buy less/sell more in the future, thereby putting downward pressure on later prices which, in turn, reduces the expected cost of future rebalancer buying. Fifth, the rebalancer trades on information about the asset value $\tilde{v}$. If $\rho > 0$, the rebalancer starts out with stock valuation information. However, even if the rebalancer is initially uninformed about $\tilde{v}$ (i.e., $\rho = 0$), he still acquires stock valuation information over time (see (2.12)) that he can use to reduce his rebalancing costs and even, possibly, to earn a trading profit. In particular, he can filter the aggregate order flow to learn about the insider's trading, and thereby learn about $\tilde{v}$, better than the market makers.

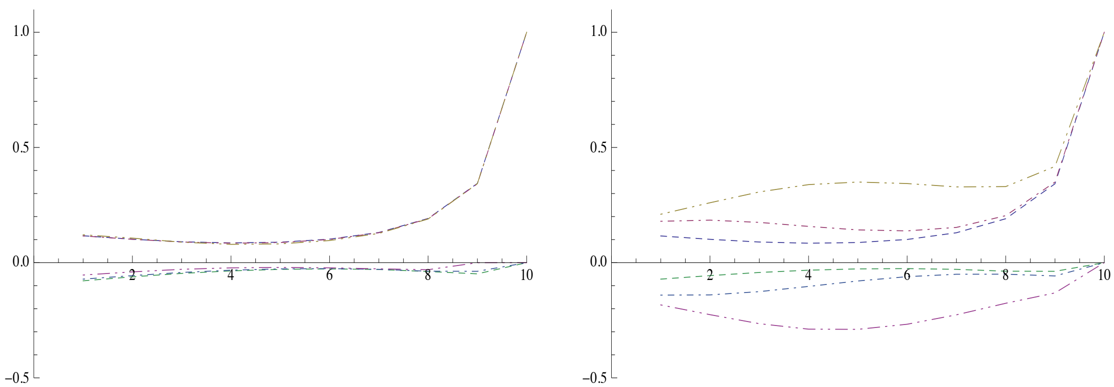
To gain further intuition, we rearrange (2.7) to decompose the rebalancer's order at time n as follows:

$$\begin{aligned}\Delta\theta_n^R &= \Big(\Delta\theta_n^R - \mathbb{E}[\Delta\theta_n^R|\sigma(y_1,\dots,y_{n-1})]\Big) + \mathbb{E}[\Delta\theta_n^R|\sigma(y_1,\dots,y_{n-1})] \\ &= \beta_n^R(\tilde{a} - \theta_{n-1}^R - q_{n-1}) + (\alpha_n^R + \beta_n^R)q_{n-1}.\end{aligned} \tag{3.1}$$

The second component, $(\alpha_n^R + \beta_n^R)q_{n-1}$, is the market maker's expectation of the rebalancer's order at time n. This is the amount the rebalancer trades at time n with no price impact. The first component, $\beta_n^R(\tilde{a} - \theta_{n-1}^R - q_{n-1})$, represents the combined effect of i) strategic trading by the rebalancer on his private information, $\tilde{a} - \theta_{n-1}^R - q_{n-1}$, which is informative about $\tilde{v} - p_{n-1}$ (see (2.12)), and ii) rebalancing trading given that the remaining amount that the rebalancer actually needs to trade (i.e., $\tilde{a} - \theta_{n-1}^R$) differs, in general, from the market maker's expectation q_{n-1}.

Figure 6 shows trajectories for the rebalancer's strategy coefficients β_n^R and α_n^R. We use the decomposition (3.1) to interpret them. Since $\alpha_n^R + \beta_n^R$ is positive but small until time N, the rebalancer trades a relatively small fraction of his expected trading gap q_{n-1} over time until time N at which time $\alpha_N^R + \beta_N^R = 1$ and then he trades the full remaining gap. In addition, the fact that β_n^R is positive means that the rebalancer trades in the direction of his private information. He does this for two reasons: First, the larger $\tilde{a}$ is relative to θ_{n-1}^R (given q_{n-1}) the more the rebalancer needs to trade to achieve his target compared to the market maker's expectation of his trading gap. Second, the smaller θ_{n-1}^R is relative to q_{n-1} (given $\tilde{a}$) the less the rebalancer has actually bought relative to the market maker's expectation, which, in turn, implies that, given the prior observed aggregate order flows, the more the insider has bought in expectation given the rebalancer's information. This implies, then, that the rebalancer believes that the market maker has underpriced the stock and, therefore, strategically buys more/sells less stock.

Figure 6: Plot of $(\alpha_n^R)_{n=1}^N$ (below the x-axis) and of $(\beta_n^R)_{n=1}^N$ (above the x-axis) for $n = 1, 2, ..., 10$. The parameters are $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 4$, $N = 10$, $\sigma_{\tilde{a}}^2 = 1$ (right only), and $\rho = 0$ (left only).

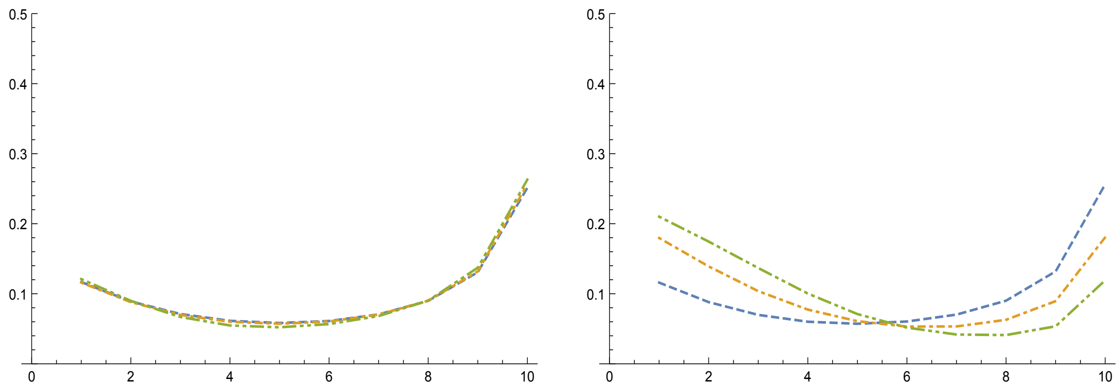


A: $\sigma_{\tilde{a}}^2 = 0.48$ $(- - -)$, $\sigma_{\tilde{a}}^2 = 1$ $(- \cdot - \cdot -)$, $\sigma_{\tilde{a}}^2 = 3.7$ $(- \cdot \cdot - \cdot \cdot -)$. B: $\rho = 0$ $(- - -)$, $\rho = 0.25$ $(- \cdot - \cdot -)$, $\rho = 0.47$ $(- \cdot \cdot - \cdot \cdot -)$.

Figure 7 shows the rebalancer's ex ante expected orders over the day for the particular realization of the trading target $\tilde{a}$ being equal to 1. These expectations are taken over the terminal stock price $\tilde{v}$ and the noise trader order path w. These expectations depend linearly on the realization of the trading target $\tilde{a}$. The graphs show that the rebalancer's trading strategy also has a U–shaped pattern over the day. Degryse, de Jong, and van Kervel (2014) obtain a similar result in their model with short–lived information for the insiders and static trading for the rebalancer. In particular, with short-lived information, their insider is unable to trade dynamically over time, which allows the rebalancer to (imperfectly) separate his order from those of the insider. In contrast, in our model, the insider is able to trade dynamically too. Thus, the U-shaped pattern of rebalancing trading does not depend on the assumption of short-lived information.

The literature on optimal order execution includes many models that produce U-shaped optimal strategies, see, e.g., Predoiu, Shaikhet, and Shreve (2011) and the many references therein. However, sunshine trading in that literature stems from exogenously specified liquidity resilience and replenishment dynamics. In contrast, liquidity in our equilibrium model is endogenously determined. In our model, there

Figure 7: Plot of $\mathbb{E}[\Delta\theta_n^R|\sigma(\tilde{a})]$ for $n = 1, 2, ..., 10$. The parameters are $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 4$, $N = 10$, $\sigma_{\tilde{a}}^2 = 1$ (right only), $\rho = 0$ (left only), and the realization of $\tilde{a}$ equals 1.



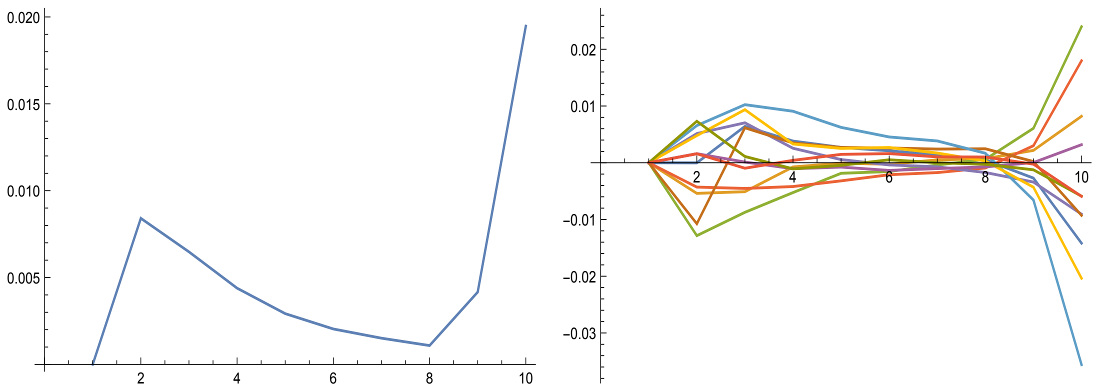
A: $\sigma_{\tilde{a}}^2 = 0.48$ (— — —), $\sigma_{\tilde{a}}^2 = 1$ (— · — · —), $\sigma_{\tilde{a}}^2 = 3.7$ (— · · — · · —).

B: $\rho = 0$ (— — —), $\rho = 0.25$ (— · — · —), $\rho = 0.47$ (— · · — · · —).

are two sources of U-shaped rebalancer trading volume. First, sunshine orders from the rebalancer early in the day can signal to the market maker the size of the predictable component of his orders at the end of the day. Second, there are also U-shaped patterns in the standard deviation of rebalancer orders. In particular, because the rebalancer's trades depend on the aggregated order flow history via q_n, there is variability across the rebalancer's order flow paths. Figure 8A shows the ex ante standard deviation of the rebalancer's orders over the day given randomness in $\tilde{v}$ and w conditional on the rebalancer's target $\tilde{a}$. Here again, we see a U-shaped pattern.

Figure 8B plots a few paths of the rebalancer's order flows over time. Here the realized stock value $\tilde{v}$ is 1, and the realized trading target $\tilde{a}$ is 0. There are 10 different randomly selected path realizations of the noise traders' orders. Along these paths, we see that the rebalancer buys/sells more than his trading target $\tilde{a}$ at early dates ($n > 1$) and then unwinds his position at later dates to achieve his trading target. This is not manipulation. Rather, the rebalancer's orders reflect a combination of informed trading motives (about $\tilde{v}$) and uninformed rebalancing motives (due to $\tilde{a}$). The rebalancer does not trade at time 1 because he does not need to rebalance and because, initially, he does not have any stock valuation information. However, at time 2 the rebalancer trades based on whether — given the value information he gleans from being able to filter the order flow y_1 better than the market makers – he thinks the stock is over– or under–valued. Eventually, however, he must unwind these earlier positions in order to achieve his realized trading target constraint $\theta_N^R = \tilde{a} = 0$. The dispersion in the paths is consistent with the trajectory of the rebalancer order flow standard deviation. Paths for non-zero values of $\tilde{a}$ involve shifting the means of these paths from zero to the appropriate ex ante conditional means given $\tilde{a}$ (e.g., Figure 7 illustrates one such conditional mean order flow tragectory for $\tilde{a} = 1$).

Figure 8: Properties of the rebalancer's orders. The parameters are $N := 10$, $\sigma_w^2 := 4$, $\sigma_{\tilde{v}}^2 := 1$, $\sigma_{\tilde{a}}^2 := 1$, and $\rho := 0$.



A: $\mathbb{E}\left[\left(\Delta\theta_n^R - \mathbb{E}[\Delta\theta_n^R|\sigma(\tilde{a})]\right)^2\big|\sigma(\tilde{a})\right]^{\frac{1}{2}}$, for the realization of $\tilde{a} = 1$.

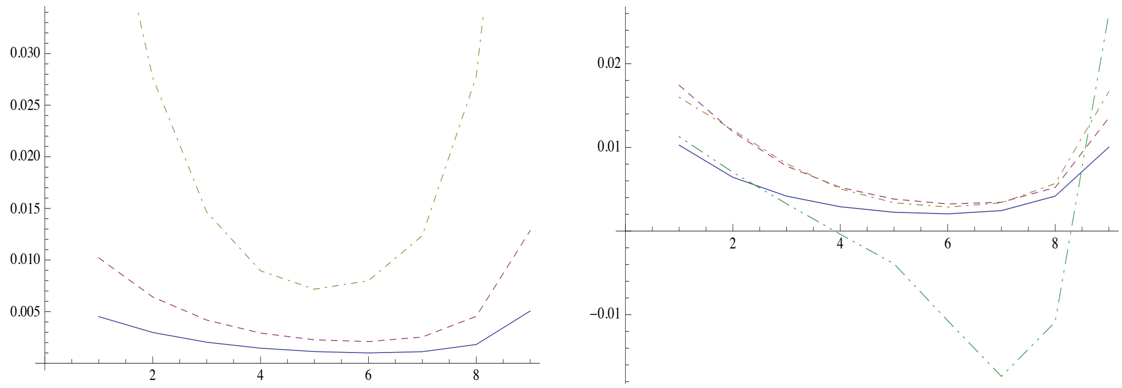
B: 10 paths of $\Delta\theta_n^R$ for the realizations $\tilde{a} = 0$ and $\tilde{v} = 1$.

Figure 9 shows the unconditional autocorrelation of the aggregate order flow over time for different values of $\sigma_{\tilde{a}}^2$ and ρ. Although the absolute level of autocorrelation is low, there is a clear U-shaped pattern of higher order flow autocorrelation at the beginning and the end of the day (when, from Figure 7, the rebalancer is trading more) with lower autocorrelation during the middle of the day (when the rebalancer trades less). Somewhat surprisingly, order flow autocorrelation can be negative in the middle of the day when the target-information correlation ρ is high.

Figure 10 shows the unconditional standard deviation for the price changes over time. Kyle's model is the solid (blue) line, which is monotonically increasing, whereas our model produces the U-shaped dotted lines (for various correlation parameters ρ and target variances $\sigma_{\tilde{a}}^2$). In other words, our model produces equilibrium prices which are more volatile at the beginning and at the end of the trading day relative to the middle of the trading day.

The rebalancer's trading strategy takes into account two types of predictability in his orders. One part of his orders is predictable to the market maker based on the

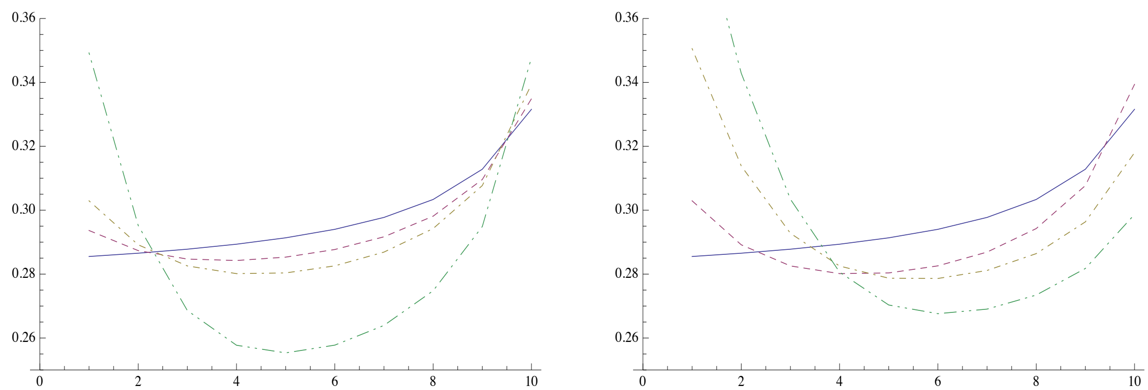
Figure 9: Plot of $\frac{\mathbb{E}[y_n y_{n+1}]}{\sqrt{\mathbb{E}[y_n^2]\mathbb{E}[y_{n+1}^2]}}$ for $n = 1, 2, ..., 9$. The parameters are $N := 10$, $\sigma_w^2 := 4$, $\sigma_{\tilde{v}}^2 := 1$, $\sigma_{\tilde{a}}^2 = 1$ (right only), and $\rho = 0$ (left only).



A: $\sigma_{\tilde{a}}^2 = 0.48$ (———), $\sigma_{\tilde{a}}^2 = 1$ (− − −), .$\sigma_{\tilde{a}}^2 = 3.7$ (− · − · −).

B: $\rho = 0$(———), $\rho = .25$ (− − −), $\rho = .47$ (− · − · −), $\rho = .86$ (− · · − · · −)

Figure 10: Plot of $\sqrt{\mathbb{E}[(p_n - p_{n-1})^2]}$ for $n = 1, 2, ..., 10$. The parameters are $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 4$, $N = 10$, $\sigma_{\tilde{a}}^2 = 1$ (right only), and $\rho = 0$ (left only).



A: Kyle (———), $\sigma_{\tilde{a}}^2 = 0.48$ (− − −), $\sigma_{\tilde{a}}^2 = 1$ (− · − · −), $\sigma_{\tilde{a}}^2 = 3.7$ (− · · − · · −).

B: Kyle (———), $\rho = 0$ (− − −), $\rho = 0.25$ (− · − · −), $\rho = 0.47$ (− · · − · · −).

prior aggregate order flow. This sunshine trading component of his order at time n is

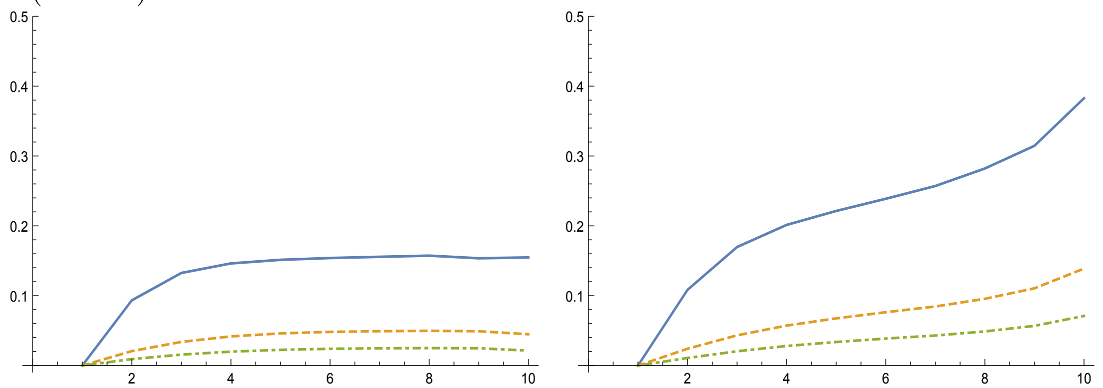
$$\begin{aligned}\mathbb{E}[\Delta\theta_n^R|\sigma(y_1, \dots, y_{n-1})] &= \mathbb{E}[\beta_n^R(\tilde{a} - \theta_{n-1}^R) + \alpha_n^R q_{n-1}|\sigma(y_1, \dots, y_{n-1})] \\ &= (\beta_n^R + \alpha_n^R)q_{n-1}.\end{aligned} \tag{3.2}$$

The advantage to the rebalancer of sunshine trading predictability is that, from (2.25), this part of his trades has no price impact. Another part of the rebalancer's orders is predictable to the insider. In particular, as shown in (2.13), the insider can filter the aggregate order flow better than the market maker to identify rebalancing orders. The part that is predictable to the insider is

$$\begin{aligned}
&\mathbb{E}[\Delta\theta_n^R|\sigma(\tilde{v}, y_1, \ldots, y_{n-1})] \\
&\quad = \beta_n^R \mathbb{E}[(\tilde{a} - \theta_{n-1}^R)|\sigma(\tilde{v}, y_1, \ldots, y_{n-1})] + \alpha^R q_{n-1} \\
&\quad = \beta_n^R \mathbb{E}[(\tilde{a} - \theta_{n-1}^R - q_{n-1})|\sigma(\tilde{v} - p_{n-1})] + (\alpha^R + \beta_n^R) q_{n-1} \\
&\quad = \beta_n^R \frac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}(\tilde{v} - p_{n-1}) + (\alpha^R + \beta_n^R) q_{n-1}.
\end{aligned} \tag{3.3}$$

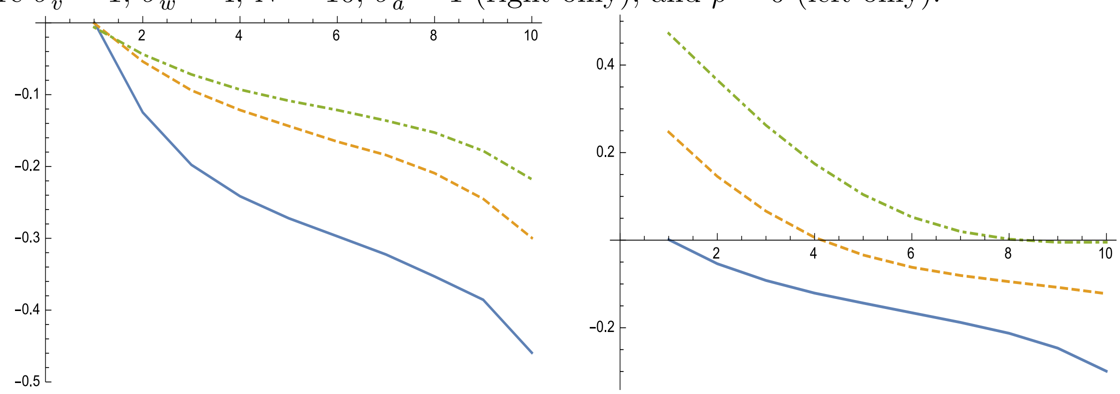
Figure 11A shows that the sunshine trading components, while present, is not particularly large. In contrast, Figure 11B shows that a substantial part of the rebalancer's orders are predictable to the insider, or, put differently, is correlated with the insider's information. In this example, this second type of predictability is beneficial for the rebalancer because, as shown in Figure 12, the resulting conditional correlation of the insider's orders and the rebalancer's orders is negative. This is intuitive since the price impact from early buying by the rebalancer raises prices, which, on average, tends to increase future informed trader selling/reduce informed trader buying, which, in turn, tends to lower future expected prices for the rebalance. In other words, the insider's and rebalancer's orders partial offset each other in expectation, which benefits them both by canceling out part of their price impacts.

Figure 11: Plot of conditional expectations of the predictable parts of the rebalancer's trades (left is the market maker's estimate and right is the insider's estimate). The parameters are $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 4$, $N = 10$, $\rho = 0$ whereas $\tilde{a}$ is realized to be 1. The variance of the trading target varies: $\sigma_{\tilde{a}}^2 = 0.48$ $(- \cdot - \cdot -)$, $\sigma_{\tilde{a}}^2 = 1$ $(- - -)$, $\sigma_{\tilde{a}}^2 = 3.7$ (———).



A: $\mathbb{E}\left[\frac{(\beta_n^R+\alpha_n^R)q_{n-1}}{\Delta\theta_n^R}\Big|\sigma(\tilde{a})\right]$ B: $\mathbb{E}\left[\frac{\beta_n^R\frac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}(\tilde{v}-p_{n-1})+(\beta_n^R+\alpha_n^R)q_{n-1}}{\Delta\theta_n^R}\Big|\sigma(\tilde{a})\right]$

Figure 12: Plot of $\mathrm{corr}(\Delta\theta_n^R, \Delta\theta_n^R)$ for $n = 1, 2, ..., 10$ (unconditional). The parameters are $\sigma_{\tilde{v}}^2 = 1$, $\sigma_w^2 = 4$, $N = 10$, $\sigma_{\tilde{a}}^2 = 1$ (right only), and $\rho = 0$ (left only).



A: $\sigma_{\tilde{a}}^2 = 0.48$ $(- \cdot - \cdot -)$, $\sigma_{\tilde{a}}^2 = 1$ $(- - -)$, $\sigma_{\tilde{a}}^2 = 3.7$ (———). B: $\rho = 0$ (———), $\rho = 0.25$ $(- - -)$, $\rho = 0.47$ $(- \cdot - \cdot -)$.

4 Conclusion

This paper has explored the equilibrium interactions between strategic dynamic informed trading, strategic dynamic portfolio rebalancing, price discovery, and liquidity in a multi–period Kyle (1985) market. To the best of our knowledge, our paper is the first to investigate these issues with both long-lived information and dynamic rebalancing given a terminal trading target.

There are many interesting possible extensions for future work. One possible extension is to model trading in continuous-time. Another possibility is to consider other forms of portfolio rebalancing constraints. A third extension is to relax the assumption that all investors are risk-neutral. For this extension, it would be natural to consider exponential utilities with different coefficients of absolute risk aversion. Finally, it would be interesting to extend the model to include multiple insiders and rebalancers.

A Proofs

A.1 Kalman filtering

Lemma A.1. *If Condition 2.2 and Condition 2.3 hold, then for $n = 1, ..., N$ we have*

$$\hat{p}_n = \mathbb{E}[\tilde{v}|\sigma(\hat{y}_1, ..., \hat{y}_n)], \tag{A.1}$$

$$\hat{q}_n = \mathbb{E}[(\tilde{a} - \hat{\theta}^R_n)|\sigma(\hat{y}_1, ..., \hat{y}_n)], \tag{A.2}$$

where $\hat{p}$ is defined by (2.19) *and $\hat{q}$ is defined by* (2.20)*.*

Proof. For $n = 1, ..., N$, we have the moment definitions in (2.21)-(2.23) where the starting values are in (2.24). We then define the process $\hat{z}_n$ as

$$\begin{aligned} \hat{z}_n :=& \hat{y}_n - (\alpha^I_n + \alpha^R_n + \beta^R_n)\hat{q}_{n-1} \\ =& \beta^I_n(\tilde{v} - \hat{p}_{n-1}) + \beta^R_n(\tilde{a} - \hat{\theta}^R_{n-1} - \hat{q}_{n-1}) + \Delta w_n. \end{aligned} \tag{A.3}$$

These variables $\hat{z}_1, \hat{z}_2, ..., \hat{z}_N$ are independent and satisfy $\sigma(\hat{z}_1, ...\hat{z}_n) = \sigma(\hat{y}_1, ...\hat{y}_n)$.

The projection theorem for Gaussian random variables produces

$$
\begin{aligned}
\Delta\hat{p}_n =& \mathbb{E}[\tilde{v}|\sigma(\hat{z}_1,...,\hat{z}_n)] - \mathbb{E}[\tilde{v}|\sigma(\hat{z}_1,...,\hat{z}_{n-1})] \\
=& \frac{\mathbb{E}[\tilde{v}\,\hat{z}_n]}{\mathbb{V}[\hat{z}_n]}\hat{z}_n, \\
\Delta\hat{q}_n =& \mathbb{E}[\tilde{a} - \hat{\theta}^R_n|\sigma(\hat{z}_1,...\hat{z}_n)] - \mathbb{E}[\tilde{a} - \hat{\theta}^R_{n-1}|\sigma(\hat{z}_1,...,\hat{z}_{n-1})] \\
=& \mathbb{E}[\tilde{a} - \hat{\theta}^R_{n-1}|\sigma(\hat{z}_1,...\hat{z}_n)] - \mathbb{E}[\tilde{a} - \hat{\theta}^R_{n-1}|\sigma(\hat{z}_1,...,\hat{z}_{n-1})] - \mathbb{E}[\Delta\hat{\theta}^R_n|\sigma(\hat{z}_1,...,\hat{z}_n)] \\
=& \frac{\mathbb{E}[(\tilde{a} - \hat{\theta}^R_{n-1})\hat{z}_n]}{\mathbb{V}[\hat{z}_n]}\hat{z}_n - \mathbb{E}\big[\beta^R_n(\tilde{a} - \hat{\theta}^R_{n-1} - \hat{q}_{n-1}) + (\alpha^R_n + \beta^R_n)\hat{q}_{n-1}\big|\sigma(\hat{z}_1,...,\hat{z}_n)\big] \\
=& \frac{\mathbb{E}[(\tilde{a} - \hat{\theta}^R_{n-1} - \hat{q}_{n-1})\hat{z}_n]}{\mathbb{V}[\hat{z}_n]}\hat{z}_n - \beta^R_n\mathbb{E}[\tilde{a} - \hat{\theta}^R_{n-1} - \hat{q}_{n-1}|\sigma(\hat{z}_n)] - (\alpha^R_n + \beta^R_n)\hat{q}_{n-1} \\
=& (1 - \beta^R_n)\frac{\mathbb{E}[(\tilde{a} - \hat{\theta}^R_{n-1} - \hat{q}_{n-1})\hat{z}_n]}{\mathbb{V}[\hat{z}_n]}\hat{z}_n - (\alpha^R_n + \beta^R_n)\hat{q}_{n-1}.
\end{aligned}
$$

To proceed, we first need to compute

$$
\begin{aligned}
\mathbb{V}[\hat{z}_n] =& \mathbb{E}\Big[\Big(\beta^I_n(\tilde{v} - \hat{p}_{n-1}) + \beta^R_n(\tilde{a} - \hat{\theta}^R_{n-1} - \hat{q}_{n-1}) + \Delta w_n\Big)^2\Big] \\
=& (\beta^I_n)^2\Sigma^{(2)}_n + (\beta^R_n)^2\Sigma^{(1)}_n + 2\beta^I_n\beta^R_n\Sigma^{(3)}_n + \sigma^2_w\Delta, \\
\mathbb{E}[\tilde{v}\hat{z}_n] =& \mathbb{E}[(\tilde{v} - \hat{p}_{n-1})\hat{z}_n] \\
=& \mathbb{E}\Big[(\tilde{v} - \hat{p}_{n-1})\Big(\beta^I_n(\tilde{v} - \hat{p}_{n-1}) + \beta^R_n(\tilde{a} - \hat{\theta}^R_{n-1} - \hat{q}_{n-1}) + \Delta w_n\Big)\Big] \\
=& \beta^I_n\Sigma^{(2)}_n + \beta^R_n\Sigma^{(3)}_n, \\
\mathbb{E}[(\tilde{a} - \hat{\theta}^R_{n-1} - \hat{q}_{n-1})\hat{z}_n] =& \mathbb{E}\Big[(\tilde{a} - \hat{\theta}^R_{n-1} - \hat{q}_{n-1})\Big(\beta^I_n(\tilde{v} - \hat{p}_{n-1}) + \beta^R_n(\tilde{a} - \hat{\theta}^R_{n-1} - \hat{q}_{n-1}) + \Delta w_n\Big)\Big] \\
=& \beta^I_n\Sigma^{(3)}_n + \beta^R_n\Sigma^{(1)}_n.
\end{aligned}
$$

By combining these expressions and by matching coefficients with (2.19) and (2.20) we find the lemma's statement equivalent to the following restrictions

$$\lambda_n = \frac{\beta_n^I \Sigma_n^{(2)} + \beta_n^R \Sigma_n^{(3)}}{(\beta_n^I)^2 \Sigma_n^{(2)} + (\beta_n^R)^2 \Sigma_n^{(1)} + 2\beta_n^I \beta_n^R \Sigma_n^{(3)} + \sigma_w^2 \Delta}, \tag{A.4}$$

$$\begin{aligned}\mu_n &= -\frac{(\alpha_n^I + \alpha_n^R + \beta_n^R)(\beta_n^I \Sigma_n^{(2)} + \beta_n^R \Sigma_n^{(3)})}{(\beta_n^I)^2 \Sigma_n^{(2)} + (\beta_n^R)^2 \Sigma_n^{(1)} + 2\beta_n^I \beta_n^R \Sigma_n^{(3)} + \sigma_w^2 \Delta} \\ &= -(\alpha_n^I + \alpha_n^R + \beta_n^R)\lambda_n,\end{aligned} \tag{A.5}$$

$$r_n = \frac{(1-\beta_n^R)(\beta_n^I \Sigma_n^{(3)} + \beta_n^R \Sigma_n^{(1)})}{(\beta_n^I)^2 \Sigma_n^{(2)} + (\beta_n^R)^2 \Sigma_n^{(1)} + 2\beta_n^I \beta_n^R \Sigma_n^{(3)} + \sigma_w^2 \Delta}, \tag{A.6}$$

$$\begin{aligned}s_n &= -\frac{(\alpha_n^I + \alpha_n^R + \beta_n^R)(1-\beta_n^R)(\beta_n^I \Sigma_n^{(3)} + \beta_n^R \Sigma_n^{(1)})}{(\beta_n^I)^2 \Sigma_n^{(2)} + (\beta_n^R)^2 \Sigma_n^{(1)} + 2\beta_n^I \beta_n^R \Sigma_n^{(3)} + \sigma_w^2 \Delta} - (\alpha_n^R + \beta_n^R) \\ &= -(\alpha_n^I + \alpha_n^R + \beta_n^R) r_n - (\alpha_n^R + \beta_n^R).\end{aligned} \tag{A.7}$$

Based on these expressions we can find the recursion for $\Sigma_n^{(1)}$, $n = 1, ..., N$, to be

$$\begin{aligned}\Sigma_{n+1}^{(1)} &= \mathbb{V}[\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1} - \Delta\hat{\theta}_n^R - \Delta\hat{q}_n], \\ &= \mathbb{V}[\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1} - \Delta\hat{\theta}_n^R - r_n\hat{y}_n - s_n\hat{q}_{n-1}], \\ &= \mathbb{V}\Big[\tilde{a} - \hat{\theta}_{n-1}^R - (1+s_n)\hat{q}_{n-1} - (1+r_n)(\beta_n^R(\tilde{a} - \hat{\theta}_{n-1}^R) + \alpha_n^R\hat{q}_{n-1}) \\ &\qquad - r_n(\beta_n^I(\tilde{v} - \hat{p}_{n-1}) + \alpha_n^I\hat{q}_{n-1}) - r_n\Delta w_n\Big], \\ &= \mathbb{V}\Big[(1-(1+r_n)\beta_n^R)(\tilde{a} - \hat{\theta}_{n-1}^R) - (1 + s_n + (1+r_n)\alpha_n^R + r_n\alpha_n^I)\hat{q}_{n-1} \\ &\qquad - r_n\beta_n^I(\tilde{v} - \hat{p}_{n-1}) - r_n\Delta w_n\Big] \\ &= \mathbb{V}\Big[(1-(1+r_n)\beta_n^R)(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}) - r_n\beta_n^I(\tilde{v} - \hat{p}_{n-1}) - r_n\Delta w_n\Big] \\ &= (1-(1+r_n)\beta_n^R)^2\Sigma_n^{(1)} + (r_n\beta_n^I)^2\Sigma_n^{(2)} + r_n^2\sigma_w^2\Delta - 2(1-(1+r_n)\beta_n^R)r_n\beta_n^I\Sigma_n^{(3)} \\ &= (1-\beta_n^R)\big((1-\beta_n^R - r_n\beta_n^R)\Sigma_n^{(1)} - r_n\beta_n^I\Sigma_n^{(3)}\big).\end{aligned} \tag{A.8}$$

Similarly, we find

$$\Sigma_{n+1}^{(2)} = (1-\lambda_n\beta_n^I)\Sigma_n^{(2)} - \lambda_n\beta_n^R\Sigma_n^{(3)}, \tag{A.9}$$

$$\Sigma_{n+1}^{(3)} = (1-\beta_n^R)\big((1-\lambda_n\beta_n^I)\Sigma_n^{(3)} - \lambda_n\beta_n^R\Sigma_n^{(1)}\big). \tag{A.10}$$

◇

A.2 Insider's optimization problem

In this section, we assume that Condition 2.2 and Condition 2.3 hold so we can define the "hat"-processes (2.16)-(2.20) as well as the insider's state variable processes (2.34). We let p_n be defined by (2.9) and we fix the rebalancer's strategy $\Delta\theta_n^R$ by (2.7). We note that $\Delta\theta_n^R$ depends on the insider's historical demands θ_k^I for $k = 1, 2, ..., n-1$ even though the strategy (2.7) for the rebalancer is fixed.

We start with the following lemma which contains most of the calculations we will need later.

Lemma A.2. *Assume that Condition 2.2 and Condition 2.3 hold. Fix* $\Delta\theta_n^R$ *by* (2.7) *and let* $\Delta\theta_n^I \in \sigma(\tilde{v}, y_1, ..., y_{n-1})$, $n = 1, ..., N$, *be arbitrary for the insider. Then for* $n = 1, ..., N$ *we have the following two measurability properties*

$$\sigma(\tilde{v}, y_1, ..., y_n) = \sigma(\tilde{v}, \hat{y}_1, ..., \hat{y}_n) \quad \textit{and} \quad \hat{\theta}_n^R - \theta_n^R \in \sigma(\tilde{v}, y_1, ..., y_{n-1}). \tag{A.11}$$

Based on this, we get the following expectations

$$\mathbb{E}[\hat{y}_n|\sigma(\tilde{v},y_1,...,y_{n-1})] = \big(\beta_n^I + \frac{\beta_n^R\Sigma_n^{(3)}}{\Sigma_n^{(2)}}\big)X_{n-1}^{(1)} + (\alpha_n^I + \alpha_n^R + \beta_n^R)X_{n-1}^{(2)},$$

$$\mathbb{E}[y_n|\sigma(\tilde{v},y_1,...,y_{n-1})] = \Delta\theta_n^I + \frac{\beta_n^R\Sigma_n^{(3)}}{\Sigma_n^{(2)}}X_{n-1}^{(1)} + (\alpha_n^R + \beta_n^R)X_{n-1}^{(2)} + \beta_n^R X_{n-1}^{(3)} - \alpha_n^R X_{n-1}^{(4)},$$

$$\mathbb{E}[X_n^{(1)}|\sigma(\tilde{v},y_1,...,y_{n-1})] = \big(1 - \lambda_n(\beta_n^I + \frac{\beta_n^R\Sigma_n^{(3)}}{\Sigma_n^{(2)}})\big)X_{n-1}^{(1)} - \big(\lambda_n(\alpha_n^I + \alpha_n^R + \beta_n^R) + \mu_n\big)X_{n-1}^{(2)},$$

$$\mathbb{E}[X_n^{(2)}|\sigma(\tilde{v},y_1,...,y_{n-1})] = r_n(\beta_n^I + \frac{\beta_n^R\Sigma_n^{(3)}}{\Sigma_n^{(2)}})X_{n-1}^{(1)} + \big(1 + r_n(\alpha_n^I + \alpha_n^R + \beta_n^R) + s_n\big)X_{n-1}^{(2)},$$

$$\mathbb{E}[(X_n^{(1)})^2|\sigma(\tilde{v},y_1,...,y_{n-1})] = \big(\mathbb{E}[X_n^{(1)}|\sigma(\tilde{v},y_1,...,y_{n-1})]\big)^2 + \lambda_n^2(\beta_n^R)^2\Big(\Sigma_n^{(1)} - \frac{(\Sigma_n^{(3)})^2}{\Sigma_n^{(2)}}\Big) + \lambda_n^2\sigma_w^2\Delta,$$

$$\mathbb{E}[(X_n^{(2)})^2|\sigma(\tilde{v},y_1,...,y_{n-1})] = \big(\mathbb{E}[X_n^{(2)}|\sigma(\tilde{v},y_1,...,y_{n-1})]\big)^2 + r_n^2(\beta_n^R)^2\Big(\Sigma_n^{(1)} - \frac{(\Sigma_n^{(3)})^2}{\Sigma_n^{(2)}}\Big) + r_n^2\sigma_w^2\Delta,$$

$$\begin{aligned}\mathbb{E}[X_n^{(1)}X_n^{(2)}|\sigma(\tilde{v},y_1,...,y_{n-1})] &= \mathbb{E}[X_n^{(1)}|\sigma(\tilde{v},y_1,...,y_{n-1})]\mathbb{E}[X_n^{(2)}|\sigma(\tilde{v},y_1,...,y_{n-1})] \\ &\quad - \lambda_n r_n\sigma_w^2\Delta - \lambda_n r_n(\beta_n^R)^2\Big(\Sigma_n^{(1)} - \frac{(\Sigma_n^{(3)})^2}{\Sigma_n^{(2)}}\Big),\end{aligned}$$

$$\begin{aligned}\mathbb{E}[\tilde{v} - p_n|\sigma(\tilde{v},y_1,...,y_{n-1})] &= -\lambda_n\Delta\theta_n^I + \big(1 - \frac{\lambda_n\beta_n^R\Sigma_n^{(3)}}{\Sigma_n^{(2)}}\big)X_{n-1}^{(1)} - \big(\lambda_n(\alpha_n^R + \beta_n^R) + \mu_n\big)X_{n-1}^{(2)} \\ &\quad - \lambda_n\beta_n^R X_{n-1}^{(3)} + (\lambda_n\alpha_n^R + \mu_n)X_{n-1}^{(4)} + X_{n-1}^{(5)}.\end{aligned}$$

Furthermore, $X_n^{(3)}, X_n^{(4)}, X_n^{(5)} \in \sigma(\tilde{v},y_1,...,y_{n-1})$ *and we have the dynamics*

$$\begin{aligned}X_n^{(3)} &= (1-\beta_n^R)X_{n-1}^{(3)} + \alpha_n^R X_{n-1}^{(4)}, \\ X_n^{(4)} &= -r_n\Delta\theta_n^I + r_n\beta_n^I X_{n-1}^{(1)} + r_n\alpha_n^I X_{n-1}^{(2)} - r_n\beta_n^R X_{n-1}^{(3)} + (r_n\alpha_n^R + s_n + 1)X_{n-1}^{(4)}, \\ X_n^{(5)} &= -\lambda_n\Delta\theta_n^I + \lambda_n\beta_n^I X_{n-1}^{(1)} + \lambda_n\alpha_n^I X_{n-1}^{(2)} - \lambda_n\beta_n^R X_{n-1}^{(3)} + (\lambda_n\alpha_n^R + \mu_n)X_{n-1}^{(4)} + X_{n-1}^{(5)}.\end{aligned}$$

Proof. We first prove (A.11) by induction. We observe that

$$\begin{aligned}&\sigma(\tilde{v},y_1) = \sigma(\tilde{v},\beta_1^R\tilde{a} + \Delta w_1) = \sigma(\tilde{v},\hat{y}_1), \\ &\hat{\theta}_1^R - \theta_1^R = 0,\end{aligned}$$

which follows from $\hat{\theta}_1^I, \theta_1^I \in \sigma(\tilde{v})$. Suppose that (A.11) holds for n. Then,

$$\begin{aligned}
\hat{\theta}_{n+1}^R - \theta_{n+1}^R &= (1-\beta_{n+1}^R)(\hat{\theta}_n^R - \theta_n^R) + \alpha_{n+1}^R(\hat{q}_n - q_n) \\
&\in \sigma(\tilde{v}, y_1, ..., y_n), \\
\sigma(\tilde{v}, \hat{y}_1, ..., \hat{y}_{n+1}) &= \sigma(\tilde{v}, y_1, ..., y_n, \hat{y}_{n+1}) \\
&= \sigma(\tilde{v}, y_1, ..., y_n, y_{n+1} + \Delta\hat{\theta}_{n+1}^I - \Delta\theta_{n+1}^I + \Delta\hat{\theta}_{n+1}^R - \Delta\theta_{n+1}^R) \\
&= \sigma(\tilde{v}, y_1, ..., y_{n+1}).
\end{aligned}$$

This proves (A.11). Next, we define the sequence of independent random variables $\hat{z}_n$ by (A.3) and we recall the property $\sigma(\hat{z}_1, ..., \hat{z}_n) = \sigma(\hat{y}_1, ..., \hat{y}_n)$. We can then compute the conditional expectations of $\hat{y}_n$ and y_n as follows:

$$\begin{aligned}
&\mathbb{E}[\hat{y}_n|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&\quad = \mathbb{E}[\Delta\hat{\theta}_n^I + \Delta\hat{\theta}_n^R|\sigma(\tilde{v}, \hat{y}_1, ..., \hat{y}_{n-1})] \\
&\quad = \mathbb{E}[\Delta\hat{\theta}_n^I + \Delta\hat{\theta}_n^R|\sigma(\tilde{v}, \hat{z}_1, ..., \hat{z}_{n-1})] \\
&\quad = \beta_n^I(\tilde{v} - \hat{p}_{n-1}) + (\alpha_n^I + \alpha_n^R + \beta_n^R)\hat{q}_{n-1} + \beta_n^R\mathbb{E}[\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}|\sigma(\tilde{v} - \hat{p}_{n-1}, \hat{z}_1, ..., \hat{z}_{n-1})] \\
&\quad = \beta_n^I(\tilde{v} - \hat{p}_{n-1}) + (\alpha_n^I + \alpha_n^R + \beta_n^R)\hat{q}_{n-1} + \beta_n^R\frac{\mathbb{E}[(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1})(\tilde{v} - \hat{p}_{n-1})]}{\mathbb{V}[\tilde{v} - \hat{p}_{n-1}]}(\tilde{v} - \hat{p}_{n-1}) \\
&\quad = \big(\beta_n^I + \frac{\beta_n^R\Sigma_n^{(3)}}{\Sigma_n^{(2)}}\big)X_{n-1}^{(1)} + (\alpha_n^I + \alpha_n^R + \beta_n^R)X_{n-1}^{(2)}, \\
&\mathbb{E}[y_n|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&\quad = \mathbb{E}[\Delta\theta_n^I + \Delta\theta_n^R|\sigma(\tilde{v}, \hat{y}_1, ..., \hat{y}_{n-1})] \\
&\quad = \mathbb{E}[\Delta\theta_n^I + \Delta\theta_n^R|\sigma(\tilde{v}, \hat{z}_1, ..., \hat{z}_{n-1})] \\
&\quad = \Delta\theta_n^I + \mathbb{E}[\beta_n^R(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1})|\sigma(\tilde{v}, \hat{z}_1, ..., \hat{z}_{n-1})] + \beta_n^R\hat{q}_{n-1} + \beta_n^R(\hat{\theta}_{n-1}^R - \theta_{n-1}^R) + \alpha_n^R q_{n-1} \\
&\quad = \Delta\theta_n^I + \beta_n^R\hat{q}_{n-1} + \beta_n^R(\hat{\theta}_{n-1}^R - \theta_{n-1}^R) + \alpha_n^R q_{n-1} + \beta_n^R\mathbb{E}[\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}|\sigma(\tilde{v} - \hat{p}_{n-1})] \\
&\quad = \Delta\theta_n^I + \frac{\beta_n^R\Sigma_n^{(3)}}{\Sigma_n^{(2)}}X_{n-1}^{(1)} + (\alpha_n^R + \beta_n^R)X_{n-1}^{(2)} + \beta_n^R X_{n-1}^{(3)} - \alpha_n^R X_{n-1}^{(4)}.
\end{aligned}$$

For the second equality in the second conditional expectation, we have used $\hat{\theta}_{n-1}^R - \theta_{n-1}^R \in \sigma(\tilde{v}, y_1, ..., y_{n-2})$ which we established in (A.11). By using the property

$$\hat{y}_n - y_n = -\Delta\theta_n^I + \beta_n^I X_{n-1}^{(1)} + \alpha_n^I X_{n-1}^{(2)} - \beta_n^R X_{n-1}^{(3)} + \alpha_n^R X_{n-1}^{(4)},$$

we find

$$
\begin{aligned}
X_n^{(3)} &= X_{n-1}^{(3)} + \Delta X_n^{(3)} = (1-\beta_n^R)X_{n-1}^{(3)} + \alpha_n^R X_{n-1}^{(4)},\\
X_n^{(4)} &= X_{n-1}^{(4)} + \Delta X_n^{(4)} = X_{n-1}^{(4)} + r_n(\hat{y}_n - y_n) + s_n X_{n-1}^{(4)},\\
&= -r_n\Delta\theta_n^I + r_n\beta_n^I X_{n-1}^{(1)} + r_n\alpha_n^I X_{n-1}^{(2)} - r_n\beta_n^R X_{n-1}^{(3)} + (r_n\alpha_n^R + s_n + 1)X_{n-1}^{(4)},\\
X_n^{(5)} &= X_{n-1}^{(5)} + \Delta X_n^{(5)} = X_{n-1}^{(5)} + \lambda_n(\hat{y}_n - y_n) + \mu_n X_{n-1}^{(4)},\\
&= -\lambda_n\Delta\theta_n^I + \lambda_n\beta_n^I X_{n-1}^{(1)} + \lambda_n\alpha_n^I X_{n-1}^{(2)} - \lambda_n\beta_n^R X_{n-1}^{(3)} + (\lambda_n\alpha_n^R + \mu_n)X_{n-1}^{(4)} + X_{n-1}^{(5)}.
\end{aligned}
$$

Therefore, we have $X_n^{(3)}, X_n^{(4)}, X_n^{(5)} \in \sigma(\tilde{v}, y_1, ..., y_{n-1})$. Furthermore, by the above we have

$$
\begin{aligned}
&\mathbb{E}[X_n^{(1)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\\
&= X_{n-1}^{(1)} + \mathbb{E}[\Delta X_n^{(1)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\\
&= X_{n-1}^{(1)} - \lambda_n\mathbb{E}[\hat{y}_n|\sigma(\tilde{v}, y_1, ..., y_{n-1})] - \mu_n\hat{q}_{n-1}\\
&\quad = \big(1 - \lambda_n(\beta_n^I + \frac{\beta_n^R\Sigma_n^{(3)}}{\Sigma_n^{(2)}})\big)X_{n-1}^{(1)} - \big(\lambda_n(\alpha_n^I + \alpha_n^R + \beta_n^R) + \mu_n\big)X_{n-1}^{(2)},\\
&\mathbb{E}[X_n^{(2)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\\
&= X_{n-1}^{(2)} + \mathbb{E}[\Delta X_n^{(2)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\\
&= X_{n-1}^{(2)} + r_n\mathbb{E}[\hat{y}_n|\sigma(\tilde{v}, y_1, ..., y_{n-1})] + s_n\hat{q}_{n-1}\\
&= r_n(\beta_n^I + \frac{\beta_n^R\Sigma_n^{(3)}}{\Sigma_n^{(2)}})X_{n-1}^{(1)} + \big(1 + r_n(\alpha_n^I + \alpha_n^R + \beta_n^R) + s_n\big)X_{n-1}^{(2)}.
\end{aligned}
$$

Since all involved random variables are jointly normal, we have the formula

$$
\begin{aligned}
\mathbb{E}[X_n^{(i)}X_n^{(j)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})] &= \mathbb{E}[X_n^{(i)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\mathbb{E}[X_n^{(j)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\\
&+ \mathbb{E}\Big[\big(X_n^{(i)} - \mathbb{E}[X_n^{(i)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\big)\big(X_n^{(j)} - \mathbb{E}[X_n^{(j)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\big)\Big].
\end{aligned}
\tag{A.12}
$$

By applying this formula we produce

$$
\begin{aligned}
&\mathbb{E}[(X_n^{(1)})^2|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&= \big(\mathbb{E}[X_n^{(1)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\big)^2 + \mathbb{V}\big[X_n^{(1)} - \mathbb{E}[X_n^{(1)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\big] \\
&= \big(\mathbb{E}[X_n^{(1)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\big)^2 + \lambda_n^2(\beta_n^R)^2\mathbb{V}\Big[(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}) - \frac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}X_{n-1}^{(1)}\Big] + \lambda_n^2\sigma_w^2\Delta \\
&= \big(\mathbb{E}[X_n^{(1)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\big)^2 + \lambda_n^2(\beta_n^R)^2\Big(\Sigma_n^{(1)} - \frac{(\Sigma_n^{(3)})^2}{\Sigma_n^{(2)}}\Big) + \lambda_n^2\sigma_w^2\Delta, \\
&\mathbb{E}[(X_n^{(2)})^2|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&= \big(\mathbb{E}[X_n^{(2)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\big)^2 + \mathbb{V}\big[X_n^{(2)} - \mathbb{E}[X_n^{(2)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\big] \\
&= \big(\mathbb{E}[X_n^{(2)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\big)^2 + r_n^2(\beta_n^R)^2\mathbb{V}\Big[(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}) - \frac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}X_{n-1}^{(1)}\Big] + r_n^2\sigma_w^2\Delta \\
&= \big(\mathbb{E}[X_n^{(2)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\big)^2 + r_n^2(\beta_n^R)^2\Big(\Sigma_n^{(1)} - \frac{(\Sigma_n^{(3)})^2}{\Sigma_n^{(2)}}\Big) + r_n^2\sigma_w^2\Delta.
\end{aligned}
$$

Likewise we find via (A.12)

$$
\begin{aligned}
&\mathbb{E}[X_n^{(1)}X_n^{(2)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&= \mathbb{E}[\big(X_n^{(1)} - \mathbb{E}[X_n^{(1)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\big)\big(X_n^{(2)} - \mathbb{E}[X_n^{(2)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\big)] \\
&\quad + \mathbb{E}[X_n^{(1)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\mathbb{E}[X_n^{(2)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&= \mathbb{E}[X_n^{(1)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\mathbb{E}[X_n^{(2)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})] - \lambda_n r_n\sigma_w^2\Delta - \lambda_n r_n(\beta_n^R)^2\Big(\Sigma_n^{(1)} - \frac{(\Sigma_n^{(3)})^2}{\Sigma_n^{(2)}}\Big), \\
&\mathbb{E}[\tilde{v} - p_n|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&= \mathbb{E}[X_{n-1}^{(1)} + X_{n-1}^{(5)} - \Delta p_n|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&= X_{n-1}^{(1)} + X_{n-1}^{(5)} - \mathbb{E}[\Delta p_n|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&= X_{n-1}^{(1)} + X_{n-1}^{(5)} - \lambda_n\mathbb{E}[y_n|\sigma(\tilde{v}, y_1, ..., y_{n-1})] - \mu_n X_{n-1}^{(2)} + \mu_n X_{n-1}^{(4)} \\
&= -\lambda_n\Delta\theta_n^I + \big(1 - \frac{\lambda_n\beta_n^R\Sigma_n^{(3)}}{\Sigma_n^{(2)}}\big)X_{n-1}^{(1)} - \big(\lambda_n(\alpha_n^R + \beta_n^R) + \mu_n\big)X_{n-1}^{(2)} \\
&\quad - \lambda_n\beta_n^R X_{n-1}^{(3)} + (\lambda_n\alpha_n^R + \mu_n)X_{n-1}^{(4)} + X_{n-1}^{(5)}.
\end{aligned}
$$

$\diamondsuit$

Theorem A.3. *If Conditions 2.2 and 2.3 hold, then for $n = 0, ..., N-1$, the insider's value function has the following quadratic form*

$$\max_{\substack{\Delta\theta_k^I \in \sigma(\tilde{v}, y_1, ..., y_{k-1}) \\ n+1 \le k \le N}} \mathbb{E}\Big[\sum_{k=n+1}^{N} (\tilde{v} - p_k)\Delta\theta_k^I \Big| \sigma(\tilde{v}, y_1, ..., y_n)\Big] = I_n^{(0)} + \sum_{1 \le i \le j \le 5} I_n^{(i,j)} X_n^{(i)} X_n^{(j)}, \tag{A.13}$$

where $X_n^{(1)}, ..., X_n^{(5)}$ are defined in (2.34) *and Δp_n is defined by* (2.9) *for $\Delta\theta_n^R$ defined by* (2.7)*. Furthermore, the insider's optimal trading strategy is given by* (2.55)*.*

Proof. We prove the theorem by the backward induction.o we suppose that (A.13) holds for $n + 1$. The n'th iteration then becomes

$$\begin{aligned} &\max_{\substack{\Delta\theta_k^I \in \sigma(\tilde{v}, y_1, ..., y_{k-1}) \\ n \le k \le N}} \mathbb{E}\Big[\sum_{k=n}^{N} (\tilde{v} - p_k)\Delta\theta_k^I \Big| \sigma(\tilde{v}, y_1, ..., y_{n-1})\Big] \\ &= \max_{\Delta\theta_n^I \in \sigma(\tilde{v}, y_1, ..., y_{n-1})} \mathbb{E}\Big[(\tilde{v} - p_n)\Delta\theta_n^I + I_n^{(0)} + \sum_{1 \le i \le j \le 5} I_n^{(i,j)} X_n^{(i)} X_n^{(j)} \Big| \sigma(\tilde{v}, y_1, ..., y_{n-1})\Big]. \end{aligned} \tag{A.14}$$

Lemma A.2 shows that

$$\mathbb{E}\Big[(\tilde{v} - p_n)\Delta\theta_n^I + I_n^{(0)} + \sum_{1 \le i \le j \le 5} I_n^{(i,j)} X_n^{(i)} X_n^{(j)} \Big| \sigma(\tilde{v}, y_1, ..., y_{n-1})\Big] \tag{A.15}$$

is quadratic in $\Delta\theta_n^I$ and that the coefficient in front of $(\Delta\theta_n^R)^2$ is given by the left-hand-side of (2.43). Consequently, since (2.43) holds as part of Condition 2.3, the first-order condition is sufficient for optimality. This shows that (2.55) holds.

$\diamondsuit$

A.3 Rebalancer's optimization problem

In this section, we assume that Condition 2.2 and Condition 2.3 hold so we can define the "hat"-processes (2.16)-(2.20) as well as the rebalancer's state variable processes (2.37). We let p_n be defined by (2.9) and we fix the insider's strategy $\Delta\theta_n^I$ by (2.8). Similarly to before, $\Delta\theta_n^I$ depends on the rebalancer's historical demands $(\theta_i^R)_{i=1,2,...,n-1}$, even though the insider's strategy is fixed.

We will need the following analogue of Lemma A.2:

Lemma A.4. *Assume that Condition 2.2 and Condition 2.3 hold and define* $\Delta\theta_n^I$ *by* (2.8). *For* $\Delta\theta_n^R \in \sigma(\tilde{a}, y_1, ..., y_{n-1})$, $n = 1, ..., N$, *we have the following measurability properties:* $Y_n^{(1)}, ..., Y_n^{(5)} \in \sigma(\tilde{a}, y_1, ..., y_n)$ *as well as*

$$\sigma(\tilde{a}, y_1, ..., y_n) = \sigma(\tilde{a}, \hat{y}_1, ..., \hat{y}_n) \quad \textit{and} \quad \hat{y}_n - y_n \in \sigma(\tilde{a}, y_1, ..., y_{n-1}). \tag{A.16}$$

Based on this, we get the following expectations

$$\begin{aligned}
&\mathbb{E}[\hat{y}_n|\sigma(\tilde{a}, y_1, ..., y_{n-1})] = \big(\beta_n^R + \frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}\big)Y_{n-1}^{(1)} + \big(\alpha_n^I + \alpha_n^R - \frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}\big)Y_{n-1}^{(2)},\\
&\mathbb{E}[y_n|\sigma(\tilde{a}, y_1, ..., y_{n-1})] = \Delta\theta_n^R + \frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}Y_{n-1}^{(1)} + \big(\alpha_n^I - \frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}\big)Y_{n-1}^{(2)} - \alpha_n^I Y_{n-1}^{(4)} + \beta_n^I Y_{n-1}^{(5)},\\
&\mathbb{E}[Y_n^{(2)}|\sigma(\tilde{a}, y_1, ..., y_{n-1})] = r_n\big(\beta_n^R + \frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}\big)Y_{n-1}^{(1)} + \big(r_n(\alpha_n^I + \alpha_n^R - \frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}) + s_n + 1\big)Y_{n-1}^{(2)},\\
&\mathbb{E}[(Y_n^{(2)})^2|\sigma(\tilde{a}, y_1, ..., y_{n-1})] = \big(\mathbb{E}[Y_n^{(2)}|\sigma(\tilde{a}, y_1, ..., y_{n-1})]\big)^2 + r_n^2\sigma_w^2\Delta + r_n^2(\beta_n^I)^2\big(\Sigma_n^{(2)} - \frac{(\Sigma_n^{(3)})^2}{\Sigma_n^{(1)}}\big),\\
&\mathbb{E}[-(\tilde{a} - \theta_{n-1}^R)\Delta p_n|\sigma(\tilde{a}, y_1, ..., y_{n-1})] = -\lambda_n(Y_{n-1}^{(1)} + Y_{n-1}^{(3)})\Delta\theta_n^R\\
&- (Y_{n-1}^{(1)} + Y_{n-1}^{(3)})\Big(\frac{\lambda_n\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}Y_{n-1}^{(1)} + \big(\lambda_n(\alpha_n^I - \frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}) + \mu_n\big)Y_{n-1}^{(2)} - (\lambda_n\alpha_n^I + \mu_n)Y_{n-1}^{(4)} + \lambda_n\beta_n^I Y_{n-1}^{(5)}\Big).
\end{aligned}$$

Furthermore, $Y_n^{(1)}, Y_n^{(3)}, Y_n^{(4)}, Y_n^{(5)} \in \sigma(\tilde{a}, y_1, ..., y_{n-1})$ *and we have the dynamics*

$$\begin{aligned}
Y_n^{(1)} &= (1 - \beta_n^R)Y_{n-1}^{(1)} - \alpha_n^R Y_{n-1}^{(2)},\\
Y_n^{(3)} &= -\Delta\theta_n^R + \beta_n^R Y_{n-1}^{(1)} + \alpha_n^R Y_{n-1}^{(2)} + Y_{n-1}^{(3)},\\
Y_n^{(4)} &= -r_n\Delta\theta_n^R + r_n\beta_n^R Y_{n-1}^{(1)} + r_n\alpha_n^R Y_{n-1}^{(2)} + (r_n\alpha_n^I + s_n + 1)Y_{n-1}^{(4)} - r_n\beta_n^I Y_{n-1}^{(5)},\\
Y_n^{(5)} &= -\lambda_n\Delta\theta_n^R + \lambda_n\beta_n^R Y_{n-1}^{(1)} + \lambda_n\alpha_n^R Y_{n-1}^{(2)} + (\lambda_n\alpha_n^I + \mu_n)Y_{n-1}^{(4)} + (1 - \lambda_n\beta_n^I)Y_{n-1}^{(5)}.
\end{aligned}$$

Proof. We start by proving (A.16) by induction. We observe that

$$\begin{aligned}
&\sigma(\tilde{a}, y_1) = \sigma(\tilde{a}, \beta_1^I\tilde{v} + \Delta w_1) = \sigma(\tilde{a}, \hat{y}_1),\\
&\hat{y}_1 - y_1 = \hat{\theta}_1^R - \theta_1^R \in \sigma(\tilde{a}),
\end{aligned}$$

because $\hat{\theta}_1^R, \theta_1^R \in \sigma(\tilde{a})$. We then suppose that (A.16) holds for n in which case we get

$$\begin{aligned}
\hat{y}_{n+1} - y_{n+1} &= \Delta\hat{\theta}_{n+1}^R - \Delta\theta_{n+1}^R + \Delta\hat{\theta}_{n+1}^I - \Delta\theta_{n+1}^I \\
&= \Delta\hat{\theta}_{n+1}^R - \Delta\theta_{n+1}^R - \beta_{n+1}^I(\hat{p}_n - p_n) + \alpha_{n+1}^I(\hat{q}_n - q_n) \\
&\in \sigma(\tilde{a}, y_1, ..., y_n), \\
\sigma(\tilde{a}, \hat{y}_1, ..., \hat{y}_{n+1}) &= \sigma(\tilde{a}, y_1, ..., y_n, \hat{y}_{n+1}) = \sigma(\tilde{a}, y_1, ..., y_n, y_{n+1} + \hat{y}_{n+1} - y_{n+1}) \\
&= \sigma(\tilde{a}, y_1, ..., y_{n+1}).
\end{aligned}$$

For the above inclusion we used the facts $\Delta\hat{\theta}_{n+1}^R \in \sigma(\tilde{a}, \hat{y}_1, ..., \hat{y}_n)$, $\Delta\theta_{n+1}^R \in \sigma(\tilde{a}, y_1, ..., y_n)$, $\hat{p}_n, \hat{q}_n \in \sigma(\hat{y}_1, ..., \hat{y}_n)$, $p_n, q_n \in \sigma(y_1, ..., y_n)$, and the induction hypothesis.

To compute the conditional expectations of $\hat{y}_n$ and y_n we let $\hat{z}_n$ be defined by (A.3). Then we have

$$\begin{aligned}
&\mathbb{E}[\hat{y}_n|\sigma(\tilde{a}, y_1, ..., y_{n-1})] \\
&= \mathbb{E}[\Delta\hat{\theta}_n^I + \Delta\hat{\theta}_n^R + \Delta w_n|\sigma(\tilde{a}, y_1, ..., y_{n-1})] \\
&= \mathbb{E}[\Delta\hat{\theta}_n^I + \Delta\hat{\theta}_n^R|\sigma(\tilde{a}, \hat{z}_1, ..., \hat{z}_{n-1})] \\
&= \beta_n^R(\tilde{a} - \hat{\theta}_{n-1}^R) + (\alpha_n^I + \alpha_n^R)\hat{q}_{n-1} + \beta_n^I\mathbb{E}[\tilde{v} - \hat{p}_{n-1}|\sigma(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}, \hat{z}_1, ..., \hat{z}_{n-1})] \\
&= \beta_n^R(\tilde{a} - \hat{\theta}_{n-1}^R) + (\alpha_n^I + \alpha_n^R)\hat{q}_{n-1} + \beta_n^I\frac{\mathbb{E}[(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1})(\tilde{v} - \hat{p}_{n-1})]}{\mathbb{V}[\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}]}(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}) \\
&= \big(\beta_n^R + \frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}\big)Y_{n-1}^{(1)} + \big(\alpha_n^I + \alpha_n^R - \frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}\big)Y_{n-1}^{(2)}, \\
&\mathbb{E}[y_n|\sigma(\tilde{a}, y_1, ..., y_{n-1})] \\
&= \mathbb{E}[\Delta\theta_n^I + \Delta\theta_n^R|\sigma(\tilde{a}, \hat{z}_1, ..., \hat{z}_{n-1})] \\
&= \Delta\theta_n^R + \beta_n^I\mathbb{E}[\tilde{v} - \hat{p}_{n-1}|\sigma(\tilde{a}, \hat{z}_1, ..., \hat{z}_{n-1})] + \beta_n^I(\hat{p}_{n-1} - p_{n-1}) + \alpha_n^I q_{n-1} \\
&= \Delta\theta_n^R + \beta_n^I\mathbb{E}[\tilde{v} - \hat{p}_{n-1}|\sigma(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1})] + \beta_n^I(\hat{p}_{n-1} - p_{n-1}) + \alpha_n^I\hat{q}_{n-1} - \alpha_n^I(\hat{q}_{n-1} - q_{n-1}) \\
&= \Delta\theta_n^R + \frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}Y_{n-1}^{(1)} + \big(\alpha_n^I - \frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}\big)Y_{n-1}^{(2)} - \alpha_n^I Y_{n-1}^{(4)} + \beta_n^I Y_{n-1}^{(5)}.
\end{aligned}$$

We find

$$
\begin{aligned}
Y_n^{(1)} &= Y_{n-1}^{(1)} + \Delta Y_n^{(1)} = (1-\beta_n^R)Y_{n-1}^{(1)} - \alpha_n^R Y_{n-1}^{(2)},\\
Y_n^{(3)} &= Y_{n-1}^{(3)} + \Delta Y_n^{(3)} = -\Delta\theta_n^R + \beta_n^R Y_{n-1}^{(1)} + \alpha_n^R Y_{n-1}^{(2)} + Y_{n-1}^{(3)},\\
Y_n^{(4)} &= Y_{n-1}^{(4)} + \Delta Y_n^{(4)} = Y_{n-1}^{(4)} + r_n(\hat{y}_n - y_n) + s_n Y_{n-1}^{(4)},\\
&= -r_n\Delta\theta_n^R + r_n\beta_n^R Y_{n-1}^{(1)} + r_n\alpha_n^R Y_{n-1}^{(2)} + (r_n\alpha_n^I + s_n + 1)Y_{n-1}^{(4)} - r_n\beta_n^I Y_{n-1}^{(5)},\\
Y_n^{(5)} &= Y_{n-1}^{(5)} + \Delta Y_n^{(5)} = Y_{n-1}^{(5)} + \lambda_n(\hat{y}_n - y_n) + \mu_n Y_{n-1}^{(4)},\\
&= -\lambda_n\Delta\theta_n^R + \lambda_n\beta_n^R Y_{n-1}^{(1)} + \lambda_n\alpha_n^R Y_{n-1}^{(2)} + (\lambda_n\alpha_n^I + \mu_n)Y_{n-1}^{(4)} + (1-\lambda_n\beta_n^I)Y_{n-1}^{(5)}.
\end{aligned}
$$

Because $Y_{n-1}^{(1)}, ..., Y_{n-1}^{(5)} \in \sigma(\tilde{v}, y_1, ..., y_{n-1})$ we see that $Y_n^{(1)}, Y_n^{(3)}, Y_n^{(4)}, Y_n^{(5)} \in \sigma(\tilde{a}, y_1, ..., y_{n-1})$. The conditional expectation of $Y^{(2)}$ and $(Y^{(2)})^2$ can be seen as follows:

$$
\begin{aligned}
&\mathbb{E}[Y_n^{(2)}|\sigma(\tilde{a}, y_1, ..., y_{n-1})]\\
&= Y_{n-1}^{(2)} + r_n\mathbb{E}[\hat{y}_n|\sigma(\tilde{a}, y_1, ..., y_{n-1})] + s_n Y_{n-1}^{(2)}\\
&= r_n\big(\beta_n^R + \frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}\big)Y_{n-1}^{(1)} + \big(r_n(\alpha_n^I + \alpha_n^R - \frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}) + s_n + 1\big)Y_{n-1}^{(2)},\\
&\mathbb{E}[(Y_n^{(2)})^2|\sigma(\tilde{a}, y_1, ..., y_{n-1})]\\
&= \big(\mathbb{E}[Y_n^{(2)}|\sigma(\tilde{a}, y_1, ..., y_{n-1})]\big)^2 + \mathbb{E}\Big[\big(Y_n^{(2)} - \mathbb{E}[Y_n^{(2)}|\sigma(\tilde{a}, y_1, ..., y_{n-1})]\big)^2\Big]\\
&= \big(\mathbb{E}[Y_n^{(2)}|\sigma(\tilde{a}, y_1, ..., y_{n-1})]\big)^2 + r_n^2\mathbb{V}\Big[\Delta w_n + \beta_n^I(\tilde{v} - \hat{p}_{n-1}) - \frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1})\Big]\\
&= \big(\mathbb{E}[Y_n^{(2)}|\sigma(\tilde{a}, y_1, ..., y_{n-1})]\big)^2 + r_n^2\sigma_w^2\Delta + r_n^2(\beta_n^I)^2\big(\Sigma_n^{(2)} - \frac{(\Sigma_n^{(3)})^2}{\Sigma_n^{(1)}}\big),
\end{aligned}
$$

where we for the latter expectation have used (A.12). Finally, the following computation produces the last claim:

$$
\begin{aligned}
&\tilde{a} - \theta_{n-1}^R = Y_{n-1}^{(1)} + Y_{n-1}^{(3)},\\
&\mathbb{E}[\Delta p_n|\sigma(\tilde{a}, y_1, ..., y_{n-1})]\\
&= \lambda_n\mathbb{E}[y_n|\sigma(\tilde{a}, y_1, ..., y_{n-1})] + \mu_n Y_{n-1}^{(2)} - \mu_n Y_{n-1}^{(4)}\\
&= \lambda_n\Delta\theta_n^R + \frac{\lambda_n\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}Y_{n-1}^{(1)} + \big(\lambda_n(\alpha_n^I - \frac{\beta_n^I\Sigma_n^{(3)}}{\Sigma_n^{(1)}}) + \mu_n\big)Y_{n-1}^{(2)} - (\lambda_n\alpha_n^I + \mu_n)Y_{n-1}^{(4)} + \lambda_n\beta_n^I Y_{n-1}^{(5)}.
\end{aligned}
$$

◊

Theorem A.5. *If Conditions 2.2 and 2.3 hold, then for $n = 0, 1, ..., N-1$ the rebalancer's value function has the following quadratic form*

$$\max_{\substack{\Delta\theta^R_k \in \sigma(\tilde{a}, y_1, ..., y_{k-1}) \\ n+1 \leq k \leq N-1}} -\mathbb{E}\Big[\sum_{k=n+1}^{N} (\tilde{a} - \theta^R_{k-1})\Delta p_k \Big| \tilde{a}, y_1, ..., y_n\Big] = L^{(0)}_n + \sum_{1\leq i \leq j \leq 5} L^{(i,j)}_n Y^{(i)}_n Y^{(j)}_n, \tag{A.17}$$

where $Y^{(1)}_n, ..., Y^{(5)}_n$ are defined in (2.37) *and Δp_n is defined by* (2.9) *for $\Delta\theta^I_n$ defined by* (2.8)*. Furthermore, the rebalancer's optimal trading strategy is given by* (2.61)*.*

Proof. The proof is similar to the proof of Theorem A.3 and is therefore omitted.

♢

A.4 Remaining proofs

Proof of Theorem 2.4. Part (iii) of Definition 2.1 holds from Lemma A.1. Parts (i)-(ii) of Definition 2.1 hold from Theorem A.3 and Theorem A.5 as soon as we show that the optimizers (2.55) and (2.61) agree with (2.16) and (2.17). Equations (2.39) and (2.40) ensure that

$$\beta^I_n = \gamma^{(1)}_n, \quad \beta^R_n = \delta^{(1)}_n, \quad n = 1, ..., N.$$

So for $n = 1$ the optimal strategies (2.55) and (2.61) agree with $(\hat{\theta}^I_1, \hat{\theta}^R_1)$ stated in (2.16) and (2.17). Consequently, (p_1, q_1, y_1) corresponding to the optimal strategies are equal to $(\hat{p}_1, \hat{q}_1, \hat{y}_1)$. This implies that the deviation state variables satisfy

$$X^{(3)}_1 = X^{(4)}_1 = X^{(5)}_1 = Y^{(3)}_1 = Y^{(4)}_1 = Y^{(5)}_1 = 0.$$

To inductively show that the same holds true for $n = 2, ..., N$ we use that (2.41)-(2.42) define (α^R_n, α^I_n) and that (2.29)-(2.30) define (μ_n, s_n). Since the deviation state variables at time $n-1$ are all zero by hypothesis and since the strategy coefficients are identical for (θ^I_n, θ^R_n) and $(\hat{\theta}^I_n, \hat{\theta}^R_n)$, the realization of the strategies at time n are identical too.

The coefficient restriction in the last part of the theorem is proven by the induction. From the definition of the insider's value function we have $I^{(1,2)}_N = I^{(2,2)}_N = I^{(2,3)}_N = I^{(2,4)}_N = I^{(2,5)}_N = 0$. By using that $\beta^R_N = 1, \alpha^R_N = 0$, and equations (2.42) and

(2.57) we conclude that $\alpha_N^I = \gamma_N^{(2)} = 0$.

For the induction step we suppose that $\alpha_{n+1}^I = I_{n+1}^{(1,2)} = I_{n+1}^{(2,2)} = I_{n+1}^{(2,3)} = I_{n+1}^{(2,4)} = I_{n+1}^{(2,5)} = \gamma_{n+1}^{(2)} = 0$. Then the following recursion for $I^{(i,j)}$,

$$I_n^{(0)} + \sum_{1\le i\le j\le 5} I_n^{(i,j)} X_n^{(i)} X_n^{(j)} = \mathbb{E}\Big[I_{n+1}^{(0)} + \sum_{1\le i\le j\le 5} I_{n+1}^{(i,j)} X_{n+1}^{(i)} X_{n+1}^{(j)} \Big| \sigma(\tilde{v}, \hat{y}_1, ..., \hat{y}_n) \Big],$$

produces $I_n^{(1,2)} = I_n^{(2,2)} = I_n^{(2,3)} = I_n^{(2,4)} = I_n^{(2,5)} = 0$. By again using equations (2.42) and (2.57) we conclude that $\alpha_n^I = \gamma_n^{(2)} = 0$.

$\diamondsuit$

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