Trading in Crowded Markets

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We study crowded markets using a symmetric continuous-time model with strategic informed traders. We model crowdedness by assuming that traders may have incorrect beliefs about the number of smart traders in the market and the correlation among private signals, which distort their inference, trading strategies, and market prices. If traders underestimate the crowdedness, then markets are more liquid, both permanent and temporary market depths tend to be higher, traders take larger positions and trade more on short-run profit opportunities. In contrast, if traders overestimate the crowdedness, then traders believe markets to be less liquid, they are more cautious in both trading on their information and supplying liquidity to others; fears of crowded markets may also lead to "illusion of liquidity" so that the actual endogenous market depth is even lower than what traders believe it to be. Crowdedness makes markets fragile, because flash crashes, triggered whenever some traders liquidate large positions at fire-sale rates, tend to be more pronounced.

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With a dramatic growth in the asset management industry, financial markets have become platforms where the sophisticated institutional players trade intensively with each other, while retail investors are of much less importance. Even though traders usually seek for overlooked opportunities and try to add diversity into their portfolios, many trading strategies often become crowded. Traders are concerned about crowded markets, because these markets tend to be more fragile and prone to crashes.

Traders come up with investment ideas for generating alphas, evaluate transaction costs of implementing these strategies in real markets, and try to assess, often informally, to what extent strategies might be crowded, i.e., how many other traders might be simultaneously entering the same strategy space and to what extent their private signals might be correlated. While there has been lots of academic research in finance on anomalies in asset returns and liquidity, until recently the question about crowding has received little formal attention. In the 2009-presidential address, Jeremy Stein emphasizes this point and notes that "for a broad class of quantitative trading strategies, an important consideration for each individual arbitrageur is that he cannot know in real time exactly how many others are using the same model and taking the same position as him." Recognizing the importance of this issue, some firms started to provide tools for identifying and measuring crowdedness of trades and strategies, for example, such as the "crowding scorecard" offered by the MSCI. In this paper, we fill the gap and study theoretically the crowded-market problem, analyzing how thinking about crowdedness interacts with other aspects of trading, such as private information and liquidity.

We consider a stationary continuous-time model of trading among oligopolistic traders. Traders observe flows of private information about asset's fundamental value and trade on their disagreement about the precision of private information. Traders are of two types. "Smart" traders observe private information with high precision and other traders observe private information with low precision; yet, each trader believes that he observes private information with high precision. All traders trade strategically. They take into account how their trades affect prices and smooth out the execution of their bets over time. This modelling structure is borrowed from the smooth trading model of Kyle, Obizhaeva and Wang (2017) due to its convenience and tractability.

We model crowding by assuming that traders make informed guesses about how many of

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their peers might be investing in the same trading strategies, how correlated their private signals might be, and how many of them are smart traders. The perceived subjective characteristics about the number of traders, the correlation among private signals, and the number of smart traders can differ from true characteristics defining the market. Actual characteristics are hard to observe, and trader may either underestimate or overestimate these parameters. Our approach differs from the approach in Callahan (2004) and Stein (2009), who propose to model crowding as the uncertainty about the number of traders, but assume that market participants have unbiased estimates about model parameters.

Each trader trades toward a target inventory, which is proportional to the difference between his own valuation and the average valuation of other market participants, inferred from prices and dividends. The price-based mechanism works properly in our model, as traders do learn from history of prices and condition their strategies on their estimates of fundamental values. Trading strategies are not required to be "unanchored," this is in sharp contrast with Stein (2009). In the equilibrium, since traders optimally choose their consumption path together with trading strategies using their subjective beliefs, strategies depend only on traders' subjective parameters, not the actual model parameters. Yet, the equilibrium price also reflects the true number of traders, since it is obtained through the actual market-clearing mechanism, which aggregates demand functions of all traders.

Can traders learn about their mistakes by observing price dynamics? For the case when traders might mis-estimate the total number of traders, traders can learn the average of other traders' signals from prices, but it is impossible for them to figure out the average of exactly how many signals get into the pricing formula. We show that under the *consistency condition* that imposes a restriction on the relationship between traders' beliefs about the number of peers and the correlation among private signals, traders cannot learn about their possibly wrong beliefs from observable prices and price volatility. The consistency condition requires that traders either simultaneously underestimate or overestimate both the number of traders and correlation among private signals, though the adjustment in correlation estimates satisfying the consistency condition tends to be very small. The main impact on market liquidity and trading strategies is coming from mis-estimation of the number of traders. Thus, we view this consistency condition as a reasonable one for real-world markets. The intuition is simple. For example, if traders simultaneously overestimate the number of peers and correlation among private signals, i.e., they overestimate the crowdedness of the market, then traders would expect a relatively lower volatility due to a larger number of peers and a relatively higher volatility due to a higher correlation among private signals. When both effects perfectly balance each other, traders can not learn from price dynamics about their mistakes. Similar arguments apply for the case when traders underestimate the crowdedness.

In fact, traders can learn about their mistakes only by experimenting and deviating from equilibrium strategies or from one-time off-equilibrium events, which may allow traders to learn about actual slope of residual demand function. In practice, this type of experiment can be expensive to implement. Even if traders could learn about the actual total number of traders by obtaining some data on residual demand schedules, they still cannot know in real time exactly how many smart traders are trading in the same direction. We study market properties in this situation as well.

In our model, there is a temporary market depth and a permanent market depth that depend on the execution speed and the size of executed orders, respectively. Since traders build their calculations based on *subjective* market-clearing condition, the perceived market depth may differ from actual market depth in the market. Perceived market depth differs from actual market depth by a factor approximately equal to the ratio of the perceived number of traders to the actual number of traders.

Fear of a crowded market may lead to *illusion of liquidity*. We show that when traders overestimate how crowded the market is, they overestimate both temporary and permanent market depth in comparison with actual market depth. However, fear of crowded markets tends to decrease both perceived and actual market depth. Traders trade less intensively, take smaller positions, and are less willing to supply liquidity to other traders. In contrast, when traders underestimate how crowded the market is, they trade more aggressively, take larger positions, and readily supply liquidity to others.

Crowded markets dominated by institutional investors are often blamed for increased fragility and instability of financial markets, see for example Basak and Pavlova (2013). Market crashes often occur when some market participants are liquidating substantial positions at a fast pace (e.g., Kyle and Obizhaeva (2016)). We model one-time off-equilibrium execution of large orders and study how the market reaction changes depending on traders' beliefs about market crowdedness. The more traders overestimate the number of their peers, the less they are willing to provide liquidity to others, and the more pronounced are flash-crash patterns.

The crowded-trade hypothesis is often mentioned in discussions about some important finance episodes. During the market-neutral "quant meltdown" in August of 2007, some of the most successful hedge funds suddenly experienced massive losses, even though the overall market itself did not move much. Khandani and Lo (2010) and Pedersen (2009) discuss a popular hypothesis that attributes this event to unprecedentedly large number of hedge funds investing in similar quantitative strategies. The anecdotal evidence shows that crowding in strategies may play roles during the unwinding of carry trades as well as during the momentum crashes. Stein (2009) also illustrates the effect of crowding using a case study about announced changes in the construction of MSCI indices in 2001-2002 that created a profit opportunity for arbitrageurs. In anticipation of trading by index fund managers in response to changes in index weights, arbitrageurs could in theory buy stocks whose weights were known to increase and sell stocks whose weights were known to decrease. This strategy though did not result in predicted profits in practice, perhaps because too many arbitrageurs rushed into this opportunity at the beginning and this led to price overshooting followed by correction.

Our paper contributes to the existing literature on crowded markets. Stein (2009) proposes a one-period model, in which some traders underreact to their private signals, and uncertain number of arbitrageurs chase to profit on this opportunity. To keep things simple, he makes a number of simplifying assumptions by hard-wiring existence of anomalies, restricting strategies, and considering limiting cases. Arbitrageurs do not condition their strategies on their own estimates of fundamental values and their demand functions may be a non-decreasing functions of asset prices. In contrast, in our model, except for overconfidence, traders apply Bayes Law consistently, optimize correctly, and dynamically update their estimates of both alphas and target inventories. Stein (2009) suggests that the effect of crowding among arbitrageurs on market efficiency is likely to exhibit complicated patterns. When there is uncertainty about the degree of crowding, in some cases prices might be pushed further away from fundamentals.

Another related paper is Callahan (2004), who analyze the model of Kyle (1985) with added uncertainty about the number of informed traders. Under specific assumptions about signals of informed traders, he obtains a solution for the case when the total number of informed traders is some unknown number less or equal to two. In contrast, we model crowded markets in oligopolistic setting and the number of strategic informed traders can be any number greater than two. Kondor and Zawadowski (2016) study another issue related to crowding. They analyze how learning induced by competition affects capital allocation and welfare. They find that additional potential entrants do not improve efficiency of capital allocation and decrease social welfare.

Thinking about crowded markets has recently become important in public policy discussions. Regulators are increasingly concerned about whether some strategies and market segments become crowded and whether any of them are currently at risk of unwinding. For example, crowded trades and concentration on a small set of risk factors may create a systemic risk for a central clearing party and financial system, when some traders are forced to liquidate their positions, as discussed in Menkveld (2017). Our model suggests that market turns to be more vulnerable of crashes when traders overestimate the fraction of traders who are trading in the same direction.

It is difficult to identify and track crowded trades. A number of studies propose and test some measures of crowdedness. Pojarliev and Levich (2011) measure the style crowdedness in currency trades as the percentage of funds with significant positive exposure to a given style less the percentage of funds with significant negative exposure to the same style. Polk and Lou (2013) gauge the level of arbitrageurs crowdedness in momentum strategies from high-frequency (daily or weekly) abnormal returns correlations among stocks in the winner and/or loser portfolios. Sokolovski (2016) applies both measures to analyze dynamics in returns of carry trades. Hong et al. (2013) suggest using days-to-cover metrics, defined as the ratio of a stock's short interest to trading volume, which is expected to be a proxy for the cost of exiting crowded trade. Yan (2013) measures the crowdedness by combining the short interest ratio and the exit rate of institutional investors, defined as the number of shares liquidated; he shows that momentum losses can often be avoided by shorting only non-crowded losers. Usually researchers find empirically that these measures provide useful information about following up performance of strategies. Strategies may work well as long as they are not crowded, and they tend to crash or revert when crowdedness increases.

This paper is structured as follows. Section 1 describes a continuous-time model of crowded markets. Section 2 presents some comparative statics and studies the implications of crowding. Section 3 examines how crowdedness may affect the magnitude of flash crashes and implementation shortfalls. Section 4 concludes. All proofs are in the Appendix.

1. A Model of Crowded Market

We consider a dynamic model of trading among N oligopolistic traders. There is a risky security with zero net supply, which pays out dividends at continuous rate $D(t)$. The dividend $D(t)$ is publicly observable and follows a stochastic process with mean-reverting stochastic growth rate $G^*(t)$. The dividend has a constant instantaneous volatility $\sigma_D > 0$ and constant rate of mean reversion $\alpha_D > 0$,

$$dD(t) := -\alpha_D\, D(t)\, dt + G^*(t)\, dt + \sigma_D\, dB_D(t), \tag{1}$$

where $G^*(t)$ is unobservable growth rate. The growth rate $G^*(t)$ follows an AR-1 process with mean reversion $\alpha_G > 0$ and volatility $\sigma_G > 0$,

$$dG^*(t) := -\alpha_G\, G^*(t)\, dt + \sigma_G\, dB_G(t). \tag{2}$$

Each trader n observes a continuous stream of private information $I_n(t)$ defined by

$$dI_n(t) := \tau_n^{1/2}\, \frac{G^*(t)}{\sigma_G\, \Omega^{1/2}}\, dt + \rho^{1/2} dZ(t) + (1-\rho)^{1/2} dB_n(t). \tag{3}$$

Since its drift is proportional to $G^*(t)$, each increment $dI_n(t)$ in the process $I_n(t)$ is a noisy observation of $G^*(t)$. The denominator $\sigma_G\, \Omega^{1/2}$ scales $G^*(t)$ so that its conditional variance is one. The parameter Ω measures the steady-state error variance of the trader's estimate of $G^*(t)$ in units of time; it is defined algebraically below (see equation (8)). The precision parameter τ_n measures the informativeness of the signal $dI_n(t)$ as a signal-to-noise ratio describing how fast new information flows into the market. The error terms are correlated, and $Cov(dI_n, dI_m) = \rho dt$ for $m \neq n$, where $\rho < 1$.

The stream of dividends contains some information about the growth rate as well. Define $dI_0(t) := [\alpha_D\, D(t)\, dt + dD(t)] / \sigma_D$ and $dB_0 := dB_D$. Then, $dI_0(t)$ can be written

$$dI_0(t) := \tau_0^{1/2}\, \frac{G^*(t)}{\sigma_G\, \Omega^{1/2}}\, dt + dB_0(t), \quad \text{where} \quad \tau_0 := \frac{\Omega\, \sigma_G^2}{\sigma_D^2}, \tag{4}$$

so that public information $dI_0(t)$ in the divided stream $D(t)$ has a form similar to the notation for private information. The process $I_0(t)$ is informationally equivalent to the dividend process $D(t)$. The quantity τ_0 measures the precision of the dividend process. The Brownian motions $dB_0(t)$, $dZ(t)$, $dB_1(t)$,…, $dB_N(t)$ are independently distributed.

To model trading, we assume that all traders agree about the precision of the public signal τ_0, but agree to disagree about the precisions of private signals τ_n. Each trader n is certain that his own private information has high precision $\tau_n = \tau_H$ and $N-1$ other traders can be of two types: $N_I - 1$ traders have private information with high precision τ_H and the other $N_U := N - N_I$ traders have private information with low precision τ_L, where $\tau_H > \tau_L \geq 0$.

Denote the fraction of other traders (except trader n himself) with high precision in the market as

$$\theta := \frac{N_I - 1}{N_U + N_I - 1}. \tag{5}$$

This implies that $1 - \theta$ fraction of other traders' private information has low precision. Traders do not know each others' type.

To model crowded markets, we make the following two assumptions that capture two different aspects of these markets. First, traders might make incorrect estimates about the total number of traders; we assume that all traders symmetrically think that there are $N_s := N_{Is} + N_{Us}$ participants. Second, traders might have incorrect beliefs about correlations in private signals (3); we assume that traders symmetrically believe that $Cov(dI_n, dI_m) = \rho_s dt$ for $m \neq n$. Trader may also have subjective beliefs θ_s about the fraction of informed traders. We use subscripts s to denote subjective beliefs to differentiate them from the actually correct parameters N, ρ, and θ. We assume that traders' beliefs about the number of traders and the correlation of signals are some known constants. There is no uncertainty about the number of traders and correlation. We study how mistakes in traders' views about these parameters affect trading, prices, and liquidity.

We refer to the model with crowding as $(N_{Is}, N_{Us}, \rho_s; N_I, N_U, \rho)$-model, where N_I, N_U, and ρ are objective parameters describing the environment, and N_{Is}, N_{Us}, and ρ_s are subjective parameters describing traders' beliefs. We refer to the model without crowding as $(N_I, N_U, \rho; N_I, N_U, \rho)$-model, where traders have correct beliefs about the correlation in private signals of market participants and the number of traders. For any $(N_{Is}, N_{Us}, \rho_s; N_I, N_U, \rho)$-model, equilibrium strategies depend only on the parameters N_{Is}, N_{Us}, and ρ_s, since traders make their decisions based only on subjective beliefs, not the actual parameters. The equilibrium price though is a result of the correct market clearing based on the actual total number of traders in the market $N = N_I + N_U$. In spite of the fact that the equilibrium strategies depend only on the parameters N_{Is}, N_{Us}, and ρ_s.

Let $S_n(t)$ denote the inventory of trader n at time t. Each trader n chooses a consumption intensity $c_n(t)$ and trading intensity $x_n(t)$ to maximize an expected constant-absolute-risk-aversion (CARA) utility function $U(c_n(s)) := -e^{-A\, c_n(s)}$ with risk aversion parameter

A. Letting $\beta > 0$ denote a time preference parameter, trader n solves the maximization problem

$$\max_{\{c_n(t), x_n(t)\}} \quad E_t^n \left\{ \int_{s=t}^{\infty} e^{-\beta(s-t)}\, U(c_n(s))\, ds \right\}, \tag{6}$$

where trader n's inventories follow the process $dS_n(t) = x_n(t)\, dt$ and his money holdings $M_n(t)$ follow the stochastic process

$$dM_n(t) = (r\, M_n(t) + S_n(t)\, D(t) - c_n(t) - P(t)\, x_n(t))\, dt. \tag{7}$$

Each trader trades "smoothly" in the sense that $S_n(t)$ is a differentiable function of time with trading intensity $x_n(t) = dS_n(t)/dt$. Each trader explicitly takes into account how both the level of his inventory $S_n(t)$ and the derivative of his inventory $x_n(t)$ affect the price of a risky asset $P(t)$.

Each trader dynamically adjusts his estimates and their error variance. We use $E_t^n\{\ldots\}$ to denote the expectation of trader n calculated with respect to his information at time t. The superscript n indicates that the expectation is taken with respect to the beliefs of trader n. The subscript t indicates that the expectation is taken with respect to trader n's information set at time t, which consists of both private information as well as public information extracted from the history of dividends and prices.

Let $G_n(t) := E_t^n\{G^*(t)\}$ denote trader n's estimate of the growth rate. Let Ω denote the steady state error variance of the estimate of $G^*(t)$, scaled in units of the standard deviation of its innovation σ_G. Stratonovich-Kalman-Bucy filtering implies that, for the beliefs of any trader n, the total precision τ and scaled error variance Ω are constants that do not vary over time and given by

$$\Omega := Var\left\{ \frac{G^*(t) - G_n(t)}{\sigma_G} \right\} = (2\,\alpha_G + \tau)^{-1}, \tag{8}$$

$$\tau = \tau_0 + \tau_H + (N_s - 1) \frac{\left((\theta_s - \rho_s)\tau_H^{1/2} + (1 - \theta_s)\tau_L^{1/2} \right)^2}{(1 - \rho_s)(1 + (N_s - 1)\rho_s)}. \tag{9}$$

Define signals of trader n and the average signal of other traders as

$$H_n(t) := \int_{u=-\infty}^{t} e^{-(\alpha_G+\tau)\,(t-u)}\, dI_n(u), \quad n = 0, 1, \ldots, N_s, \tag{10}$$

and

$$H_{-n}(t) := \tfrac{1}{N_s-1} \sum_{\substack{m=1 \\ m\neq n}}^{N_s} H_m(t). \tag{11}$$

The importance of each bit of information dI_n about the growth rate decays exponentially at a rate $\alpha_G + \tau$, i.e., the sum of the decay rate α_G of fundamentals and the speed τ of learning about fundamentals.

Trader n's estimate $G_n(t)$ can be conveniently written as the weighted sum of three sufficient statistics $H_0(t)$, $H_n(t)$, and $H_{-n}(t)$, which summarize the information content of dividends, his private information, and other traders' private information, respectively. The filtering formulas imply that trader n's expected growth rate $G_n(t)$ is a linear combination given by

$$\begin{aligned} G_n(t) := \sigma_G\, \Omega^{1/2} \Big(\tau_0^{1/2}\, H_0(t) + (1-\theta_s) \left(\tau_H^{1/2} - \tau_L^{1/2}\right) / (1-\rho_s)\, H_n(t) \\ + \frac{(\theta_s - \rho_s)\tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2}}{(1-\rho_s)(1+(N_s-1)\rho_s)}\, \left(H_n(t) + (N_s-1)H_{-n}(t)\right)\Big). \end{aligned} \tag{12}$$

This equation has a simple intuition. Each trader places the same weight $\tau_0^{1/2}$ on the dividend-information signal $H_0(t)$, assigns a larger weight to his own signal $H_n(t)$ and a lower weight to signals of presumably $N_s - 1$ other traders, aggregated in variable $H_{-n}(t)$.

We focus on a symmetric linear equilibrium. To reduce the number of state variables, it is convenient to replace the three state variables $H_0(t)$, $H_n(t)$, $H_{-n}(t)$ with two composite state variables $\hat{H}_n(t)$ and $\hat{H}_{-n}(t)$ defined using a constant $\hat{a}$ by

$$\hat{H}_n(t) := H_n(t) + \hat{a}\, H_0(t), \quad \hat{H}_{-n}(t) := H_{-n}(t) + \hat{a}\, H_0(t), \tag{13}$$

$$\hat{a} := \frac{(1+(N_s-1)\rho_s)\tau_0^{1/2}}{(1+(N_s-1)\theta_s)\tau_H^{1/2} + (N_s-1)(1-\theta_s)\tau_L^{1/2}}. \tag{14}$$

The trader n conjectures that the symmetric linear demand schedules for other traders m, $m \neq n, m = 1, \ldots, N_s$ is given by

$$x_m(t) = \frac{dS_m(t)}{dt} = \gamma_D\, D(t) + \gamma_H\, \hat{H}_m(t) - \gamma_S\, S_m(t) - \gamma_P\, P(t). \tag{15}$$

Each trader thinks that his flow-demand $x_n(t) = dS_n(t)/dt$ must satisfy the following market clearing

$$x_n(t) + \sum_{\substack{m=1 \\ m \neq n}}^{N_s} \Big(\gamma_D\, D(t) + \gamma_H\, \hat{H}_m(t) - \gamma_S\, S_m(t) - \gamma_P\, P(t) \Big) = 0, \tag{16}$$

which depends on his estimate N_s about the number of traders in the market. Using zero net supply restriction $\sum_{m=1}^{N_s} S_m(t) = 0$, he solves this equation for $P(t)$ as a function of his own trading speed $x_n(t)$ to obtain his estimate about the residual supply function,

$$P(x_n(t)) = \frac{\gamma_D}{\gamma_P}\, D(t) + \frac{\gamma_H}{\gamma_P}\, \hat{H}_{-n}(t) + \frac{\gamma_S}{(N_s - 1)\gamma_P}\, S_n(t) + \frac{1}{(N_s - 1)\gamma_P}\, x_n(t). \tag{17}$$

Then, each trader n exercises monopoly power in choosing how fast to demand liquidity from other traders to profit from private information. He also exercises monopoly power in choosing how fast to provide liquidity to the other $N_s - 1$ traders. Trader n solves for his optimal consumption and trading strategy by plugging the price impact function (17) into his dynamic optimization problem (6). Although strategies are defined in terms of the average of other traders' signals $H_{-n}(t)$, each trader believes that equilibrium prices reveal the average private signal, which enables him to implement his equilibrium strategy by conditioning his trading speed on market prices.

1.1. Prices and Consistency Condition

The equilibrium price is determined based on the *actual* market clearing condition which sums up demands of the actual number of traders N in the market,

$$\sum_{m=1}^{N} x_m(t) = 0, \quad \text{and} \quad \sum_{m=1}^{N} S_m(t) = 0. \tag{18}$$

Using equations (15) and (17), we obtain the actual equilibrium price

(19)

$$P(t)=\frac{\gamma_D(N_s,\theta_s,\rho_s)}{\gamma_P(N_s,\theta_s,\rho_s)}D(t)+\frac{\gamma_H(N_s,\theta_s,\rho_s)}{\gamma_P(N_s,\theta_s,\rho_s)}\hat{a}(N_s,\theta_s,\rho_s)H_0(t)+\frac{\gamma_H(N_s,\theta_s,\rho_s)}{N\gamma_P(N_s,\theta_s,\rho_s)}\sum_{m=1}^{N}H_m(t).$$

By contrast, each trader uses in his calculations the *subjective* market clearing condition by summing up demands of the perceived number of traders N_s,

(20)
$$\sum_{m=1}^{N_s}x_m(t)=0,\quad \text{and}\quad \sum_{m=1}^{N_s}S_m(t)=0.$$

Each trader believes that the equilibrium price is determined by

(21)

$$P_s(t)=\frac{\gamma_D(N_s,\theta_s,\rho_s)}{\gamma_P(N_s,\theta_s,\rho_s)}D(t)+\frac{\gamma_H(N_s,\theta_s,\rho_s)}{\gamma_P(N_s,\theta_s,\rho_s)}\hat{a}(N_s,\theta_s,\rho_s)H_0(t)+\frac{\gamma_H(N_s,\theta_s,\rho_s)}{N_s\gamma_P(N_s,\theta_s,\rho_s)}\sum_{m=1}^{N_s}H_m(t).$$

The only difference between the two pricing equations (19) and (21) are indices N and N_s over which the summation of private signals is done. In the equilibrium, all calculations are done from the perspective of traders, so only their subjective parameters enter equilibrium demands and prices. This is, for example, why parameters ρ and θ are not in the formulas. Among all objective parameters, only the objective number of traders N sneaks into the pricing formula through the actual market clearing mechanism (18).

Each trader observes the market price $P(t)$ but thinks that it is his conjectured price $P_s(t)$. He infers the average signal of all traders in the model, and (potentially incorrectly) interprets it as the average of N_s signals, rather than N signals.

In continuous time, it is not difficult to estimate accurately the diffusion variance of the process $dP(t)$ by looking at its quadratic variation. If traders simply misinterpret information about the averages in the price, then they would be able to learn about their mistakes from the price dynamics. For example, if traders underestimate the total number of participants in the market ($N_s < N$), then traders would expect to observe a relatively high price volatility comparing to what they see in the market, because errors in private signals would not average out.

Since traders cannot know in real time exactly how many other traders are investing in the same strategies, we make sure that incorrect estimates about the number of traders cannot be easily falsified by observing the price dynamics. This requires that the quadratic variation of actual price dynamics $dP(t)$ must coincide with the quadratic variation of perceived price dynamics $dP_s(t)$. Using equations (19) and (21), we obtain the consistency condition ensuring that the quadratic variation of $\rho^{1/2}dZ(t) + (1-\rho)^{1/2}\frac{1}{N}\sum_{m=1}^{N} dB_m(t)$ must coincide with the quadratic variation of $\rho_s^{1/2}dZ(t) + (1-\rho_s)^{1/2}\frac{1}{N_s}\sum_{m=1}^{N_s} dB_m(t)$.

COROLLARY 1: *Under the consistency condition*

$$\frac{1+(N-1)\rho}{N} = \frac{1+(N_s-1)\rho_s}{N_s} \tag{22}$$

such that $Var_n(dP(t)) = Var_n(dP_s(t))$*, we have*

$$Cov(dI_n(t), dP(t)) = Cov(dI_n(t), dP_s(t)). \tag{23}$$

The corollary means that, if the consistency condition (22) is satisfied, then for each trader, the correlation coefficient between his private signal and the actual price change is consistent with the subjective correlation between his private signal and price change. This condition ensures that traders can not learn about their mistakes from price dynamics.

The consistency condition imposes the restriction on N, N_s, ρ, and ρ_s. If $N_s < N$, then the condition implies that $\rho_s < \rho$, and vice versa. If traders underestimate the total number of participants in the market ($N_s < N$), they should simultaneously underestimate the correlation among their private signals ($\rho_s < \rho$) in order to bring downward the overestimated volatility of dollar price changes due to the underestimated number of traders.

1.2. Liquidity

Equation (17) defines the subjective permanent market depth $1/\lambda_s$ and temporary market depth $1/\kappa_s$, as inverse slopes of residual demand functions with respect to number of shares traded and the rate of trading,

$$1/\lambda_s := \frac{(N_s-1)\,\gamma_p}{\gamma_S}, \qquad 1/\kappa_s := (N_s-1)\,\gamma_p, \tag{24}$$

where λ_s is the permanent price impact coefficient and κ_s is the temporary price impact coefficient according to traders' views. Traders believe that markets are deeper when the number of traders is higher (N_s is high) and they tend to be more willing to provide liquidity to others (γ_p is high).

The subjective estimates of market liquidity may differ from the actual permanent market depth $1/\lambda$ and temporary market depth $1/\kappa$, because in reality the price is determined by the actual market clearing condition (16), but with N_s replaced by N. Using the market-clearing condition (16) and equation (17), subjective permanent and temporary market depth $1/\lambda_s$ and $1/\kappa_s$ are related to the actual ones as

$$1/\lambda_s = \frac{N_s - 1}{N - 1}\ 1/\lambda, \quad 1/\kappa_s = \frac{N_s - 1}{N - 1}\ 1/\kappa. \tag{25}$$

The subjective market depth is $\frac{N_s-1}{N-1}$ times of the objective one. If traders overestimate the number of traders in the market ($N_s > N$), they also overestimate both permanent and temporary market depth. We refer to this case as "illusion of liquidity." If traders underestimate the number of traders, they underestimate market depth and we refer to this case as "illusion of illiquidity." The subjective and objective market depth differ approximately by a factor of N_s/N. For example, when traders overestimate the number of total traders by 50 percent, the subjective market depth is larger than actual market depth also by about 50 percent, and vice versa.

Traders do not observe actual residual demand schedules in the equilibrium. They might be able to learn about the actual residual demand schedule's slopes by implementing a series of experiments and analyzing price responses to executions at some off-equilibrium trading rates. In practice, this type of experiments however are either infeasible or very costly to implement. Even if we assume that traders could learn about N by obtaining some data on residual demand schedules, they still can not learn about the fraction of informed traders θ.

1.3. Solution

The following theorem characterizes the equilibrium trading strategies and price. Traders calculate target inventories, defined as inventory levels such that trader n does not trade ($x_n(t) = 0$). Traders update their targets dynamically and trade toward them smoothly, thus optimizing the market impact of trading.

THEOREM 1: *There exists a steady-state equilibrium with symmetric linear flow-strategies and positive trading volume if and only if the six polynomial equations (C-38)–(C-43) have a solution satisfying the second-order condition* $\gamma_P > 0$ *and the stationarity condition* $\gamma_S > 0$*. Such an equilibrium has the following properties:*

1) *There is an endogenously determined constant* $C_L > 0$*, defined in equation (C-32), such that trader* n*'s optimal flow-strategy* $x_n(t)$ *is given by*

$$x_n(t) = \frac{dS_n(t)}{dt} = \gamma_S \, \left(S_n^{TI}(t) - S_n(t)\right), \tag{26}$$

where $S_n^{TI}(t)$ *is trader* n*'s "target inventory" defined as*

$$S_n^{TI}(t) = C_L \, \left(\hat{H}_n(t) - \hat{H}_{-n}(t)\right). \tag{27}$$

2) *There is an endogenously determined constant* $C_G > 0$*, defined in equation (C-32), such that the equilibrium price is*

$$P(t) = \frac{D(t)}{r + \alpha_D} + C_G \, \frac{\bar{G}(t)}{(r + \alpha_D)(r + \alpha_G)}, \tag{28}$$

where $\bar{G}(t) := \frac{1}{N} \sum_{n=1}^{N} G_n(t)$ *denotes the average expected growth rate.*

Trader n targets a long position if his own signal $\hat{H}_n(t)$ is greater than the average signal of other traders $\hat{H}_{-n}(t)$ and a short position vice versa. The proportionality constant C_L in equation (27) measures the sensitivity of target inventories to the difference. The parameter γ_S in equation (26) measures the speed of trading as the rate at which inventories adjust toward their target levels. The price in equation (28) immediately reveals the average of all signals. If C_G were equal to one, the price in equation (28) would equal the average of traders' risk-neutral buy-and-hold valuations, consistent with the Gordon's growth formula. Aggregation of heterogeneous beliefs in a dynamic model, which we refer to as the Keynesian beauty contest effect, makes the multiplier C_G less than one.

Obtaining an analytical solution for the equilibrium in Theorem 1 requires solving the six polynomial equations (C-38)–(C-43). While these equations have no obvious analytical solution, they can be solved numerically. Extensive numerical calculations lead us to conjecture that the existence condition for the continuous-time model is exactly the same as the existence condition for the similar one-period model presented in Appendix A:

CONJECTURE 1: ***Existence Condition.*** *A steady-state equilibrium with symmetric, linear flow-strategies exists if and only if*

$$\theta_s < 1 - \frac{N_s(1 - \rho_s)\tau_H^{1/2}}{(N_s - 1)(2 + (N_s - 2)\rho_s)(\tau_H^{1/2} - \tau_L^{1/2})} < 1. \tag{29}$$

Equation (29) implies that, for an equilibrium with positive trading volume to exist, the fraction of other traders whose private information has high precision θ_s cannot be too high. The existence condition is reduced to $\tau_H^{1/2}/\tau_L^{1/2} > 2 + \frac{N_s}{N_s-2}$ if $\rho_s = 0$ and $\theta_s = 0$, as in the setting of Kyle, Obizhaeva and Wang (2017). This condition requires $N_s \geq 3$ and $\tau_H^{1/2}$ to be sufficiently more than twice as large as $\tau_L^{1/2}$.

2. Properties of Crowded Markets

In this section, we study how changes in correlation of private signals and the number of traders whose private information has high precision affect market liquidity and traders' trading strategies.

2.1. Effects of Changes in Correlations of Private Signals

To develop intuition, we next consider the $(N_I, N_U, \rho; N_I, N_U, \rho)$-model with traders making no mistakes about the level of crowdedness, but the correlation ρ among private information potentially may take different values. We study how changes in ρ affect the market.

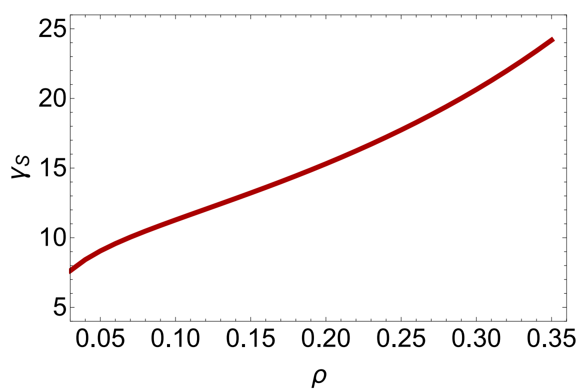


Figure 1. γ_S against ρ.

Figure 1 illustrates that the speed of trade γ_S increases with the correlation coefficient ρ.[1] When traders observe private signals with highly correlated errors, they engage in a rat race with each other, as in Foster and Vishwanathan (1996), and trade more aggressively at a higher speed γ_S toward their target inventory levels. Figure 2 shows that as ρ

[1]In Figures 1, 2, 3, 4, and 6 parameter values are $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_G = 0.02$, $\sigma_D = 0.5$, $\sigma_G = 0.1$, $\theta = 0.1$, $\tau_H = 1$, $\tau_L = 0.2$.

increases, the total precision of information τ decreases, the error variance of the growth rate estimates increases, and the coefficient γ_P increases, i.e., each trader is more willing to provide liquidity to others.[2]

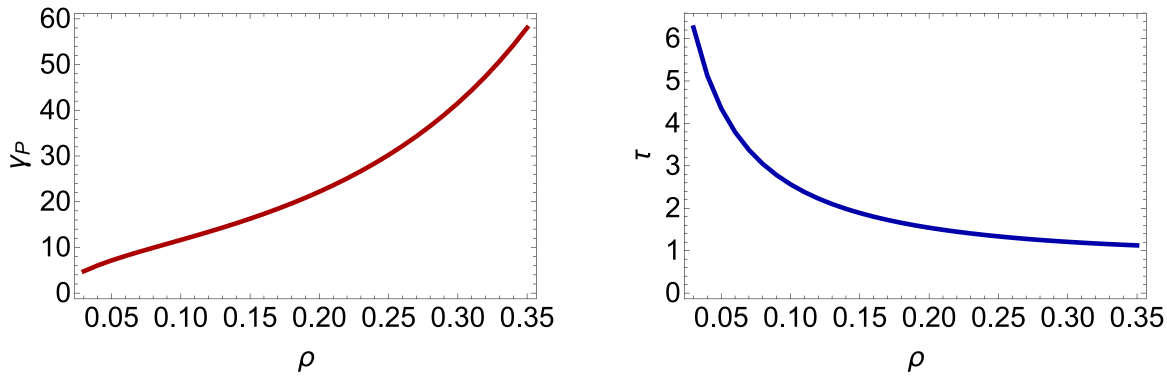


FIGURE 2. VALUES OF γ_P AND τ AGAINST ρ.

Figure 3 shows that both permanent market depth $1/\lambda$ and temporary market depth $1/\kappa$ increase, as ρ increases. The market becomes deeper. The perceived depth coincides with the actual market depth, because traders do not misestimate the number of their peers in the market.

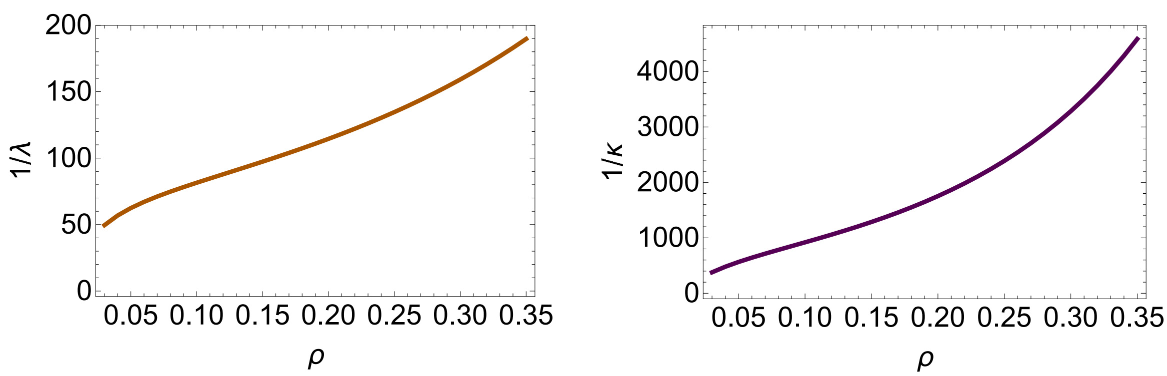


FIGURE 3. VALUES OF $1/\lambda$ AND $1/\kappa$ AGAINST ρ.

Traders believe that trading becomes more valuable and the value of trading on innovations to future information (built into the constant term $-\psi_0$ defined in trader's value function (C-20)) increases in correlation ρ, as shown in Figure 4.

[2]Total precision τ decreases with ρ as long as ρ is not very close to 1.

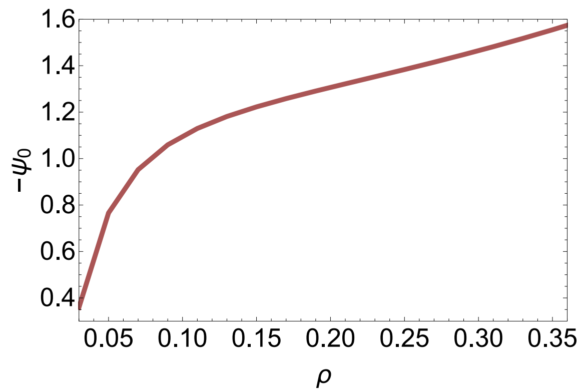
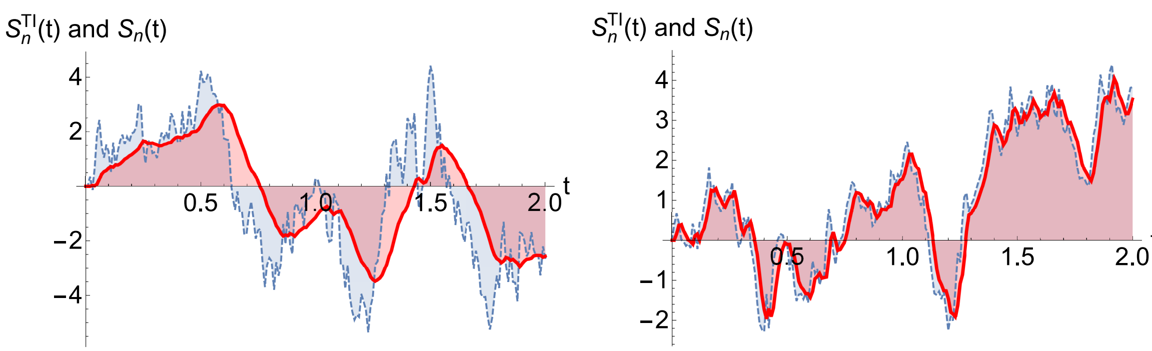


Figure 4. Value of trading on innovations to future information $-\psi_0$ against ρ.

Figure 5 presents two simulated paths for target inventories (dashed lines) and actual inventories (solid lines).[3] In panel (a) where correlation ρ is small, the market is less liquid, traders adjust their inventories at a lower rate to reduce transaction costs, and actual inventories may deviate significantly from target inventories. In panel (b) where correlation ρ is larger, the market is more liquid, traders adjust their inventories at a faster rate, and actual inventories closely track target inventories.



(a) With low correlation

(b) With high correlation

Figure 5. Simulated paths of $S_n^{TI}(t)$ (Dashed) and $S_n(t)$ (Solid).

Figure 6 shows that the coefficient C_G in the equilibrium pricing rule decreases, when the correlation coefficient ρ increases. Higher correlation among private signals leads to

[3]The paths are generated using equations (27), (C-18), and (C-19), which describe the dynamics of $\hat{H}_n(t), \hat{H}_{-n}(t)$, and $S_n^{TI}(t)$. Numerical calculations in Figure 5 are based on the exogenous parameter values $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_G = 0.02$, $\sigma_D = 0.5$, $\sigma_G = 0.1$, $\theta = 0.1$, $\tau_H = 1$, $\tau_L = 0.2$ in both (a) and (b); $\rho = 0.05$ in (a); $\rho = 0.5$ in (b).

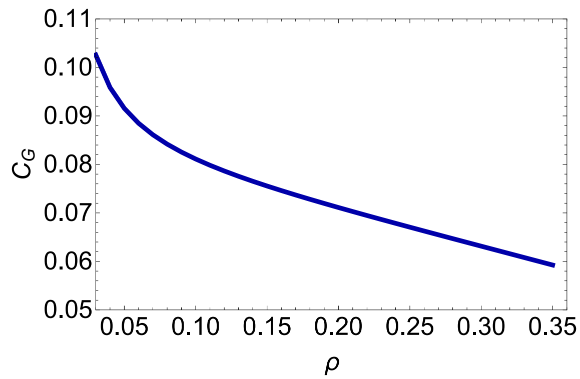


FIGURE 6. C_G AGAINST ρ.

more pronounced price dampening effect ($C_G < 1$). Indeed, in the model each trader believes that other traders make mistakes and they will revise their forecasts in the future. Due to highly correlated signals and a lot of liquidity, traders have greater incentives to engage in short-term speculative trading and take advantage of this predictability in short-term trading patterns of other traders, which could be quite different from expected price dynamics in the long run, because each trader believes that at some point in the future the price will converge to his own estimates of fundamentals.

2.2. Effects of Changes in Crowdedness

We next study properties of the $(N_{Is}, N_{Us}, \rho_s; N_I, N_U, \rho)$-markets where parameters N_I, N_U, and ρ describe the trading environment, and parameters N_{Is}, N_{Us}, ρ_s describe traders' subjective beliefs. Traders believe that there are N_{Is} and N_{Us} traders whose private information has high precision (e.g., "smart traders") and low precision, respectively, and that the correlation among innovations in private signals is equal to ρ_s. We study how beliefs of traders about the crowdedness of smart traders affect the market and its properties.

We consider two cases. In both cases, we fix the trading environment N_I, N_U, and ρ. In the first case, we vary beliefs of traders about the number of smart traders N_{Is} in the same market, but fix the number of traders whose private information has low precision $N_{Us} = N_U$. In the second case, we vary N_{Is} but fix the total number of traders $N_s = N_{Is} + N_{Us} = N$.[4]

[4]As we discussed, traders might implement a series of experiments and analyze price response to executions at some off-equilibrium trading rates to estimate the total number of traders in the market N. However, traders cannot know in real time exactly how many of these traders are smart, i.e., they don't know N_{Is}.

Since traders' estimate of the total number of traders N_s may differ from actual parameter N in the first case, we consider two subcases. In the base case, we change both N_{Is} and ρ_s in lockstep to satisfy the consistency condition (22) so that traders can not learn about their mistakes by observing price dynamics. The subjective correlation is calculated as $\rho_s = \frac{1}{N_s-1}\left(\frac{N_s}{N}(1+(N-1)\rho)-1\right)$. In another subcase, we change only N_{Is}, but keep $\rho_s = \rho$ fixed. These subcases allow us to disentangle the effects of changes in the perceived correlation ρ_s and the estimate of the number of traders N_{Is}. The first subcase is presented by solid lines, and the second subcase is presented in dashed lines in Figures 8, 9, 10, 13, 14, 15 and 16 below.

Figure 7 shows how ρ_s must change with changes in N_{Is} in order to satisfy the consistency condition. When N_{Is} is the same as the actual number of traders N_I ($N_I = N_{Is} = 30$), the subjective correlation ρ_s converges to the actual correlation ρ ($\rho = \rho_s = 0.20$). If N_{Is} drops from 30 to 10, the subjective correlation ρ_s changes only slightly from 0.20 to about 0.195. If N_{Is} raises from 30 to 50, the subjective correlation ρ_s changes from 0.20 to about 0.202.[5]

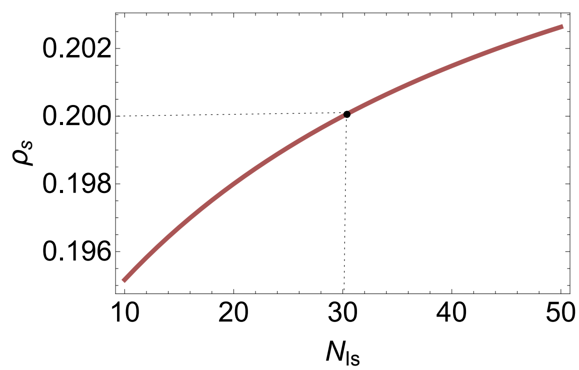
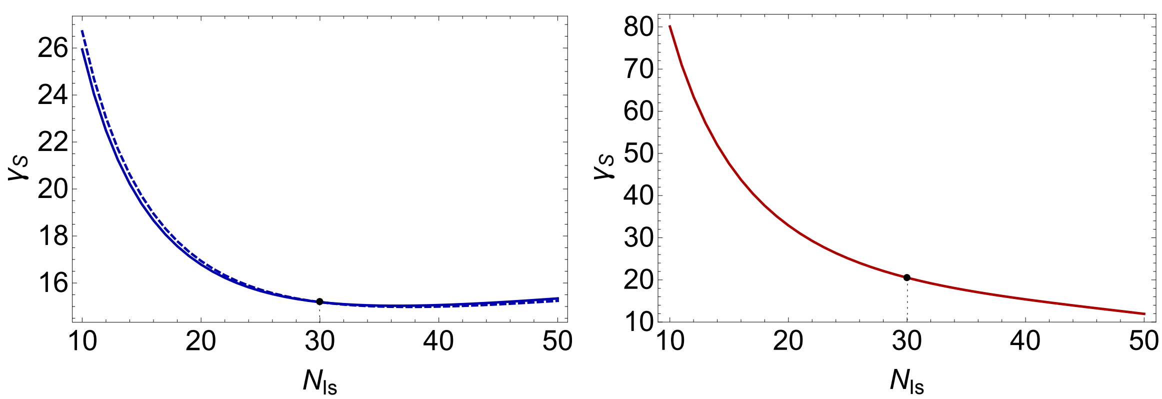


FIGURE 7. ρ_s AGAINST N_{Is}.

It means that the consistency condition requires only small changes in subjective correlations in response to large changes in subjective estimates of the number of traders. Since it is difficult to estimate the correlation among private signals in practice, this consistency condition is practically realistic, because potentially incorrect beliefs of traders cannot be easily falsified by observing the price dynamics. Also, in both subcases, all variables exhibit very similar patterns, so we do not discuss these cases separately.

[5]Parameter values are $r = 0.01$, $\beta = 0.05$, $A = 1$, $\alpha_D = 0.1$, $\alpha_G = 0.02$, $\sigma_D = 0.5$, $\sigma_G = 0.1$, $N_U = 40$, $N_I = 30$, $\rho = 0.2$, $\tau_H = 1$, $\tau_L = 0$.

Figure 8 plots the speed of trading γ_S against N_{Is} for fixed number of traders with low precision $N_{Us} = N_U$ (left panel)[6] and fixed total number of traders $N_s = N$ (right panel).[7] When traders overestimate the number of smart traders in the market, they tend to trade less aggressively. If $N_s = N$ is fixed in panel (b), then traders also underestimate the number of traders with low-precision signals, which makes them to trade less aggressively as well. If $N_{Us} = N_U$ is fixed, then there are two effects. More smart traders imply that competition among traders becomes more fierce and information decays at a faster rate which also increases traders' trading speed. However, more traders with high precision also imply that adverse price impact increases, this tends to slow down traders' trading speed. These two opposite effects explain why γ_S may first decrease and then increase slightly when N_{Is} is getting larger with fixed N_U, since first the adverse price impact effect dominates and then the competition effect dominates. In panel (a) of Figure 8 the difference between dashed and solid lines is hardly noticeable, this suggests that the decrease in trading speed mainly comes from overestimating the number of smart traders, not from misestimating the correlation.



(a) With fixed $N_{Us} = N_U$ (b) With fixed $N_s = N$

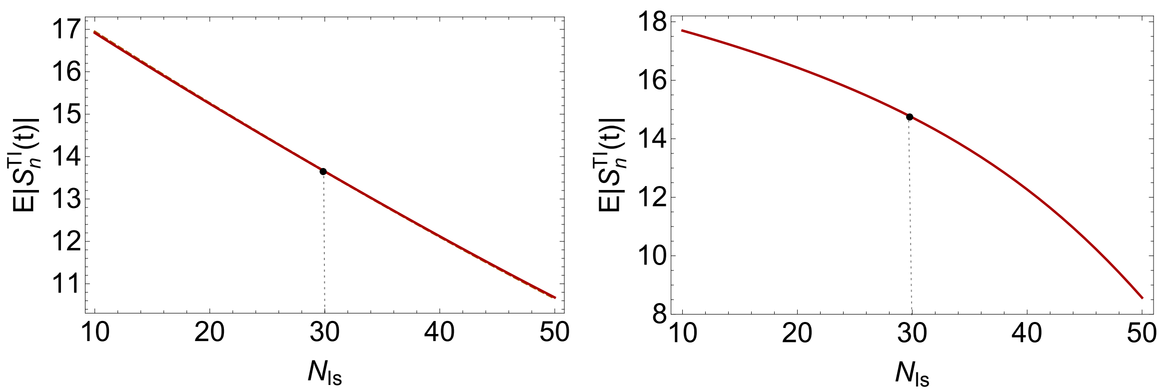
FIGURE 8. VALUES OF γ_S AGAINST N_{Is}.

Figure 9 shows that what traders think about crowdedness also affects how large positions they are willing to take. Traders target smaller positions when they overestimate the crowdedness of the smart traders, since the profit opportunities get smaller. The effect is

[6]In Figure 10 and in the left panel of Figures 8, 9, 13, 14, 15 and 16, parameter values are $r = 0.01$, $\beta = 0.05$, $A = 1$, $\alpha_D = 0.1$, $\alpha_G = 0.02$, $\sigma_D = 0.5$, $\sigma_G = 0.1$, $N_U = 40$, $N_I = 30$, $\rho = 0.2$, $\tau_H = 1$, $\tau_L = 0$.

[7]In Figure 11 and in the right panel of Figures 8, 9, 13, 14, 15, and 16, parameter values are $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_G = 0.02$, $\sigma_D = 0.5$, $\sigma_G = 0.1$, $N = N_s = 70$, $\tau_H = 1$, $\tau_L = 0$.

slightly more pronounced when the total number of traders is fixed, because the adverse price impact is more significant with an increased fraction of smart traders while fixing the total number of traders in the market.



(a) With fixed N_U (b) With fixed $N_s = N$

FIGURE 9. VALUES OF $E|S_n^{TI}(t)|$ AGAINST N_{Is}.

Figure 10 plots permanent market depth $1/\lambda$ and temporary market depth $1/\kappa$ against N_{Is} using $\rho_s = \rho$ (dashed curve) and ρ_s (solid curve) satisfying the consistency condition (22). It also plots subjective estimates of market depths $1/\lambda_s$ and $1/\kappa_s$. As before, the figure suggests that the change in market depth comes mainly from misestimation of the number of traders, not correlation among private signals.

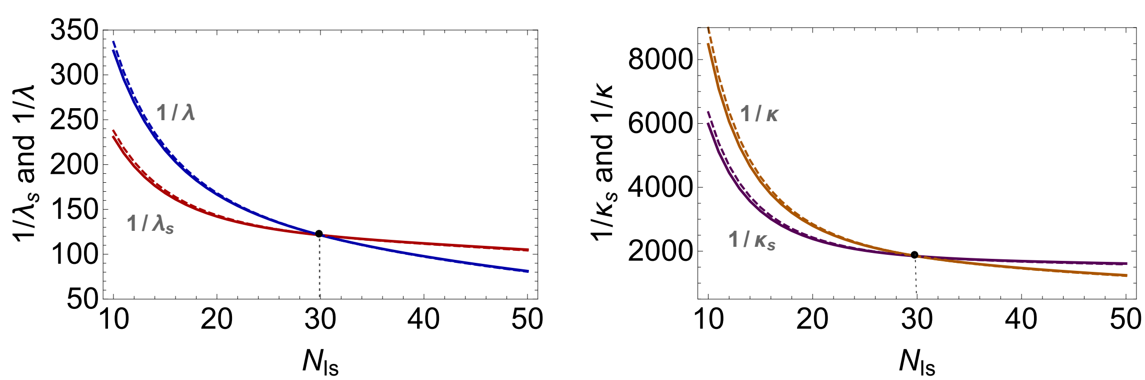


FIGURE 10. VALUES OF $1/\lambda$, $1/\kappa$, $1/\lambda_s$, AND $1/\kappa_s$ AGAINST N_{Is} FOR THE CASE WITH FIXED N_U.

Fear of crowding of smart traders reduces market liquidity. Indeed, when traders overestimate the crowdedness of smart traders ($N_{Is} > N_I$), they also expect that the market depth is somewhat low, because everybody is less willing to provide liquidity to each other.

In reality, the actual market depth is even lower than what traders think, $1/\lambda < 1/\lambda_s$ and $1/\kappa < 1/\kappa_s$. In contrast, when traders underestimate the number of smart traders ($N_{Is} < N_I$ and $\rho_s < \rho$), all types of market depth increase, because traders are more aggressive in trading on private information and providing liquidity to others. The actual market depth is even higher than the perceived one ($1/\lambda > 1/\lambda_s$ and $1/\kappa > 1/\kappa_s$).

Figure 11 plots permanent market depth $1/\lambda$ and temporary market depth $1/\kappa$ against N_{Is} with fixed N. For this case, the perceived market depth is the same as the actual market depth since traders correctly estimate the total number of market participants. This figure shows that underestimating the number of smart traders N_{Is} with fixed N tends to increase market liquidity by a larger magnitude comparing to the case with fixed N_U.

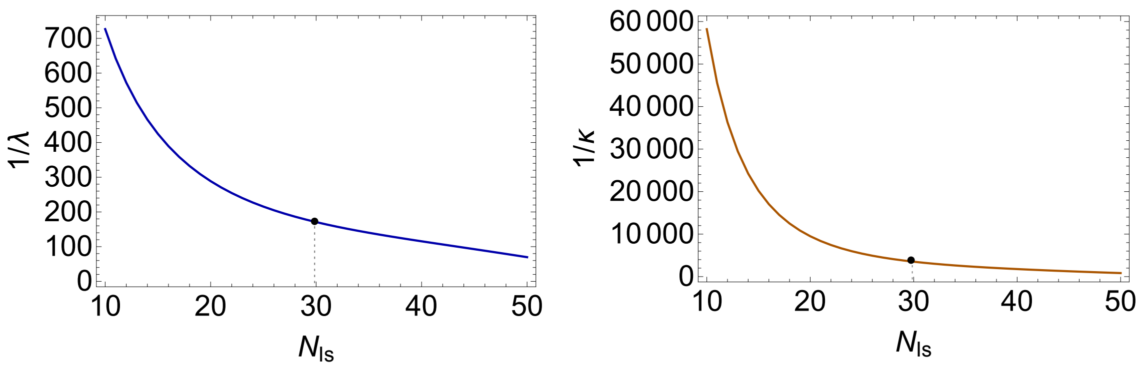
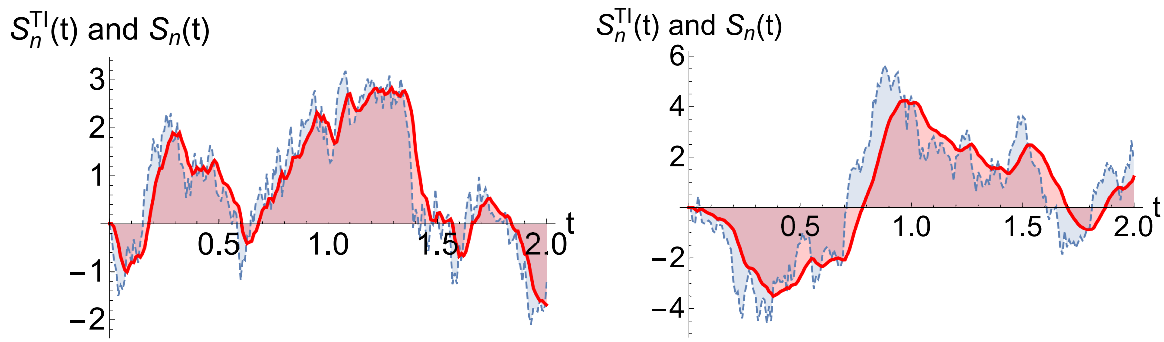


FIGURE 11. VALUES OF $1/\lambda$, $1/\kappa$, $1/\lambda_s$, AND $1/\kappa_s$ AGAINST N_{Is} FOR THE CASE WITH FIXED $N_s = N$.

Figure 12 presents two simulated paths for target inventories (dashed curve) and actual inventories (solid curve).[8] When traders underestimate the number of smart traders—and the market is more liquid—actual inventories deviate less significantly from target inventories since traders trade at a higher rate, as in panel (a). When traders overestimate the number of smart traders —and the market is less liquid—actual inventories deviate more significantly from target inventories, as in panel (b).

The left panel of Figure 13 plots γ_P against N_{Is} for fixed N_U using $\rho_s = \rho$ (dashed curve) and ρ_s (solid curve) satisfying the consistency condition (22). The right panel of Figure 13 plots γ_P against N_{Is} for fixed N. As we can see from Figure 13, γ_P is lower (higher) when

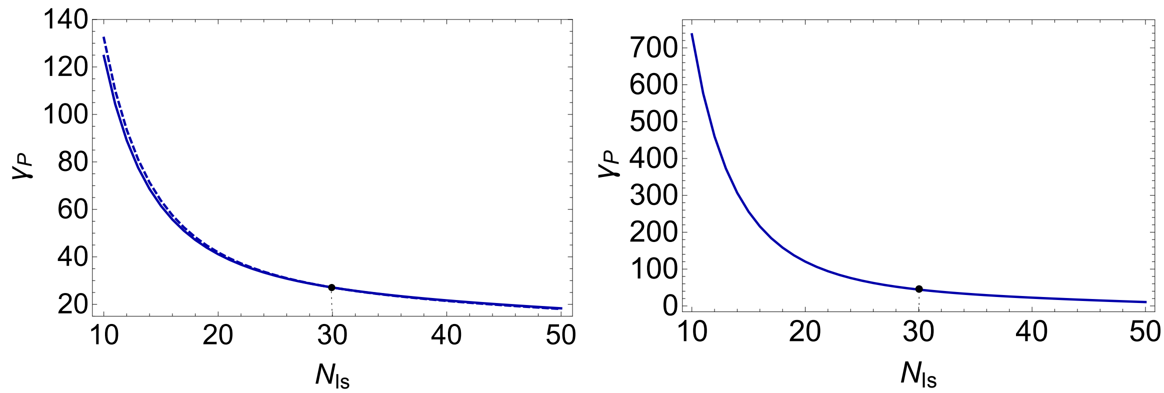
[8]The paths are generated using equations (27), (C-18), and (C-19), which describe the dynamics of $\hat{H}_n(t), \hat{H}_{-n}(t)$, and $S_n^{TI}(t)$. Numerical calculations in Figure 12 are based on the exogenous parameter values $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_G = 0.02$, $\sigma_D = 0.5$, $\sigma_G = 0.1$, $\tau_H = 1$, $\tau_L = 0$, $N_I = 30$, $N_U = 40$, $\rho_s = \rho = 0.2$ in both (a) and (b); $N_{Is} = 20$ and $N_{Us} = 50$ in (a); $N_{Is} = 40$ and $N_{Us} = 30$ in (b).



(a) Underestimation of N_{Is}. (b) Overestimation of N_{Is}.

FIGURE 12. SIMULATED PATHS OF $S_n^{TI}(t)$ (DASHED) AND $S_n(t)$ (SOLID).

traders overestimate (underestimate) the number of traders whose private information has low precision.

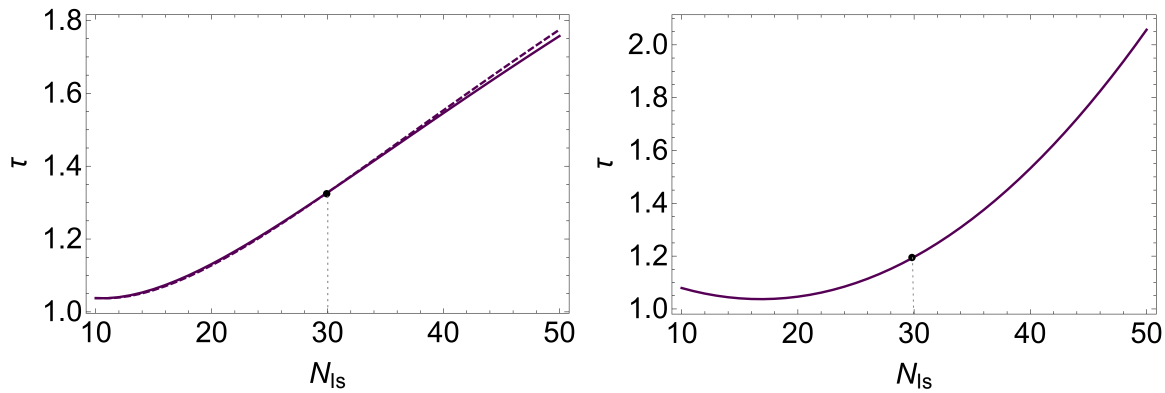


(a) With fixed N_U (b) With fixed $N_s = N$

FIGURE 13. VALUES OF γ_P AGAINST N_{Is}.

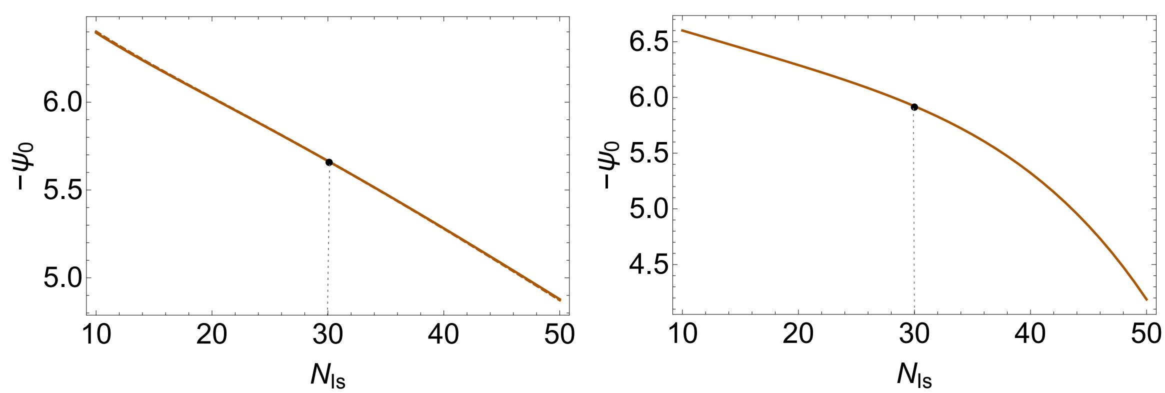
Figure 14 illustrates changes in the total precision τ. A higher ρ tends to decrease the total precision. Overestimating the number of traders tends to increase total precision, as illustrated by the dashed curve in Figure 14. The net effect of overestimating the number of traders and correlation increases the total precision (as shown by the solid curve in Figure 14) and decreases the error variance of the estimate of the growth rate. This makes trading due to agreeing to disagreement less valuable, as shown in Figure 15, the value of trading on innovations to future information ($-\psi_0$) decreases.

Figure 16 presents how C_G changes with N_{Is} with fixed N_U and fixed N. Figure 16



(a) With fixed N_U (b) With fixed $N_s = N$

FIGURE 14. VALUES OF τ AGAINST N_{Is}.

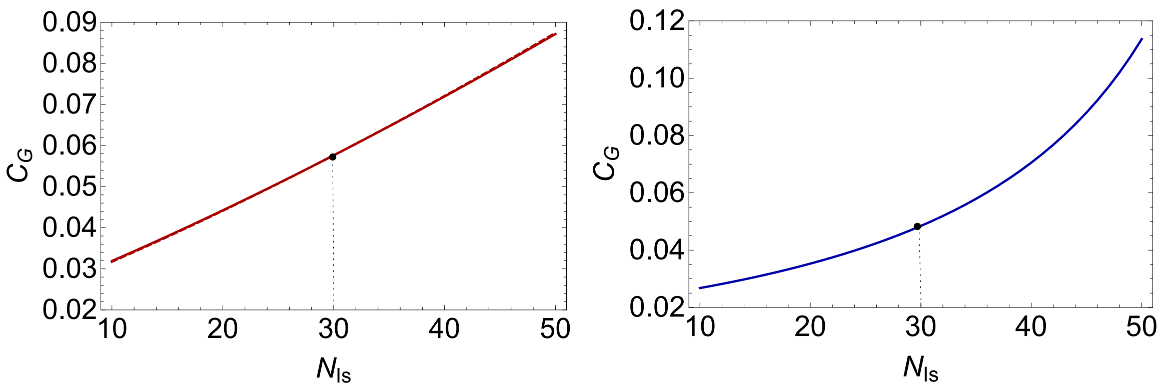


(a) With fixed N_U (b) With fixed $N_s = N$

FIGURE 15. VALUES OF $-\psi_0$ AGAINST N_{Is}.

illustrates that C_G is higher when traders overestimate the number of smart traders. Overestimating the number of smart traders results in less pronounced price dampening (a larger C_G), as traders are less willing to engage in short-term speculation due to greater adverse price impacts.

To summarize, when traders overestimate the crowdedness of the smart traders, they tend to have smaller target inventories, trade less aggressively toward target levels, trade less on short-run opportunities, expect less liquidity, and believe that trading is less valuable. When traders underestimate how many other smart traders who are trading in the same direction as them, they tend to have larger target inventories, adjust actual inventories faster toward target levels, trade more on short-run profit opportunities, expect higher liquidity, and are more willing to provide liquidity to others.



(a) With fixed N_U (b) With fixed $N_s = N$

FIGURE 16. VALUES OF C_G AGAINST N_{Is}.

3. Crowding and Fire Sales

It is believed that crowding may make markets more fragile. As we discussed in the previous section, when traders underestimate the crowdedness of smart traders, they tend to take larger positions, trade more on short-run profit opportunities, and are more willing to provide liquidity to others. When traders are concerned that they might have underestimated the crowdedness of the traders who are trading in the same direction. They would liquidate some of their inventories and market becomes less liquid at the same time. This tends to make market more fragile.

Our model allows us to study what would happen in the crowded market if some traders suddenly have to liquidate large positions in a "fire sale" mode. For example, this analysis will help us examine theoretically how the market is expected to respond to events similar to quant meltdown in August 2007. We show that these unexpected off-equilibrium fire sales would create flash crashes. When traders are concerned about the crowding in their trading strategies, they trade less aggressively toward their targets and provide less liquidity to others. We show that this makes flash crashes more substantial.

We present a numerical example of how the market would respond to an off-equilibrium fire sale of a trader. For simplicity, suppose at time 0, a trader observes a private signal $H_n(0)$ and holds some positive inventory, which is consistent with his target inventory. We also assume that he thinks signals of other traders are at their long-term mean $H_{-n}(0) = 0$ and dividends $D(0) = 0$ (with $H_0(0) = 0$). It follows that his inventory at time 0 is

$$S_n(0) = S_n^{TI}(0) = C_L(N_{Is}, \rho_s)\, H_n(0) > 0. \tag{30}$$

We explicitly state the argument (N_{Is}, ρ_s) on which the coefficient C_L depends to emphasize that this coefficient—as well as some other coefficients—depends on subjective beliefs of a trader about the number of traders and correlations ρ_s of signals. From equations (17) and (24), we get

$$P(0) = \frac{\gamma_S(N_{Is}, \rho_s)}{(N-1)\gamma_P(N_{Is}, \rho_s)}\, S_n(0) > 0. \tag{31}$$

A trader is not able to learn from the price about mistakes. Equations (19) and (21) therefore imply that the perceived average of private signals must coincide with the actual average.[9] This is a starting consistent-with-equilibrium point of our example.

Next, assume that at time $t = 0^+$, all traders receive new private information, so that trader n's signal $H_n(0)$ and other traders' signal $H_{-n}(0^+)$ suddenly drop to zero, reducing his target inventory from $S_n^{TI}(0)$ to $S_n^{TI}(0^+) = 0$. Since $H_n(0^+) = H_{-n}(0^+) = 0$, the new equilibrium price is $\mathrm{E}_0^n[P(t)] = 0$. Suppose also that a trader has to trade toward his target inventory at a fire-sale speed $\bar{\gamma}_S$, which is much faster than the equilibrium rate γ_S,

$$\bar{x}_n(t) = \bar{\gamma}_S\, \left(S_n^{TI}(t) - \bar{S}_n(t)\right) \tag{32}$$

at each point $t > 0$. Since $\bar{\gamma}_S > \gamma_S$, the trader moves to his target inventory $S_n^{TI}(t)$ more aggressively. This captures the idea of a sudden rushed sale in the market.

After date $t = 0$, off-equilibrium inventory $\bar{S}_n(t)$ is expected to evolve according to

$$\bar{S}_n(t) = \mathrm{e}^{-\bar{\gamma}_S\, t}\, \Big(S_n(0) + \int_{u=0}^{t} \mathrm{e}^{\bar{\gamma}_S\, u}\; \bar{\gamma}_S\, C_L\, \left(H_n(u) - H_{-n}(u)\right) du\Big). \tag{33}$$

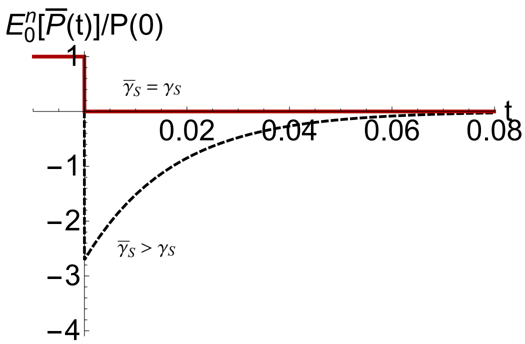
A rushed sale leads to execution at a heavy discount. Trader n can calculate the impulse-response functions of how market prices $\mathrm{E}_0^n[\bar{P}(t)]$ are expected to change in response to his sales, described by $\mathrm{E}_0^n[\bar{S}_n(t)]$,

$$\mathrm{E}_0^n[\bar{S}_n(t)] = \mathrm{e}^{-\bar{\gamma}_S\, t}\; S_n(0), \tag{34}$$

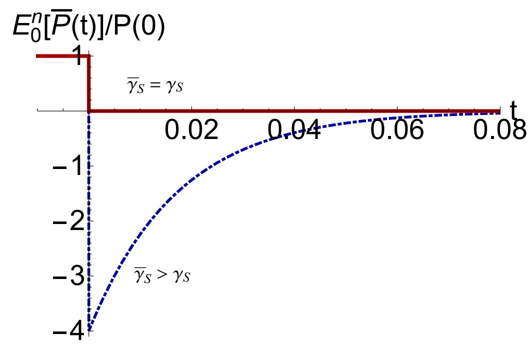
[9]For the case with fixed number of "noise traders" N_U, traders' estimate about the total number of traders is N_s while the actual number of traders is N. The actual average of all private signals is $1/N\,(H_n(0) + (N-1)\breve{H}_{-n}(0))$, whereas the trader believes that the perceived average of all private signals is equal to $1/N_s\, H_n(0)$, since there are N_s traders and signals of other traders are zero. Matching these two averages, we get the average of other traders' signals $\breve{H}_{-n}(0)$ such that a trader does not learn from the price about his misestimation of the total number of traders in the market. For the case with fixed N, traders correctly estimate the total number of traders, then $\breve{H}_{-n}(0) = H_{-n}$.

$$\text{(35)} \qquad \mathrm{E}_0^n[\bar{P}(t)] = -\frac{\bar{\gamma}_S - \gamma_S(N_{Is}, \rho_s)}{(N_s - 1)\gamma_P(N_{Is}, \rho_s)} \; \mathrm{e}^{-\bar{\gamma}_S t} \; S_n(0).$$

Figure 17 shows expected paths of future prices based on equation (35) for the case without crowding ($N_{Is} = N_I$) and with crowding ($N_{Is} > N_I$). Figure 18 shows paths of trader n' s future inventories based on equation (34) for the case without crowding ($N_{Is} = N_I$) and with crowding ($N_{Is} > N_I$).[10] There are two cases in each figure. The first baseline case is shown by the solid red lines: If trader n liquidates his inventory at an equilibrium rate ($\bar{\gamma}_S = \gamma_S$), then the price immediately drops to the equilibrium level of zero, but the trader continues to trade out of his inventories smoothly over time.



(a) No Crowding ($N_{Is} = N_I$)



(b) With Crowding ($N_{Is} > N_I$)

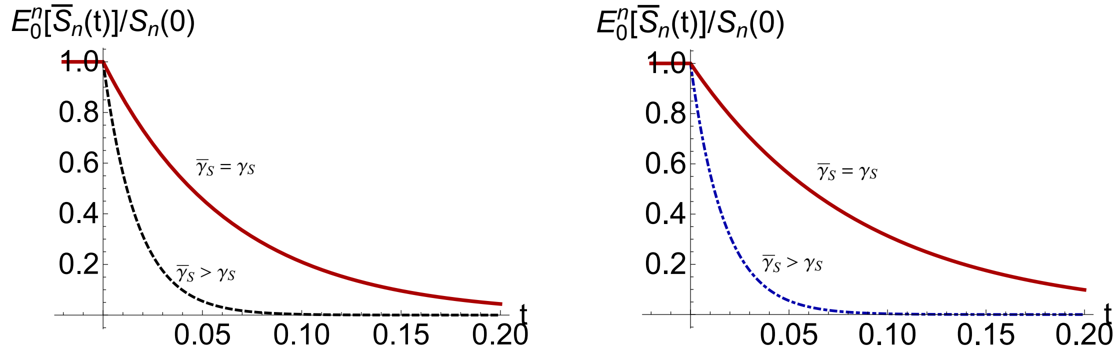
FIGURE 17. THE DYNAMICS OF EXPECTED PRICES WITH AND WITHOUT CROWDING.

The other case show what happens when trader n liquidates his position at an off-equilibrium fire-sale rate, which is five times faster than normal rate ($\bar{\gamma}_S = 5\ \gamma_S$). In panel (a) of Figure 17, black dashed line corresponds to price dynamics for the case with no crowding ($N_I = N_{Is}$), and blue dashed line in panel (b) of Figure 17 corresponds to price dynamics for the case when traders are concerned about crowding ($N_I < N_{Is}$).

In panel (a) of Figure 18, black dashed line corresponds to inventory dynamics for the case with no crowding ($N_I = N_{Is}$), and blue dashed line in panel (b) corresponds to inventory dynamics for the case when traders are concerned about crowding ($N_I < N_{Is}$).

In both cases with and without crowding, price paths exhibit distinct V-shaped patterns, i.e., after a sharp initial drop the price changes its direction and converges to the new equilibrium level. As explained in Kyle, Obizhaeva and Wang (2017), faster-than-equilibrium trading generates "flash crashes" by increasing temporary price impact.

[10]Parameter values are $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_G = 0.02$, $\sigma_D = 0.5$, $\sigma_G = 0.1$, $N = N_s = 70$, $N_I = 30$, $N_{Is} = 40$, $\rho = 0.2$, $\tau_H = 1$, $\tau_L = 0$, and $D(0^+) = 0$, $H_0(0^+) = 0$. The endogenous parameter



(a) No Crowding ($N_{Is} = N_I$) (b) With Crowding ($N_{Is} > N_I$)

Figure 18. The Dynamics of Expected Inventories With and Without Crowding.

When traders are concerned about crowding in their trading strategies, traders are more cautious and slower in trading on their information and providing liquidity to others, therefore flash crashes may be more likely to occur and their price patterns may be more pronounced, as indeed confirmed by more significant price changes in panel (b) of Figure 17 when $N_{Is} > N_I$.

When traders overestimate the number of smart traders in the market, both temporary and permanent market depth are smaller, and thus transaction costs are larger. We next present two simple examples to illustrate how overestimating the fraction of smart traders affects execution costs.

Suppose a "new" trader $n = N + 1$ silently enters the market and liquidates inventories $\bar{S}_{N+1}(t)$ at a rate $\bar{x}_{N+1}(t)$, unbeknownst to the other N traders. We can explicitly calculate the effect on prices if a trader deviates from his optimal inventory policy $S_n^*(t)$ and instead holds inventories denoted $S_n(t)$. As a result of the deviation, the old equilibrium price path $P^*(t)$ will be changed to a different price path, denoted $P(t)$, given by

$$P(t) = P^*(t) + \lambda\ (S_n(t) - S_n^*(t)) + \kappa\ (x_n(t) - x_n^*(t))\,. \tag{36}$$

Since the new trader does not trade in actual equilibrium, we assume $S^*_{N+1}(t) = x^*_{N+1}(t) = 0$

We measure his execution costs C using the concept of implementation shortfall, as described by Perold (1988). The expected price impact costs are given by

$$E\{C\} = E\left\{\int_{u=t}^{\infty} (P(u) - P^*(u))\, \bar{x}(u)\, du\right\}. \tag{37}$$

values are $\gamma_S(N_I) = 15.635$, for $N_I = 30$ and $\gamma_S(N_{Is}) = 11.6015$ for $N_{Is} = 40$, $\bar{\gamma}_S = 5\ \gamma_S(N_{Is}) = 58$.

The expected implementation shortfall depends on how the new trader trades. Here are two simple examples.

Example 1: Suppose the new trader $N+1$ enters the market at date $t=0$ and liquidates a random block of shares B, uncorrelated with signals $H_n(t)$, $n=1,\ldots,N$, by trading at the constant rate $\bar{x}(t)=B/T$ over some interval $[0,T]$. Then his expected implementation shortfall is given by

$$E\{C_1\} = \Big(\lambda + \frac{\kappa}{T/2}\Big)\,\frac{B^2}{2}. \tag{38}$$

Example 2: Suppose instead that the new trader enters the market at date $t=0$ and liquidates the random inventory B by trading at rate $x_{N+1}(t)=\gamma_S\,(B-\bar{S}_{N+1}(t))$. Then his inventory evolves as $\bar{S}(t)=B(1-e^{-\gamma_S\,t})$, with $\bar{S}(t)\to B$ as $t\to\infty$, and the implementation shortfall is given by

$$E\{C_2\} = \Big(\lambda + \kappa\,\gamma_S\Big)\,\frac{B^2}{2}. \tag{39}$$

When traders are concerned about crowdedness of their trading strategies, market becomes less liquid and the implementation shortfall increases for a trader who enters the market and acquires certain shares of the stock. Since faster execution leads to larger temporary price impact, overestimating the fraction of smart traders tends to have bigger impact on the implementation shortfall when a trader needs to acquire or liquidate certain inventory level faster.

4. Conclusion

After the Quant Meltdown of August 2007, institutional traders are increasingly concerned about crowded markets, because this factor may impede their efforts to deliver good performance and make them vulnerable to externalities imposed by other market participants.

In this paper, we develop a continuous-time model with strategic informed traders to study the phenomenon of crowded markets. Traders may have incorrect views about the correlation among traders' private signals and the number of traders chasing similar investment strategies.

Even though equilibrium trading strategies depend only on traders' subjective beliefs, the equilibrium prices are determined by the actual market clearing condition, and thus the perceived market depth may differ from the actual market depth available in the market.

Underestimation of the crowdedness of smart traders in the market increases both the perceived and actual market depth. Traders trade more intensively, take larger positions, and are more willing to supply liquidity to other traders. Overestimation of the crowdedness of the market tends to increase both temporary and permanent price impact and thus increase traders' implementation shortfall. Traders trade less aggressively, take smaller positions, and are less willing to supply liquidity to others.

When some traders are forced to liquidate large positions at a suboptimal fire-sale pace, then flash crashes happen. Our paper suggests that flash-crash price patterns may be more pronounced when traders become more concerned about the crowdedness of the market. To reduce the risks, it is important to understand the mechanisms that drive these patterns in crowded markets. Our analysis also implies that it is important that regulators carefully monitor the crowding risk of many investment strategies.

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A. One-period Model

There are two assets. A risk free asset and a risky asset that has random liquidation value $v \sim N(0, 1/\tau_v)$. Both assets are in zero net supply. Trader n is endowed with inventory S_n with $\sum_{n=1}^{N} S_n = 0$. Traders observe signals about the normalized liquidation value $\tau_v^{1/2} v \sim N(0,1)$. All traders observe a public signal $i_0 := \tau_0^{1/2}\,(\tau_v^{1/2} v) + e_0$ with $e_0 \sim N(0,1)$. Each trader n observes a private signal $i_n := \tau_n^{1/2}\,(\tau_v^{1/2} v) + \rho^{1/2} z + (1-\rho)^{1/2} e_n$ with $e_n \sim N(0,1)$, where v, z, e_0, $e_1, \ldots, e_N$ are independently distributed.

Traders agree about the precision of the public signal τ_0 and agree to disagree about the precisions of private signals τ_n. Each trader is certain that his own private information has a high precision $\tau_n = \tau_H$ and $N-1$ other traders can be of two types: $N_I - 1$ traders' private information has high precision τ_H and the other $N_U := N - N_I$ traders' private information has low precision τ_L, with $\tau_H > \tau_L \geq 0$.

Denote the fraction of other traders (except trader n himself) with high precision in the market as

$$\theta := \frac{N_I - 1}{N_U + N_I - 1}. \tag{A-1}$$

Each trader submits a demand schedule $X_n(p) := X_n(i_0, i_n, S_n, p)$ to a single-price auction. An auctioneer clears the market at price $p := p[X_1, \ldots, X_N]$. Trader n's terminal wealth is

$$W_n := v\;(S_n + X_n(p)) - p\,X_n(p). \tag{A-2}$$

Each trader n maximizes the same expected exponential utility function of wealth $\mathrm{E}^n[-\,\mathrm{e}^{-A\,W_n}]$ using his own beliefs to calculate the expectation.

An *equilibrium* is a set of trading strategies $X_1, \ldots, X_N$ such that each trader's strategy maximizes his expected utility, taking as given the trading strategies of other traders. Let $i_{-n} := \frac{1}{N-1} \sum_{m=1, m\neq n}^{N} i_m$ denote the average of other traders' signals. When trader n conjectures that other traders submit symmetric linear demand schedules

$$X_m(i_0, i_m, S_m, p) = \alpha\, i_0 + \beta\, i_m - \gamma\, p - \delta\, S_m, \qquad m = 1, \ldots, N, \quad m \neq n, \tag{A-3}$$

he infers from the market-clearing condition

$$x_n + \sum_{\substack{m=1 \\ m\neq n}}^{N} (\alpha\, i_0 + \beta\, i_m - \gamma\, p - \delta\, S_m) = 0 \tag{A-4}$$

that his residual supply schedule $P(x_n)$ is a function of his quantity x_n given by

$$P(x_n) = \frac{\alpha}{\gamma}\, i_0 + \frac{\beta}{\gamma}\, i_{-n} + \frac{\delta}{(N-1)\gamma}\, S_n + \frac{1}{(N-1)\gamma}\, x_n. \tag{A-5}$$

Let $\mathrm{E}^n[\ldots]$ and $\mathrm{Var}^n[\ldots]$ denote trader n's expectation and variance operators conditional on all signals $i_0, i_1, \ldots, i_N$. Define "total precision" τ by

$$\tau := (\mathrm{Var}^n[v])^{-1} = \tau_v \left(1 + \tau_0 + \tau_H + (N-1) \frac{\left((\theta-\rho)\tau_H^{1/2} + (1-\theta)\tau_L^{1/2}\right)^2}{(1-\rho)(1+(N-1)\rho)} \right). \tag{A-6}$$

The projection theorem for jointly normally distributed random variables implies

$$\mathrm{E}^n[v] = \frac{\tau_v^{1/2}}{\tau} \left(\tau_0^{1/2}\, i_0 + \frac{1-\theta}{1-\rho} \left(\tau_H^{1/2} - \tau_L^{1/2}\right) i_n + \frac{(\theta-\rho)\tau_H^{1/2} + (1-\theta)\tau_L^{1/2}}{(1-\rho)(1+(N-1)\rho)}\; (i_n + (N-1) i_{-n}) \right). \tag{A-7}$$

Conditional on all information, trader n's terminal wealth W_n is a normally distributed random variable with mean and variance given by

$$\mathrm{E}^n[W_n] = \mathrm{E}^n[v]\, (S_n + x_n) - P(x_n)\, x_n, \qquad \mathrm{Var}^n[W_n] = (S_n + x_n)^2\, \mathrm{Var}^n[v]. \tag{A-8}$$

Maximizing this function is equivalent to maximizing $\mathrm{E}^n[W_n] - \frac{1}{2} A\, \mathrm{Var}^n[W_n]$. Oligopolistic trader n exercises market power by taking into account how his quantity x_n affects the price $P(x_n)$ on his residual supply schedule (A-5). The following Theorem characterizes the equilibrium in this one-period model.

THEOREM 2: *There exists a unique symmetric equilibrium with linear trading strategies and nonzero trade if and only if the second-order condition*

$$\theta < 1 - \frac{N(1-\rho)\tau_H^{1/2}}{(N-1)(2+(N-2)\rho)(\tau_H^{1/2} - \tau_L^{1/2})} \tag{A-9}$$

holds. The equilibrium satisfies the following:

1. Trader n trades the quantity x_n^* given by

(A-10) $$x_n^* = \delta \left(\tfrac{1}{A} \, \frac{1-\theta}{1-\rho} \left(1 - \tfrac{1}{N}\right) \tau_v^{1/2} \left(\tau_H^{1/2} - \tau_L^{1/2}\right) \left(i_n - i_{-n}\right) - \; S_n \right),$$

where the inventory adjustment factor δ is

(A-11) $$0 < \delta = \frac{2+(N-2)\rho}{1+(N-1)\rho} - \frac{N(1-\rho)\tau_H^{1/2}}{(N-1)(1-\theta)(1+(N-1)\rho)(\tau_H^{1/2} - \tau_L^{1/2})} < 1.$$

2. The price p^* is the average of traders' valuations:

(A-12)
$$p^* = \frac{1}{N} \sum_{n=1}^{N} \mathrm{E}^n[v] = \frac{\tau_v^{1/2}}{\tau} \left(\tau_0^{1/2} \, i_0 + \frac{(1+(N-1)\theta)\tau_H^{1/2} + (N-1)(1-\theta)\tau_L^{1/2}}{N \; (1+(N-1)\rho)} \sum_{n=1}^{N} i_n \right).$$

3. The parameters $\alpha > 0$, $\beta > 0$, and $\gamma > 0$, defining the linear trading strategies in equation (A-3), have unique closed-form solutions defined in (C-2).

For an equilibrium with positive trading volume to exist, the fraction of traders whose private information has high precision must satisfy condition (A-9). Each trader trades in the direction of his private signal i_n, trades against the average of other traders' signals i_{-n}, and hedges a fraction δ of his initial inventory. Equation (A-12) implies that the equilibrium price is a weighted average of traders' valuations with weights summing to one.

Define a trader's "target inventory" S_n^{TI} as the inventory such that he would not want to trade ($x_n^* = 0$). From equation (A-10), it is equal to

(A-13) $$S_n^{TI} = \tfrac{1}{A} \, \frac{1-\theta}{1-\rho} \left(1 - \tfrac{1}{N}\right) \tau_v^{1/2} \left(\tau_H^{1/2} - \tau_L^{1/2}\right) \left(i_n - i_{-n}\right).$$

Then trader n's optimal quantity traded can be written

(A-14) $$x_n^* = \delta \, (S_n^{TI} - S_n).$$

The parameter δ, defined in equation (A-11), is the fraction by which traders adjust positions toward target levels. It can be proved analytically that δ increases in correlation ρ and decreases in θ while fixing everything else.

From the perspective of trader n, equation (A-5) implies that price impact can be written as a function of both x_n and S_n,

(A-15) $$P(x_n, S_n) := p_{0,n} + \lambda \, S_n + \kappa \, x_n,$$

where $p_{0,n}$ is a linear combination of random variables i_0 and i_{-n}, and constants λ and κ are given by

$$\lambda := \frac{\delta}{(N-1)\gamma} = \frac{A(1-\rho)\left((1+(N-1)\theta)\tau_H^{1/2} + (N-1)(1-\theta)\tau_L^{1/2}\right)}{\tau(N-1)(1+(N-1)\rho)(1-\theta)(\tau_H^{1/2} - \tau_L^{1/2})}, \tag{A-16}$$

$$\kappa := \frac{\lambda}{\delta} = \frac{1}{(N-1)\gamma}. \tag{A-17}$$

It can be proved analytically that both λ and κ both decrease in correlation ρ and increase in the fraction of traders whose private information has high precision while fixing everything the same. Market becomes more liquid if traders' private information are highly correlated and less liquid if the fraction of traders whose private information has high precision increases.

B. Effects of Changes in Crowding of the Total Market

In this section we focus on the case when traders might misestimate the total number of traders while correctly estimate the fraction of smart traders in the market. We consider two situations: (1) the base case when both N_s and ρ_s are changing in lockstep satisfying the consistency condition so that traders can not learn about their mistakes by observing price dynamics, and (2) another case when only N_s is changing, but ρ_s remains fixed. The first case is presented by solid lines and the second case is presented in dashes lines below.

Figure B.1 shows that how ρ_s changes with changes in N_s in order to satisfy the consistency condition. When N_s is the same as the actual number of traders N ($N = N_s = 80$), the subjective correlation ρ_s converges to the actual correlation ρ ($\rho = \rho_s = 0.20$). If N_s drops by a half to 40, the subjective correlation ρ_s changes only slightly to about 0.19.[11]

Figure B.1 shows that satisfying the consistency condition only requires small changes in subjective correlations in response to large changes in subjective estimates of the number of traders. Since it is difficult to estimate the correlation among private signals in actual financial markets, this consistency condition is a very reasonable one to ensure that traders' potentially incorrect beliefs cannot be easily falsified by observing the price dynamics.

[11] In Figures B.1, B.2, B.3 and B.7, parameter values are $r = 0.01$, $\beta = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_G = 0.02$, $\sigma_D = 0.5$, $\sigma_G = 0.1$, $\theta = 0.1$, $N = 80$, $\rho = 0.2$, $\tau_H = 1$, $\tau_L = 0.1$.

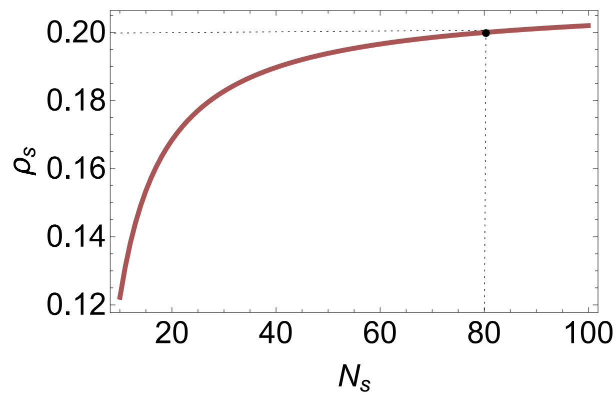


FIGURE B.1. ρ_s AGAINST N_s.

We now study how crowding affects market liquidity and traders' trading strategies. Figure B.2 plots the speed of trading γ_S against traders' subjective belief about the number of total market participants N_s. To separate the impact of the number of traders from the impact of correlation among private signals on the trading speed, we first plot γ_S against changing N_s while keeping $\rho_s = \rho$ fixed. The dashed curve illustrates that traders trade less aggressively toward target inventory when there are fewer traders in the market (fixing ρ); if traders believe that there are fewer of them in the market and the competition is less intensive, then traders trade less aggressively.

When correlation ρ_s is adjusted to satisfy the consistency condition, traders trade toward target inventories even slower, as depicted by the solid line in Figure B.2. The subjective correlation is calculated as $\rho_s = \frac{1}{N_s-1}\left(\frac{N_s}{N}(1+(N-1)\rho)-1\right)$ to satisfy the consistency condition (22), it decreases with lower N_s, and a lower subjective correlation among private signals leads to a slower trading rate γ_S, as shown previously in Figure 1. Figure B.2 also suggests that the decrease in trading speed comes mainly from underestimating the number of traders, not from underestimating of the correlation among private signals. Indeed, the difference between dashed line and solid lines are hardly noticeable. This difference is small for most of other variables, so we will next discuss only our base case when both N_s and ρ_s are changing to satisfy the consistency condition.

The left panel of Figure B.3 plots γ_P against N_s using $\rho_s = \rho$ (dashed curve) and ρ_s (solid curve) satisfying the consistency condition (22). γ_P is lower (higher) when traders underestimate (overestimate) the number of market participants. The right panel of figure B.3 illustrates changes in the total precision τ. A lower ρ tends to increase the total precision. Underestimating the number of traders tends to decrease total precision, as

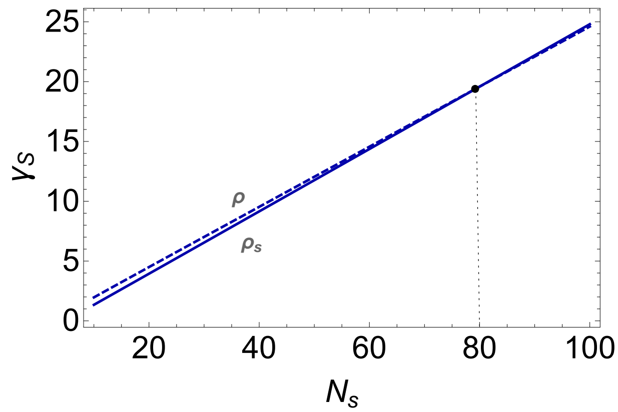


Figure B.2. Values of γ_S against N_s.

illustrated by the dashed curve in Figure B.3. The net effect of underestimating the number of traders and correlation tends to increase the total precision (as shown by the solid curve in Figure B.3) and decrease the error variance of the estimate of the growth rate. This makes trading due to agreeing to disagreement less valuable, as shown in figure B.4, the value of trading on innovations to future information ($-\psi_0$) decreases.

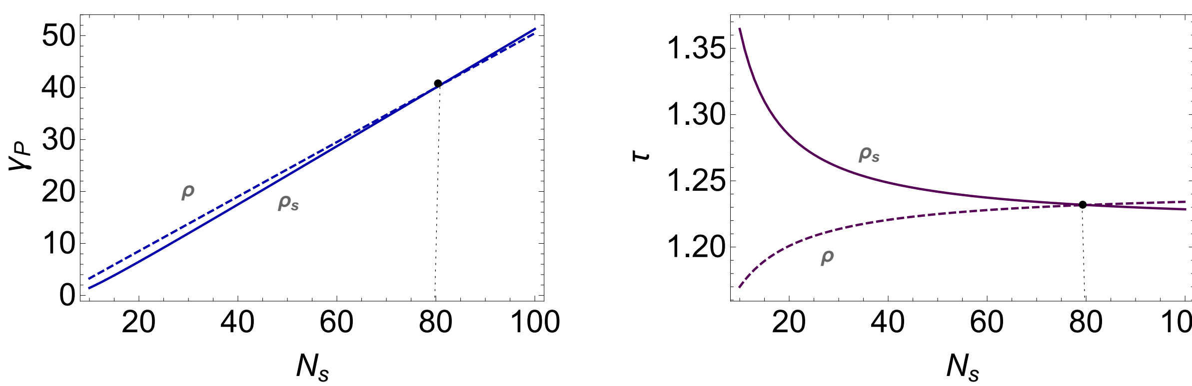


Figure B.3. Values of γ_P and τ against N_s.

Figure B.5 presents how C_L changes with N_s using $\rho_s = \rho$ (dashed curve) and ρ_s (solid curve) satisfying the consistency condition (22). It shows that C_L is lower when traders underestimate the total number of participants and correlation among private signals since traders trade less aggressively with fewer number of traders. This implies traders tend to hold smaller positions when they underestimate the number of traders.

Figure B.6 presents how C_G changes with N_s using $\rho_s = \rho$ (dashed curve) and ρ_s (solid

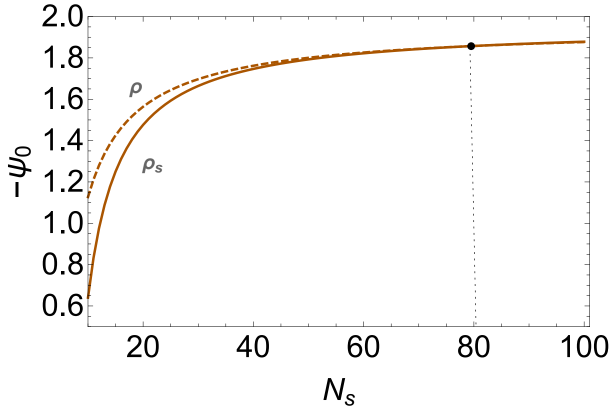


FIGURE B.4. VALUES OF $-\psi_0$ AGAINST N_s.

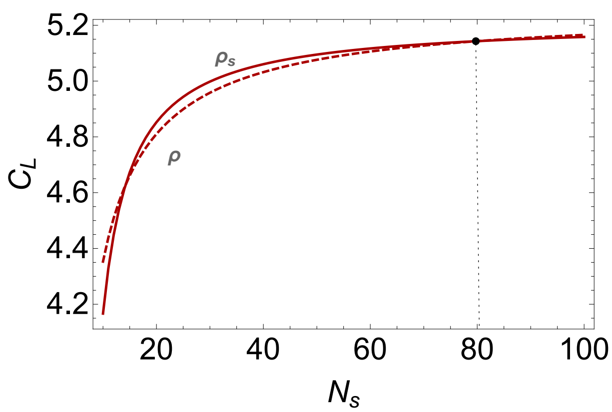


FIGURE B.5. VALUES OF C_L AGAINST N_s.

curve) satisfying the consistency condition (22). Figure B.6 illustrates that C_G is higher when traders underestimate the total number of participants and correlation among private signals. A lower ρ leads to a higher C_G since traders trade less aggressively while fewer number of traders also result in a higher C_G due to less intensive competition. Therefore, underestimating the number of traders and correlation results in less pronounced price dampening (a larger C_G) and traders are less willing to engage in short-term speculation.

Figure B.7 plots permanent market depth $1/\lambda$ and temporary market depth $1/\kappa$ against N_S using $\rho_s = \rho$ (dashed curve) and ρ_s (solid curve) satisfying the consistency condition (22). It also plots subjective estimates of market depths $1/\lambda_s$ and $1/\kappa_s$. As before, the figure suggests that the change in market depth comes mainly from misestimation of the number of traders, not correlation among private signals.

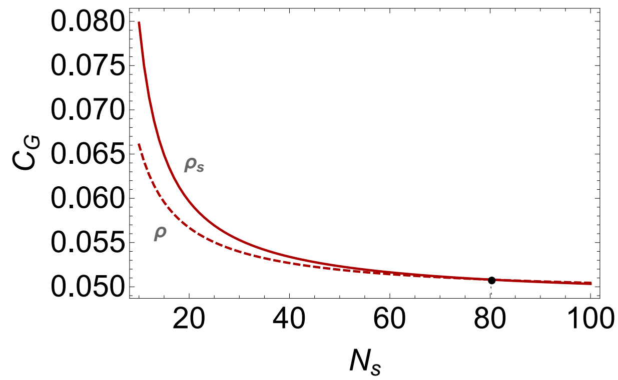


FIGURE B.6. VALUES OF C_G AGAINST N_s.

When traders overestimate the crowdedness ($N_s > N$ and $\rho_s > \rho$), traders trade more intensively. Fear of crowding leads to illusion of liquidity in the market, and indeed market depth increases. However, the actual market depth is much lower than the perceived one ($1/\lambda < 1/\lambda_s$ and $1/\kappa < 1/\kappa_s$). The actual permanent market depth $1/\lambda$ does not change much comparing to the case without crowing. When traders underestimate the crowdedness ($N_s < N$ and $\rho_s < \rho$), all types of market depth decrease, because traders trade less aggressively on their signals and are less willing to provide liquidity. Underestimating the crowdedness tends to reduce market liquidity, but the actual market depth is higher than the perceived one ($1/\lambda > 1/\lambda_s$ and $1/\kappa > 1/\kappa_s$). In this example, the drop in the actual permanent market depth is not as substantial as the drop in the actual temporary market depth. When traders underestimate the crowdedness of the market by a half (e.g., $N_s = 40$ and $N = 80$), then $1/\lambda$ changes only slightly from about 150 to 140, whereas $1/\kappa$ drops by about a half from approximately 3000 to 1500.

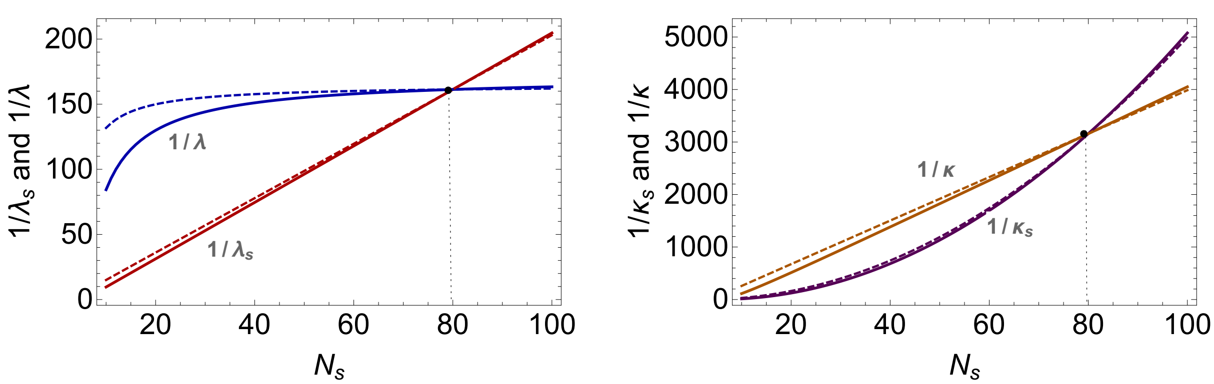
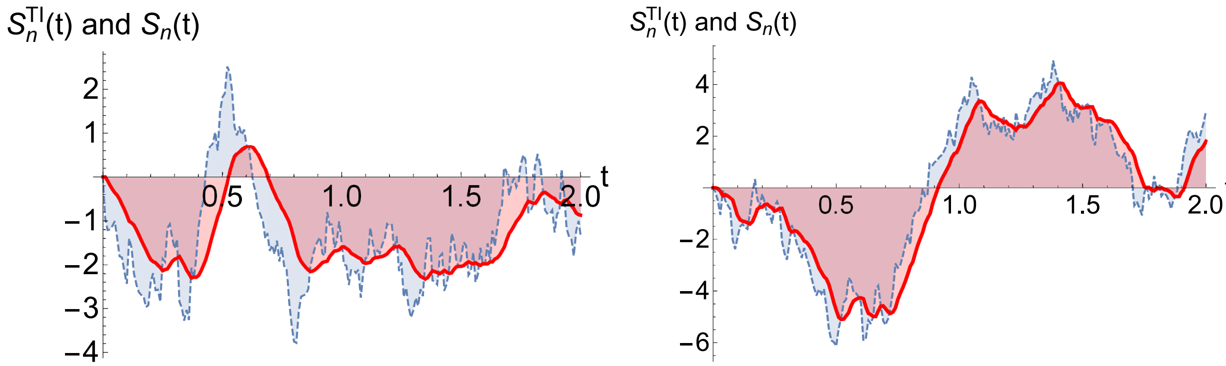


FIGURE B.7. VALUES OF $1/\lambda$, $1/\kappa$, $1/\lambda_s$, AND $1/\kappa_s$ AGAINST N_s.

Figure B.8 presents two simulated paths for target inventories (dashed curve) and actual inventories (solid curve).[12] When traders underestimate the number of traders and the correlation among private signals—and the market is less liquid—actual inventories deviate more significantly from target inventories since traders trade at a lower rate, as in panel (a). When traders correctly estimate the number of traders and correlation among private signals—and the market is more liquid—actual inventories deviate less significantly from target inventories, as in panel (b).



(a) Underestimate N and ρ (b) Correctly estimate N and ρ

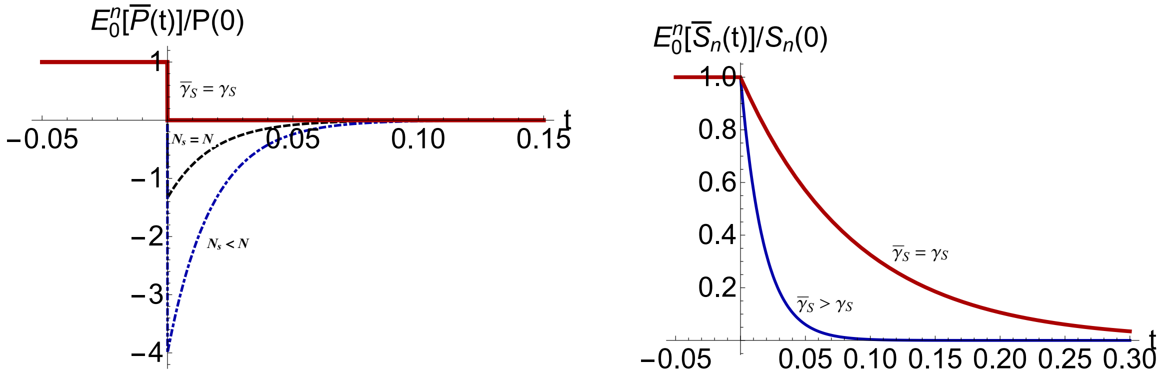
Figure B.8. Simulated paths of $S_n^{TI}(t)$ (Dashed) and $S_n(t)$ (Solid).

To summarize, when traders overestimate crowding in the market, they tend to have larger target inventories, trade more aggressively toward target levels, trade more on short-run opportunities, expect more liquidity, and believe that trading is more valuable. When traders underestimate how crowded strategies are, they tend to have smaller target inventories, adjust actual inventories more slowly toward target levels, trade less on short-run profit opportunities, expect less liquidity, and are less willing to provide liquidity to others.

Figure B.9[13] suggests that, in crowded markets, flash crashes may be more likely to occur and their price patterns may be more pronounced, as indeed confirmed by more significant price changes in Figure B.9 when $N_s < N$.

[12]The paths are generated using equations (27), (C-18), and (C-19), which describe the dynamics of $\hat{H}_n(t), \hat{H}_{-n}(t)$, and $S_n^{TI}(t)$. Numerical calculations in Figure B.8 are based on the exogenous parameter values $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_G = 0.02$, $\sigma_D = 0.5$, $\sigma_G = 0.1$, $\theta = 0.1$, $\tau_H = 1$, $\tau_L = 0.2$ in both (a) and (b); $N_s = 40 < N = 80$ and $\rho_s = 0.19 < \rho = 0.2$ in (a); $N_s = N = 80$ and $\rho_s = \rho = 0.2$ in (b).

[13]Parameter values are $r = 0.01$, $A = 1$, $\alpha_D = 0.1$, $\alpha_G = 0.02$, $\sigma_D = 0.5$, $\sigma_G = 0.1$, $\theta = \theta_s = 0$, $\tau_H = 1$, $\tau_L = 0.1$, and $D(0^+) = 0$, $H_0(0^+) = 0$. The endogenous parameter values are $\gamma_S(N, \rho) = 24.04$, for $N = 80$ and $\rho = 0.2$; and $\gamma_S(N_s, \rho_s) = 11.23$ for $N_s = 40$ and $\rho_s = 0.19$, $\bar{\gamma}_S = 5\ \gamma_S(N_s, \rho_s) = 56.14$.



(a) The Dynamics of Expected Prices

(b) The Dynamics of Inventories

FIGURE B.9. UNDERESTIMATING THE CROWDEDNESS OF THE TOTAL MARKET LEADS TO MORE PRONOUNCED FLASH-CRASH PRICE PATTERNS.

C. Proofs

C.1. Proof of Theorem 2

The first-order condition yields the optimal demand:

(C-1)
$$x_n = \frac{\mathrm{E}^n[v] - \left(\frac{\alpha}{\gamma}\, i_0 + \frac{\beta}{\gamma}\, i_{-n}\right) - \left(\frac{\delta}{(N-1)\gamma} + \frac{A}{\tau}\right)\, S_n}{\frac{2}{(N-1)\gamma} + \frac{A}{\tau}}.$$

Solving for i_{-n} instead of p in the market-clearing condition (A-4), substituting this solution into equation (C-1) above, and then solving for x_n, yields a demand schedule $X_n(i_0, i_n, S_n, p)$ for trader n as a function of price p. In a symmetric linear equilibrium, the strategy chosen by trader n must be the same as the linear strategy (A-3) conjectured for the other traders. Equating the corresponding coefficients of the variables i_0, i_n, p, and S_n yields a system of four equations in terms of the four unknowns α, β, γ, and δ. The unique solution is

(C-2)
$$\alpha = \frac{\tau_0^{1/2}\tau_v^{1/2}}{\tau}\,\gamma, \quad \beta = \frac{(1-\theta)(\tau_H^{1/2} - \tau_L^{1/2})}{A(1-\rho)}\tau_v^{1/2}\;\delta,$$

(C-3)
$$\gamma = \frac{\tau(1+(N-1)\rho)}{(1+(N-1)\theta)\tau_H^{1/2} + (N-1)(1-\theta)\tau_L^{1/2}}\,\frac{\beta}{\tau_v^{1/2}},$$

(C-4)
$$\delta = \frac{2+(N-2)\rho}{1+(N-1)\rho} - \frac{N(1-\rho)\tau_H^{1/2}}{(N-1)(1-\theta)(1+(N-1)\rho)(\tau_H^{1/2} - \tau_L^{1/2})}.$$

Substituting (C-2) into (C-1) yields trader n's optimal demand (A-10). Substituting (A-10) into (A-5) yields the equilibrium price (A-12).

The second-order condition has the correct sign if and only if $\frac{2}{(N-1)\gamma} + \frac{A}{\tau} > 0$. This is equivalent to

(C-5)
$$\theta < 1 - \frac{N(1-\rho)\tau_H^{1/2}}{(N-1)(2+(N-2)\rho)(\tau_H^{1/2} - \tau_L^{1/2})}.$$

C.2. Proof of Theorem 1

We assume that all traders believe that there are N_s traders in the market, and that their private signals are pairwise positively correlated with correlation coefficient of ρ_s.

Apply the Stratonovich–Kalman–Bucy filter to the filtering problem. This yields trader n's filtering estimate of the growth rate $G_n(t)$ defined by the Itô differential equation

(C-6)
$$\begin{aligned} dG_n(t) = &- \alpha_G\, G_n(t)\, dt + \tau_0^{1/2}\sigma_G\, \Omega^{1/2}\Big(dI_0(t) - \frac{\tau_0^{1/2} dt}{\sigma_G\, \Omega^{1/2}} G_n(t)\Big) \\ &+ \frac{\Big((1+(N_s-2)\rho_s)\tau_H^{1/2} - (N_s-1)\rho_s\tau_L^{1/2}\Big)\, \sigma_G\, \Omega^{1/2}}{(1-\rho_s)(1+(N_s-1)\rho_s)} \Big(dI_n(t) - \frac{\tau_H^{1/2} dt}{\sigma_G\, \Omega^{1/2}} G_n(t)\Big) \\ &+ \frac{(\tau_L^{1/2} - \rho_s\tau_H^{1/2})\sigma_G\, \Omega^{1/2}}{(1-\rho_s)(1+(N_s-1)\rho_s)} \Big(\sum_{\substack{m=1\\ m\neq n}}^{N_s} dI_m(t) - \frac{(N_s-1)\tau_L^{1/2} dt}{\sigma_G\, \Omega^{1/2}} G_n(t)\Big). \end{aligned}$$

Rearranging terms yields

(C-7)
$$\begin{aligned} dG_n(t) = &- (\alpha_G + \tau)\, G_n(t)\, dt + \tau_0^{1/2}\sigma_G\, \Omega^{1/2} dI_0(t) + \frac{(\tau_L^{1/2} - \rho_s\tau_H^{1/2})\sigma_G\, \Omega^{1/2}}{(1-\rho_s)(1+(N_s-1)\rho_s)} \sum_{\substack{m=1\\ m\neq n}}^{N_s} dI_m(t) \\ &+ \frac{\Big((1+(N_s-2)\rho_s)\tau_H^{1/2} - (N_s-1)\rho_s\tau_L^{1/2}\Big)\, \sigma_G\, \Omega^{1/2}}{(1-\rho_s)(1+(N_s-1)\rho_s)}\, dI_n(t). \end{aligned}$$

The mean-square filtering error of the estimate $G(t)$, denoted $\sigma_G^2\, \Omega(t)$, is defined by the

Riccati differential equation

(C-8)

$$\sigma_G^2 \frac{d\Omega(t)}{dt} = -2\alpha_G \sigma_G^2 \Omega(t) + \sigma_G^2 - \sigma_G^2 \Omega(t) \left(\tau_0 + \tau_H + (N_s - 1) \frac{\left((\theta_s - \rho_s)\tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2} \right)^2}{(1-\rho_s)(1+(N_s-1)\rho_s)} \right).$$

Let Ω denote the steady state of the function of time $\Omega(t)$. Using the steady-state assumption $d\Omega(t)/dt = 0$, solve the second equation for the steady state value $\Omega = \Omega(t)$ to obtain equation (8). The error variance Ω corresponds to a steady state that balances an increase in error variance due to innovations $dB_G(t)$ in the true growth rate with a reduction in error variance due to (1) mean reversion of the true growth rate at rate α_G and (2) arrival of new information with total precision τ.

Note that Ω is not a free parameter but is instead determined as an endogenous function of the other parameters. Equation (8) implies that Ω turns out to be the solution to the quadratic equation $\Omega^{-1} = 2\,\alpha_G + \tau$. In equations (3) and (4), we scaled the units with which precision is measured by the endogenous parameter Ω. This leads to simpler filtering expressions. The estimate $G_n(t)$ can be conveniently written as the weighted sum of $N_s + 1$ sufficient statistics $H_n(t)$ corresponding to $N_s + 1$ information flows dI_n. The sufficient statistics $H_n(t)$ is defined by equation (10). $G_n(t)$ becomes a linear combination of sufficient statistics $H_n(t)$ as given by equation (12). Using the two composite signals, $\hat{H}_n(t)$ and $\hat{H}_{-n}(t)$, defined in equation (13), trader n's estimate of the dividend growth rate can be expressed as a function of the two composite signals $\hat{H}_n(t)$ and $\hat{H}_{-n}(t)$ as

(C-9)

$$\begin{aligned} G_n(t) := \sigma_G\, \Omega^{1/2} \Bigg(\Big(\frac{(1-\theta_s)(\tau_H^{1/2} - \tau_L^{1/2})}{1-\rho_s} + \frac{(\theta_s - \rho_s)\tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2}}{(1-\rho_s)(1+(N_s-1)\rho_s)} \Big) \hat{H}_n(t) \\ + \frac{(\theta_s - \rho_s)\tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2}}{(1-\rho_s)(1+(N_s-1)\rho_s)} (N_s - 1)\hat{H}_{-n}(t) \Bigg). \end{aligned}$$

Define the processes dB_0^n, dB_n^n, and dB_m^n, $m = 1, \ldots, N_s$, $m \neq n$, by

(C-10)

$$dB_0^n(t) = \tau_0^{1/2} \frac{G^*(t) - G_n(t)}{\sigma_G\, \Omega^{1/2}}\, dt + dB_0(t),$$

(C-11)

$$dB_n^n(t) = \tau_H^{1/2} \frac{G^*(t) - G_n(t)}{\sigma_G\, \Omega^{1/2}}\, dt + \rho_s^{1/2} dZ(t) + (1-\rho_s)^{1/2} dB_n(t),$$

and

$$\text{(C-12)}\quad dB^n_m(t) = (\theta_s \tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2})\,\frac{G^*(t) - G_n(t)}{\sigma_G\,\Omega^{1/2}}\,dt + +\rho_s^{1/2} dZ(t) + (1-\rho_s)^{1/2} dB_m(t).$$

The superscript n indicates conditioning on beliefs of trader n. These $N_s + 1$ processes are correlated distributed Brownian motions from the perspective of trader n. Trader n believes that signals change as follows:

$$\text{(C-13)}\qquad dH_0(t) = -(\alpha_G + \tau)\,H_0(t)\,dt + \tau_0^{1/2}\,\frac{G_n(t)}{\sigma_G\,\Omega^{1/2}}\,dt + dB^n_0(t),$$

$$\text{(C-14)}\qquad dH_n(t) = -(\alpha_G + \tau)\,H_n(t)\,dt + \tau_H^{1/2}\,\frac{G_n(t)}{\sigma_G\,\Omega^{1/2}}\,dt + dB^n_n(t),$$

(C-15)

$$dH_{-n}(t) = -(\alpha_G + \tau)\,H_{-n}(t)\,dt + (\theta_s \tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2})\,\frac{G_n(t)}{\sigma_G\,\Omega^{1/2}}\,dt + \frac{1}{N_s - 1}\sum_{\substack{m=1\\ m\neq n}}^{N_s} dB^n_m(t).$$

Note that each signal drifts toward zero at rate $\alpha_G + \tau$ and drifts toward the optimal forecast $G_n(t)$ at a rate proportional to the square root of the signal's precision $\tau_0^{1/2}$, $\tau_H^{1/2}$, or $\theta_s \tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2}$, respectively.

In terms of the composite variables $\hat{H}_n$ and $\hat{H}_{-n}$, we conjecture (and verify below) a steady-state value function of the form $V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n})$. Letting $(c_n(t), x_n(t))$ denote consumption and investment choices, write

$$\text{(C-16)}\qquad V\left(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n}\right) := \max_{[c_n(t), x_n(t)]}\quad \mathrm{E}^n_t\left[\int_{s=t}^{\infty} -\,\mathrm{e}^{-\beta(s-t) - A\,c_n(s)}\;ds\right],$$

where $P(x_n(t))$ is given by equation (17), dividends follow equation (1), inventories follow $dS_n(t) = x_n(t)\,dt$, the change in cash holdings $dM_n(t)$ follows

$$\text{(C-17)}\qquad dM_n(t) = (r\,M_n(t) + S_n(t)\,D(t) - c_n(t) - P(x_n(t))\,x_n(t))\;dt,$$

and signals $\hat{H}_n$ and $\hat{H}_{-n}$ are given by

$$\text{(C-18)}\qquad d\hat{H}_n(t) = -(\alpha_G + \tau)\,\hat{H}_n(t)\,dt + \frac{\tau_H^{1/2} + \hat{a}\tau_0^{1/2}}{\sigma_G\,\Omega^{1/2}}\,G_n(t)\,dt + \hat{a}\,dB^n_0(t) + dB^n_n(t),$$

(C-19)

$$d\hat{H}_{-n}(t) = -(\alpha_G+\tau)\hat{H}_{-n}(t)dt + \frac{\theta_s\tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2} + \hat{a}\tau_0^{1/2}}{\sigma_G\,\Omega^{1/2}}G_n(t)dt + \hat{a}dB_0^n(t) + \frac{1}{N_s-1}\sum_{\substack{m=1\\m\neq n}}^{N_s} dB_m^n(t).$$

The dynamics of $\hat{H}_n$ and $\hat{H}_{-n}$ in equations (C-18) and (C-19) can be derived from equations (C-13), (C-14), and (C-15).

Note that the coefficient $\tau_H^{1/2} + \hat{a}\tau_0^{1/2}$ in the second line of equation (C-18) is different from the coefficient $\theta_s\tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2} + \hat{a}\tau_0^{1/2}$ in the second line of equation (C-19). This difference captures the fact that—in addition to disagreeing about the value of the asset in the present—traders also disagree about the dynamics of their future valuations.

We conjecture and verify that the value function $V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n})$ has the specific quadratic exponential form

(C-20)

$$V\left(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n}\right) = -\exp\Big(\psi_0 + \psi_M M_n + \tfrac{1}{2}\psi_{SS}S_n^2 + \psi_{SD}S_n D + \psi_{Sn}\,S_n\hat{H}_n + \psi_{Sx}\,S_n\hat{H}_{-n} + \tfrac{1}{2}\psi_{nn}\,\hat{H}_n^2 + \tfrac{1}{2}\psi_{xx}\,\hat{H}_{-n}^2 + \psi_{nx}\,\hat{H}_n\hat{H}_{-n}\Big).$$

The nine constants ψ_0, ψ_M, ψ_{SS}, ψ_{SD}, ψ_{Sn}, ψ_{Sx}, ψ_{nn}, ψ_{xx}, and ψ_{nx} have values consistent with a steady-state equilibrium. The Hamilton–Jacobi–Bellman (HJB) equation corresponding to the conjectured value function $V(M_n, S_n, D, \hat{H}_n, \hat{H}_{-n})$ in equation (C-16) is

(C-21)

$$\begin{aligned}
0 = \max_{c_n, x_n}\ & \left[U(c_n) - \beta V + \frac{\partial V}{\partial M_n}\left(rM_n + S_n D - c_n - P(x_n)\,x_n\right) + \frac{\partial V}{\partial S_n}x_n\right] \\
&+ \frac{\partial V}{\partial D}\left(-\alpha_D D + G_n(t)\right) + \frac{\partial V}{\partial \hat{H}_n}\left(-(\alpha_G+\tau)\hat{H}_n(t) + \frac{\tau_H^{1/2} + \hat{a}\tau_0^{1/2}}{\sigma_G\,\Omega^{1/2}}G_n(t)\right) + \tfrac{1}{2}\frac{\partial^2 V}{\partial D^2}\sigma_D^2 \\
&+ \frac{\partial V}{\partial \hat{H}_{-n}}\left(-(\alpha_G+\tau)\hat{H}_{-n}(t) + \frac{\theta_s\tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2} + \hat{a}\tau_0^{1/2}}{\sigma_G\,\Omega^{1/2}}G_n(t)\right) + \tfrac{1}{2}\frac{\partial^2 V}{\partial \hat{H}_n^2}\left(1+\hat{a}^2\right) \\
&+ \tfrac{1}{2}\frac{\partial^2 V}{\partial \hat{H}_{-n}^2}\left(\rho_s + \frac{1-\rho_s}{N_s-1} + \hat{a}^2\right) + \left(\frac{\partial^2 V}{\partial D\,\partial \hat{H}_n} + \frac{\partial^2 V}{\partial D\,\partial \hat{H}_{-n}}\right)\hat{a}\sigma_D + \frac{\partial^2 V}{\partial \hat{H}_n\,\partial \hat{H}_{-n}}(\rho_s + \hat{a}^2).
\end{aligned}$$

For the specific quadratic specification of the value function in equation (C-20), the HJB equation becomes

(C-22)

$$0 = \min_{c_n, x_n} \Bigg[-\frac{e^{-Ac_n}}{V} - \beta + \psi_M(rM_n + S_n\, D - c_n - P(x_n)\, x_n)$$
$$+ \, (\psi_{SS} S_n + \psi_{SD} D + \psi_{Sn} \hat{H}_n + \psi_{Sx} \hat{H}_{-n}) x_n \Bigg] + \psi_{SD} S_n(-\alpha_D D + G_n(t))$$
$$+ \Big(\psi_{Sn} S_n + \psi_{nn} \hat{H}_n + \psi_{nx} \hat{H}_{-n}\Big) \Big(-(\alpha_G + \tau)\hat{H}_n(t) + \frac{\tau_H^{1/2} + \hat{a}\tau_0^{1/2}}{\sigma_G\, \Omega^{1/2}} G_n(t)\Big)$$
$$+ \Big(\psi_{Sx} S_n + \psi_{xx} \hat{H}_{-n} + \psi_{nx} \hat{H}_n\Big) \left(-(\alpha_G + \tau)\hat{H}_{-n}(t) + \frac{\theta_s \tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2} + \hat{a}\tau_0^{1/2}}{\sigma_G\, \Omega^{1/2}} G_n(t)\right)$$
$$+ \tfrac{1}{2} \psi_{SD}^2 S_n^2 \sigma_D^2 + \tfrac{1}{2} \Big((\psi_{Sn} S_n + \psi_{nn} \hat{H}_n + \psi_{nx} \hat{H}_{-n})^2 + \psi_{nn} \Big) \, \big(1 + \hat{a}^2\big)$$
$$+ \tfrac{1}{2} \Big((\psi_{Sx} S_n + \psi_{xx} \hat{H}_{-n} + \psi_{nx} \hat{H}_n)^2 + \psi_{xx} \Big) \, \Big(\rho_s + \frac{1 - \rho_s}{N_s - 1} + \hat{a}^2 \Big)$$
$$+ \Big((\psi_{Sn} + \psi_{Sx}) S_n + (\psi_{nn} + \psi_{nx}) \hat{H}_n + (\psi_{xx} + \psi_{nx}) \hat{H}_{-n} \Big) \, \psi_{SD} S_n \hat{a} \sigma_D$$
$$+ \Big((\psi_{Sn} S_n + \psi_{nn} \hat{H}_n + \psi_{nx} \hat{H}_{-n}) \, (\psi_{Sx} S_n + \psi_{xx} \hat{H}_{-n} + \psi_{nx} \hat{H}_n) + \psi_{nx} \Big) \, (\rho_s + \hat{a}^2).$$

The solution for optimal consumption is

$$c_n(t) = -\frac{1}{A} \, \log\Big(\frac{\psi_M\, V(t)}{A}\Big). \tag{C-23}$$

The optimal trading strategy is a linear function of the state variables given by

$$x_n(t) = \frac{(N_s - 1)\gamma_P}{2\psi_M} \left(\Big(\psi_{SD} - \frac{\psi_M \gamma_D}{\gamma_P}\Big) \, D(t) + \Big(\psi_{SS} - \frac{\psi_M \gamma_S}{(N_s - 1)\gamma_P}\Big) \, S_n(t) + \psi_{Sn} \, \hat{H}_n(t) + \Big(\psi_{Sx} - \frac{\psi_M \gamma_H}{\gamma_P}\Big) \, \hat{H}_{-n}(t) \right). \tag{C-24}$$

Trader n can infer from the market-clearing condition (16) that $\hat{H}_{-n}$ is given by

$$\hat{H}_{-n}(t) = \frac{\gamma_P}{\gamma_H} \, \Big(P(t) - D(t) \, \frac{\gamma_D}{\gamma_P} \Big) - \frac{1}{(N_s - 1)\gamma_H} \, x_n(t) - \frac{\gamma_S}{(N - 1)\gamma_H} \, S_n(t). \tag{C-25}$$

Plugging equation (C-25) into equation (C-24) yields $x_n(t)$ as a linear demand schedule given by

$$x_n(t) = \frac{(N_s-1)\gamma_P}{\psi_M}\left(1+\frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1} \cdot \Bigg(\left(\psi_{SD} - \psi_{Sx}\frac{\gamma_D}{\gamma_H}\right) D(t) + \left(\psi_{SS} - \psi_{Sx}\frac{\gamma_S}{(N_s-1)\gamma_H}\right) S_n(t) + \psi_{Sn}\,\hat{H}_n(t) + \left(\psi_{Sx}\frac{\gamma_P}{\gamma_H} - \psi_M\right) P(t)\Bigg). \tag{C-26}$$

Equating the coefficients of $D(t)$, $\hat{H}_n(t)$, $S_n(t)$, and $P(t)$ in equation (C-26) to the conjectured coefficients γ_D, γ_H, $-\gamma_S$, and $-\gamma_P$ results in the following four equations:

$$\frac{(N_s-1)\gamma_P}{\psi_M}\left(1+\frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1}\left(\psi_{SD} - \psi_{Sx}\frac{\gamma_D}{\gamma_H}\right) = \gamma_D, \tag{C-27}$$

$$\frac{(N_s-1)\gamma_P}{\psi_M}\left(1+\frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1}\psi_{Sn} = \gamma_H, \tag{C-28}$$

$$\frac{(N_s-1)\gamma_P}{\psi_M}\left(1+\frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1}\left(\psi_{SS} - \psi_{Sx}\frac{\gamma_S}{(N_s-1)\gamma_H}\right) = -\gamma_S, \tag{C-29}$$

$$\frac{(N_s-1)\gamma_P}{\psi_M}\left(1+\frac{\psi_{Sx}}{\psi_M}\frac{\gamma_P}{\gamma_H}\right)^{-1}\left(\psi_{Sx}\frac{\gamma_P}{\gamma_H} - \psi_M\right) = -\gamma_P. \tag{C-30}$$

We obtain

$$\psi_{Sx} = \frac{N_s-2}{2}\psi_{Sn}, \quad \gamma_H = \frac{N_s\gamma_P}{2\psi_M}\psi_{Sn}, \quad \gamma_S = -\frac{(N_s-1)\gamma_P}{\psi_M}\psi_{SS}, \quad \gamma_D = \frac{\gamma_P}{\psi_M}\psi_{SD}. \tag{C-31}$$

Define the constants C_L and C_G by

$$C_L := -\frac{\psi_{Sn}}{2\psi_{SS}}, \qquad C_G := \frac{\psi_{Sn}}{2\psi_M}\frac{N_s(r+\alpha_D)(r+\alpha_G)\Big(1+(N_s-1)\rho_s\Big)}{\sigma_G\Omega^{1/2}\Big((1+(N_s-1)\theta_s)\,\tau_H^{1/2} + (N_s-1)(1-\theta_s)\tau_L^{1/2}\Big)}. \tag{C-32}$$

Substituting equation (C-31) into equation (C-24) yields the solution for optimal strategy.

$$x_n^*(t) = \gamma_S\Big(C_L\,(H_n(t) - H_{-n}(t)) - S_n(t)\Big). \tag{C-33}$$

Define the average of traders' expected growth rates $\bar{G}(t)$ by

$$\bar{G}(t) := \frac{1}{N_s}\sum_{n=1}^{N_s} G_n(t), \tag{C-34}$$

Then, the equilibrium price is

$$P^*(t) = \frac{D(t)}{r+\alpha_D} + \frac{C_G\,\bar{G}(t)}{(r+\alpha_D)(r+\alpha_G)}. \tag{C-35}$$

Plugging (C-23) and (C-24) back into the Bellman equation and setting the constant term and the coefficients of M_n, $S_n\,D$, S_n^2, $S_n\,\hat{H}_n$, $S_n\,\hat{H}_{-n}$, $\hat{H}_n^2$, $\hat{H}_{-n}^2$, and $\hat{H}_n\,\hat{H}_{-n}$ to be zero, we obtain nine equations. Using the first equation (C-31) to substitute ψ_{Sn} for ψ_{Sx}, there are in total nine equations in nine unknowns γ_P, $\psi_0, \psi_M, \psi_{SD}, \psi_{SS}, \psi_{Sn}, \psi_{nn}, \psi_{xx}$, and ψ_{nx}.

By setting the constant term, coefficient of M, and coefficient of S_nD to be zero, we obtain

$$\psi_M = -rA, \quad \psi_{SD} = -\frac{rA}{r+\alpha_D}, \tag{C-36}$$

(C-37)

$$\psi_0 = 1 - \ln r + \frac{1}{r}\left(-\beta + \frac{1}{2}(1+\hat{a}^2)\psi_{nn} + \frac{1}{2}\left(\hat{a}^2 + \frac{1+(N_s-2)\rho_s}{N_s-1}\right)\psi_{xx} + (\hat{a}^2+\rho_s)\psi_{nx}\right).$$

In addition, by setting the coefficients of $S_n^2, S_n\,\hat{H}_n, S_n\,\hat{H}_{-n}, \hat{H}_n^2, \hat{H}_{-n}^2$ and $\hat{H}_n\,\hat{H}_{-n}$ to be zero, we obtain six polynomial equations in the six unknowns $\gamma_P, \psi_{SS}, \psi_{Sn}, \psi_{nn}, \psi_{xx}$, and ψ_{nx}. Defining the constants a_1, a_2, a_3, and a_4 by

$$a_1 := -(\alpha_G+\tau) + (\tau_H^{1/2} + \hat{a}\tau_0^{1/2})\Big(\frac{(1-\theta_s)(\tau_H^{1/2}-\tau_L^{1/2})}{1-\rho_s} + \frac{(\theta_s-\rho_s)\tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2}}{(1-\rho_s)(1+(N_s-1)\rho_s)}\Big),$$

$$a_2 := -(\alpha_G+\tau) + (N_s-1)\left(\theta_s\tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2} + \hat{a}\tau_0^{1/2}\right)\frac{(\theta_s-\rho_s)\tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2}}{(1-\rho_s)(1+(N_s-1)\rho_s)},$$

$$a_3 := (\tau_H^{1/2} + \hat{a}\tau_0^{1/2})(N_s-1)\frac{(\theta_s-\rho_s)\tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2}}{(1-\rho_s)(1+(N_s-1)\rho_s)},$$

$$a_4 := \left(\theta_s\tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2} + \hat{a}\tau_0^{1/2}\right)\left(\frac{(1-\theta_s)(\tau_H^{1/2}-\tau_L^{1/2})}{1-\rho_s} + \frac{(\theta_s-\rho_s)\tau_H^{1/2} + (1-\theta_s)\tau_L^{1/2}}{(1-\rho_s)(1+(N_s-1)\rho_s)}\right),$$

these six equations in six unknowns can be written

(C-38)

$$0 = -\frac{1}{2}r\psi_{SS} - \frac{\gamma_P(N_s-1)}{rA}\psi_{SS}^2 + \frac{r^2A^2\sigma_D^2}{2(r+\alpha_D)^2} + \frac{1}{2}(1+\hat{a}^2)\psi_{sn}^2 + \\ +\frac{1}{2}\left(\hat{a}^2 + \frac{1+(N_s-2)\rho_s}{N_s-1}\right)\frac{(N_s-2)^2}{4}\psi_{Sn}^2 - \frac{rA}{r+\alpha_D}\hat{a}\sigma_D\frac{N_s}{2}\psi_{Sn} + \frac{N_s-2}{2}\psi_{Sn}^2(\hat{a}^2+\rho_s)$$

(C-39)

$$0 = -r\psi_{Sn} - \frac{rA}{r+\alpha_D}\sigma_G\Omega^{1/2}\Big(\frac{(1-\theta_s)(\tau_H^{1/2}-\tau_L^{1/2})}{1-\rho_s} + \frac{(\theta_s-\rho_s)\tau_H^{1/2}+(1-\theta_s)\tau_L^{1/2}}{(1-\rho_s)(1+(N_s-1)\rho_s)}\Big) + a_1\psi_{Sn} \\ -\frac{\gamma_P(N_s-1)}{rA}\psi_{SS}\psi_{Sn} + \frac{N_s-2}{2}a_4\psi_{Sn} + (1+\hat{a}^2)\psi_{nn}\psi_{Sn} + \frac{N_s-2}{2}\left(\hat{a}^2 + \frac{1+(N_s-2)\rho_s}{N_s-1}\right)\psi_{nx}\psi_{Sn} \\ -\frac{rA}{r+\alpha_D}\hat{a}\sigma_D(\psi_{nn}+\psi_{nx}) + (\hat{a}^2+\rho_s)\left(\psi_{nx}\psi_{Sn} + \frac{N_s-2}{2}\psi_{nn}\psi_{Sn}\right),$$

(C-40)

$$0 = -r\frac{N_s-2}{2}\psi_{Sn} + \frac{\gamma_P(N_s-1)}{rA}\psi_{SS}\psi_{Sn} - \frac{rA}{r+\alpha_D}\sigma_G\Omega^{1/2}(N_s-1)\frac{(\theta_s-\rho_s)\tau_H^{1/2}+(1-\theta_s)\tau_L^{1/2}}{(1-\rho_s)(1+(N_s-1)\rho_s)} \\ +(a_3 + \frac{N_s-2}{2}a_2)\psi_{Sn} + (1+\hat{a}^2)\psi_{Sn}\psi_{nx} + \frac{N_s-2}{2}\left(\hat{a}^2 + \frac{1+(N_s-2)\rho_s}{N_s-1}\right)\psi_{xx}\psi_{Sn} \\ -\frac{rA}{r+\alpha_D}\hat{a}\sigma_D(\psi_{xx}+\psi_{nx}) + (\hat{a}^2+\rho_s)\left(\psi_{xx}\psi_{Sn} + \frac{N_s-2}{2}\psi_{nx}\psi_{Sn}\right),$$

(C-41)

$$0 = -\frac{r}{2}\psi_{nn} - \frac{\gamma_P(N_s-1)}{4rA}\psi_{Sn}^2 + a_1\psi_{nn} + a_4\psi_{nx} + \frac{1}{2}(1+\hat{a}^2)\psi_{nn}^2 \\ +\frac{1}{2}\left(\hat{a}^2 + \frac{1+(N_s-2)\rho_s}{N_s-1}\right)\psi_{nx}^2 + (\hat{a}^2+\rho_s)\psi_{nn}\psi_{nx},$$

(C-42)

$$0 = -\frac{r}{2}\psi_{xx} - \frac{\gamma_P(N_s-1)}{4rA}\psi_{Sn}^2 + a_2\psi_{xx} + a_3\psi_{nx} + \frac{1+\hat{a}^2}{2}\psi_{nx}^2 \\ +\frac{1}{2}\left(\hat{a}^2 + \frac{1+(N_s-2)\rho_s}{N_s-1}\right)\psi_{xx}^2 + (\hat{a}^2+\rho_s)\psi_{xx}\psi_{nx},$$

$$
\text{(C-43)}\quad
\begin{aligned}
0 = & - r\psi_{nx} + \frac{\gamma_P(N_s-1)}{2rA}\psi_{Sn}^2 + a_3\psi_{nn} + a_4\psi_{xx} + (a_1+a_2)\psi_{nx} \\
& + (1+\hat{a}^2)\psi_{nn}\psi_{nx} + \left(\hat{a}^2 + \frac{1+(N_s-2)\rho_s}{N_s-1}\right)\psi_{xx}\psi_{nx} + (\hat{a}^2+\rho_s)(\psi_{nn}\psi_{xx}+\psi_{nx}^2).
\end{aligned}
$$

We solve equations (C-38)–(C-43) numerically. For a solution to the six polynomial equations to define a stationary equilibrium, a second-order condition implying $\gamma_P > 0$, a stationarity condition implying $\gamma_S > 0$, and a transversality condition requiring $r > 0$.

The transversality condition for the value function $V(\ldots)$ is

$$
\text{(C-44)}\qquad \lim_{T\to+\infty} \mathrm{E}_t^n\left[\mathrm{e}^{-\rho_s(T-t)}\ V\left(M_n(T), S_n(T), D(T), \hat{H}_n(T), \hat{H}_{-n}(T)\right)\right] = 0.
$$

The transversality condition (C-44) is satisfied if $r > 0$. Under the assumptions $\gamma_P > 0$ and $\gamma_S > 0$, analytical results imply $\gamma_D > 0$, $\psi_M < 0$, $\psi_{SD} < 0$, and $\psi_{SS} > 0$. The numerical results indicate that $\gamma_H > 0$, $\psi_{Sn} < 0$, $\psi_{Sx} < 0$, $\psi_{nn} < 0$, $\psi_{xx} < 0$ and the sign of ψ_{nx} is intuitively and numerically ambiguous.

C.3. Proof of Corollary 1

The consistency condition in equation (22) implies that

$$
\text{(C-45)}\qquad \rho_s - \rho = \frac{(N_s-N)(1-\rho)}{N(N_s-1)}.
$$

Equation (C-45) implies that, if $N_s < N$, then $\rho_s < \rho$.

(C-46)

$$
\begin{aligned}
Cov(dI_n(t), dP(t)) =& Cov(\rho^{1/2}dZ(t) + (1-\rho)^{1/2}dB_n(t),\ \rho^{1/2}dZ(t) + (1-\rho)^{1/2}\frac{1}{N}\sum_{m=1}^{N} dB_m(t)) \\
=& \rho + \frac{1}{N}(1-\rho).
\end{aligned}
$$

(C-47)

$$
\begin{aligned}
Cov(dI_n(t), dP_s(t)) =& Cov(\rho_s^{1/2}dZ(t) + (1-\rho_s)^{1/2}dB_n(t),\ \rho_s^{1/2}dZ(t) + (1-\rho_s)^{1/2}\frac{1}{N_s}\sum_{m=1}^{N_s} dB_m(t)) \\
=& \rho_s + \frac{1}{N_s}(1-\rho_s).
\end{aligned}
$$

Therefore, if the consistency condition (22) is satisfied, then for each trader, the correlation coefficient between his private signal and the actual price is consistent with the correlation between his private signal and his “subjective” price.