Chapter 6

Functional Form and Structural Change

**Exercises**

1. T he *F* statistic could be computed as

*F* = {[1425 ‑ (104 + 88 + ... + 211)] / [6(16)]}/[(104 + 88 + ... + 211) / (570 – 6(17))] = 0.638

The 95% critical value for the *F* distribution with 96 and 468 degrees of freedom is 1.282.

2. a. Using the hint, we seek the *c*\* which is the slope on **d** in the regression of **q** = **y** ‑ *c***d** ‑ **e** on **y** and **d**. The regression coefficients are  = . In the preceding, note that (**y′y**,**d′y**)**′** is the first column of the matrix being inverted while *c*(**y′d**,**d′d**)**′** is *c* times the second. An inverse matrix times the first column of the original matrix is the first column of an identity matrix, and likewise for the second. Also, since **d** was one of the original regressors in (1), **d′e** = 0, and, of course, **y′e** = **e′e**. If we combine all of these, the coefficient vector is

 = . We are interested in the second (lower) of the two coefficients. The matrix product at the end is **e′e** times the first column of the inverse matrix, and we wish to find its second (bottom) element. Therefore, collecting what we have thus far, the desired coefficient is *c*\* = ‑*c* ‑ **e′e** times the off diagonal element in the inverse matrix. The off diagonal element is

‑**d′y** / [(**y′y**)(**d′d**) ‑ (**y′d**)2] = ‑**d′y** / {[(**y′y**)(**d′d**)][1 ‑ (**y′d**)2/[(**y′y**)(**d′d**)]]}

= -**d′y** / [(**y′y**)(**d′d**)(1 ‑**)].

Therefore, *c*\* = [(**e′e**)(**d′y**)] / [(**y′y**)(**d′d**)(1 ‑ **)] ‑ *c*

(The two negative signs cancel.) This can be further reduced. Since all variables are in deviation form, **e′e/y′y** is (1 ‑ *R*2) in the full regression. By multiplying it out, you can show that = *P* so that

**d′d** = Σ*i* (*di* - *P*)2 = *nP*(1‑*P*)

and  **d′y** = Σ*i* (*di* - *P*)(*yi* -**) = Σ*i*(*di - P*)*yi* = *n*1(** - **)

where *n*1 is the number of observations which have *di* = 1. Combining terms once again, we have

*c*\* = {[*n*1(** - **)(1 - *R*2)} / {*nP*(1‑*P*)(1 ‑**)} ‑ *c*

Finally, since *P* = *n*1/*n*, this further simplifies to the result claimed in the problem,

*c*\* = {(** - **)(1 - *R*2)} / {(1‑*P*)(1 ‑**)} ‑ *c*

The problem this creates for the theory is that in the present setting, if, indeed, *c* is negative, (** - **) will almost surely be also. Therefore, the sign of *c*\* is ambiguous.

3. We first find the joint distribution of the observed variables. so [*y*,*x*] have a joint normal distribution with mean vector and covariance matrix, The probability limit of the slope in the linear regression of *y* on *x* is, as usual,

plim *b* = Cov[*y*,*x*]/Var[*x*] = β/(1 + σu2/σ\*2) < β.

The probability limit of the intercept is plim

*a* = *E*[*y*] - (plim *b*)*E*[*x*] = α + βμ\* - βμ\*/(1 + σu2/σ\*2)

= α + β[μ\*σu / (σ\*2 + σu2)] > α (assuming β > 0).

If *x* is regressed on *y* instead, the slope will estimate plim[*b*′] = Cov[*y*,*x*]/Var[*y*] = βσ\*2/(β2σ\*2 + σε2). Then,plim[1/*b*′] = β + σε2/β2σ\*2 > β. Therefore, *b* and *b*′ will bracket the true parameter (at least in their probability limits). Unfortunately, without more information about σu2, we have no idea how wide this bracket is. Of course, if the sample is large and the estimated bracket is narrow, the results will be strongly suggestive.

4. In the regression of **y** on **x** and **d**, if **d** and **x** are independent, we can invoke the familiar result for least squares regression. The results are the same as those obtained by two simple regressions. It is instructive to verify this. Therefore, although the coefficient on **x** is distorted, the effect of interest, namely, γ, is correctly measured. Now consider what happens if *x*\* and *d* are not independent. With the second assumption, we must replace the off diagonal zero above with plim(**x′d**/*n*). Since *u* and *d* are still uncorrelated, this equals Cov[*x*\*,*d*]. This is

Cov[*x*\*,*d*] = *E*[*x*\**d*] = π*E*[*x*\**d*|*d*=1] + (1‑π)*E*[*x*\**d*|*d*=0] = πμ1.

Also, plim[**y′d**/*n*] is now βCov[*x*\*,*d*] + γplim(**d′d**/*n*) = βπμ1 + γπ and plim[**y′x**\*/*n*] equals βplim[**x**\*′**x**\*/*n*] + γplim[**x**\*′**d**/*n*] = βσ\*2 + γπμ1. Then, the probability limits of the least squares coefficient estimators is



= 

= .

The second expression does reduce to plim *c* = γ + βπμ1σu2/[π(σ\*2 + σu2) - π2(μ1)2], but the upshot is that in the presence of measurement error, the two estimators become an unredeemable hash of the underlying parameters. Note that both expressions reduce to the true parameters if σu2 equals zero.

Finally, the two means are estimators of

*E*[*y*|*d*=1] = β*E*[*x*\*|*d*=1] + γ = βμ1 + γ

and *E*[*y*|*d*=0] = βE[*x*\*|*d*=0] = βμ0,

so the difference is β(μ1 ‑ μ0) + γ, which is a mixture of two effects. Which one will be larger is entirely indeterminate, so it is reasonable to conclude that this is *not* a good way to analyze the problem. If γ equals zero, this difference will merely reflect the differences in the values of *x*\*, which may be entirely unrelated to the issue under examination here. (This is, unfortunately, what is usually reported in the popular press.)

**Applications**

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? Application 6.1

?=======================================================================

a. Wage equation

Namelist ; X = one,educ,ability,pexp,med,fed,bh,sibs$

Regress ; Lhs = lwage ; Rhs = x $

Calc ; xb = b(1)+b(2)\*12+b(3)\*xbr(ability)+b(4)\*xbr(med)

+b(5)\*xbr(fed)+b(6)\*0+b(7)\*xbr(sibs) $

Calc ; list ; mv = exp(xb) \* b(2) $

+----------------------------------------------------+

| Ordinary least squares regression |

| LHS=LWAGE Mean = 2.296821 |

| Standard deviation = .5282364 |

| WTS=none Number of observs. = 17919 |

| Model size Parameters = 7 |

| Degrees of freedom = 17912 |

| Residuals Sum of squares = 4126.175 |

| Standard error of e = .4799564 |

| Fit R-squared = .1747197 |

| Adjusted R-squared = .1744433 |

| Model test F[ 6, 17912] (prob) = 632.02 (.0000) |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

Constant| .96950956 .03370543 28.764 .0000

EDUC | .07220350 .00225076 32.080 .0000 12.6760422

ABILITY | .07746781 .00493727 15.690 .0000 .05237402

PEXP | .03950928 .00089926 43.936 .0000 8.36268765

MED | -.00011702 .00169634 -.069 .9450 11.4719013

FED | .00545695 .00133870 4.076 .0000 11.7092472

SIBS | .00476557 .00179240 2.659 .0078 3.15620291

+------------------------------------+

| Listed Calculator Results |

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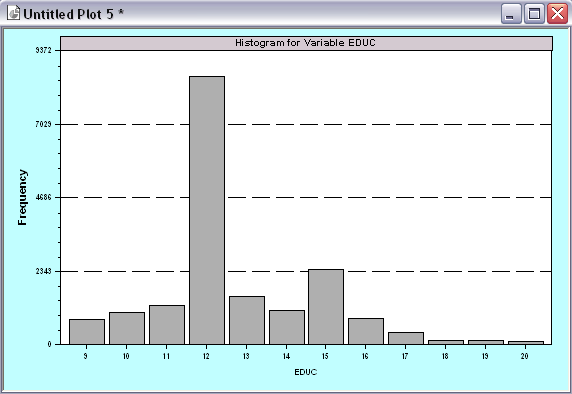
MV = .725176b. Step function

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? b.

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Histogram ; Rhs = Educ $



Create ; HS = Educ <= 12 $

Create ; Col = (Educ>12) \* (educ <=16) $

Create ; Grad = Educ > 16 $

Regress ; Lhs=lwage ; Rhs = one,Col,Grad,ability,pexp,med,fed,bh,sibs $

+----------------------------------------------------+

| Ordinary least squares regression |

| LHS=LWAGE Mean = 2.296821 |

| Standard deviation = .5282364 |

| WTS=none Number of observs. = 17919 |

| Model size Parameters = 9 |

| Degrees of freedom = 17910 |

| Residuals Sum of squares = 4215.033 |

| Standard error of e = .4851239 |

| Fit R-squared = .1569472 |

| Adjusted R-squared = .1565706 |

| Model test F[ 8, 17910] (prob) = 416.78 (.0000) |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

Constant| 1.81124933 .02069456 87.523 .0000

COL | .17467913 .00872506 20.020 .0000 .32183716

GRAD | .36244740 .02086328 17.373 .0000 .03493499

ABILITY | .10097636 .00486713 20.747 .0000 .05237402

PEXP | .03814088 .00090643 42.078 .0000 8.36268765

MED | .00081934 .00171488 .478 .6328 11.4719013

FED | .00700641 .00135096 5.186 .0000 11.7092472

BH | -.06962521 .01007870 -6.908 .0000 .15385903

SIBS | .00371191 .00181156 2.049 .0405 3.15620291

c. Education squared

Create ; educsq = educ\*educ $

Regress ; Lhs = lwage;rhs=one,educ,educsq,ability,pexp,med,fed,bh,sibs$

+----------------------------------------------------+

| Ordinary least squares regression |

| LHS=LWAGE Mean = 2.296821 |

| Standard deviation = .5282364 |

| WTS=none Number of observs. = 17919 |

| Model size Parameters = 9 |

| Degrees of freedom = 17910 |

| Residuals Sum of squares = 4114.269 |

| Standard error of e = .4792902 |

| Fit R-squared = .1771010 |

| Adjusted R-squared = .1767334 |

| Model test F[ 8, 17910] (prob) = 481.81 (.0000) |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

Constant| .42778242 .12008093 3.562 .0004

EDUC | .15590624 .01751608 8.901 .0000 12.6760422

EDUCSQ | -.00313261 .00064230 -4.877 .0000 164.377588

ABILITY | .07433494 .00496954 14.958 .0000 .05237402

PEXP | .03962214 .00089830 44.108 .0000 8.36268765

MED | .00030520 .00169504 .180 .8571 11.4719013

FED | .00519423 .00133734 3.884 .0001 11.7092472

BH | -.04957434 .01000691 -4.954 .0000 .15385903

SIBS | .00499325 .00179020 2.789 .0053 3.15620291

Namelist ; x1 = one,educ,educsq,ability,pexp,med,fed,bh,sibs $

Matrix ; means = mean(x1)$

Matrix ; means(2)=0 $

Matrix ; means(3)=0$

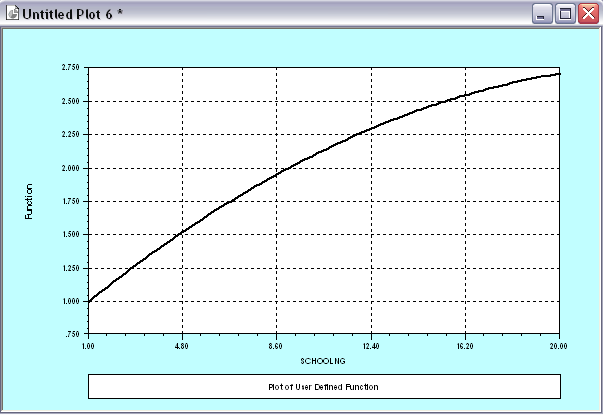
Calc ; a=means'b $

Calc ; b2=b(2) ; b3=b(3) $

Sample ; 1 $

Fplot ; fcn = a + b2\*schoolng + b3\*schoolgn^2 ; pts=100

; start = 12 ; limits = 1,20 ; labels=schoolng ; plot(schoolng) $



d. Interaction.

Sample ; All $

Create ; EA = Educ\*ability $

Regress ; Lhs = lwage;rhs=one,educ,ability,ea,pexp,med,fed,bh,sibs$

Calc ; abar =xbr(ability) $

Calc ; list ; me = b(2)+b(4)\*abar $

Calc ; sdme = sqr(varb(2,2)+abar^2\*varb(4,4) + 2\*abar\*varb(2,4))$

Calc ; list ; lower = me - 1.96\*sdme ; upper = me + 1.96\*sdme $

+----------------------------------------------------+

| Ordinary least squares regression |

| LHS=LWAGE Mean = 2.296821 |

| Standard deviation = .5282364 |

| WTS=none Number of observs. = 17919 |

| Model size Parameters = 9 |

| Degrees of freedom = 17910 |

| Residuals Sum of squares = 4119.377 |

| Standard error of e = .4795877 |

| Fit R-squared = .1760794 |

| Adjusted R-squared = .1757113 |

| Model test F[ 8, 17910] (prob) = 478.44 (.0000) |

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+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

Constant| 1.00190276 .03529335 28.388 .0000

EDUC | .07006221 .00243183 28.811 .0000 12.6760422

ABILITY | .04693108 .02494471 1.881 .0599 .05237402

EA | .00253975 .00204029 1.245 .2132 1.60372621

PEXP | .03947437 .00089903 43.908 .0000 8.36268765

MED | .542277D-04 .00169546 .032 .9745 11.4719013

FED | .00534599 .00133813 3.995 .0001 11.7092472

BH | -.05314420 .00999271 -5.318 .0000 .15385903

SIBS | .00479076 .00179231 2.673 .0075 3.15620291

+------------------------------------+

| Listed Calculator Results |

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ME = .070195

LOWER = .065503

UPPER = .074888

e.

Regress ; Lhs = lwage;rhs=one,educ,educsq,ability,ea,pexp,med,fed,bh,sibs$

+----------------------------------------------------+

| Ordinary least squares regression |

| LHS=LWAGE Mean = 2.296821 |

| Standard deviation = .5282364 |

| WTS=none Number of observs. = 17919 |

| Model size Parameters = 10 |

| Degrees of freedom = 17909 |

| Residuals Sum of squares = 4106.031 |

| Standard error of e = .4788235 |

| Fit R-squared = .1787487 |

| Adjusted R-squared = .1783360 |

| Model test F[ 9, 17909] (prob) = 433.11 (.0000) |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

Constant| -.10514525 .14931731 -.704 .4813

EDUC | .24088793 .02252126 10.696 .0000 12.6760422

EDUCSQ | -.00654261 .00085754 -7.630 .0000 164.377588

ABILITY | -.12453442 .03354596 -3.712 .0002 .05237402

EA | .01631824 .00272231 5.994 .0000 1.60372621

PEXP | .03951247 .00089761 44.020 .0000 8.36268765

MED | .00045246 .00169356 .267 .7893 11.4719013

FED | .00524829 .00133606 3.928 .0001 11.7092472

BH | -.04775208 .01000179 -4.774 .0000 .15385903

SIBS | .00460796 .00178961 2.575 .0100 3.15620291

+------------------------------------+

| Listed Calculator Results |

+------------------------------------+

AVGLOW = -.798563

AVGHIGH = .717891

Create ; lowa = ability < xbr(ability) ; higha = 1 - lowa $

Calc ; list ; avglow= lowa'ability / lowa'lowa ; avghigh=higha'ability/higha'higha $

Calc ; a = b(1) + b(6)\*xbr(pexp)+b(7)\*xbr(med)+

b(8)\*xbr(fed)+b(9)\*xbr(bh)+b(10)\*xbr(sibs)$

Calc ; al=a+b(4)\*avglow ; ah = a+b(4)\*avghigh$

Samp;1-120$

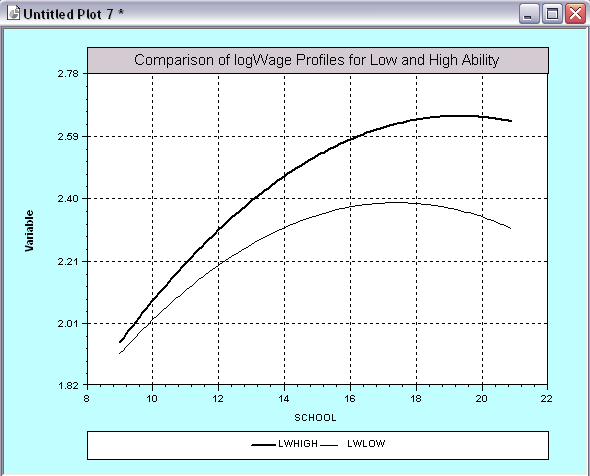
Create ; school = trn(9,.1)$

Create ; lwlow = al + b(2)\*school+b(3)\*school^2 + b(5)\*avglow\*school $

Create ; lwhigh = ah + b(2)\*school+b(3)\*school^2 + b(5)\*avghigh\*school $

Plot ; lhs = school ; rhs =lwhigh,lwlow ;fill ;grid

;Title=Comparison of logWage Profiles for Low and High Ability$



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? Application 6.2

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Sample ; All $

Namelist ; X = one,educ,ability,pexp,med,fed,sibs$

Regress ; For [bh=0] ; Lhs = lwage ; Rhs = x $

Calc ; ee0=sumsqdev $

Matrix ; b0=b ; v0=varb $

Regress ; For [bh=1] ; Lhs = lwage ; Rhs = x $

Calc ; ee1=sumsqdev $

Matrix ; b1=b ; v1=varb $

Regress ; Lhs = lwage ; Rhs = x $

Calc ; ee=sumsqdev $

Calc ; list ; chow = ((ee-ee0-ee1)/col(x))/ ((ee0+ee1)/(n-2\*col(x))) $

+------------------------------------+

| Listed Calculator Results |

+------------------------------------+

CHOW = 7.348379

Matrix ; db=b0-b1 ; vdb=v0+v1 $

Matrix ; list ; Wald = db'<vdb>db $

Matrix WALD has 1 rows and 1 columns.

1

+--------------

1| 50.57114

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? Application 6.3

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a. The least squares estimates of the four models are

*q*/*A* = .45237 + .23815ln*k*

*q*/*A* = .91967 ‑ .61863/*k*

ln(*q*/*A*) = ‑.72274 + .35160ln*k*

ln(*q*/*A*) = ‑.032194 ‑ .91496/*k*

At these parameter values, the four functions are nearly identical. A plot of the four sets of predictions from the regressions and the actual values appears below.



b. The scatter diagram is shown below. The last seven years of the data set show clearly the effect observed by Solow.



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c. The regression results for the various models are listed below. (d is the dummy variable equal to 1 for the last seven years of the data set. Standard errors for parameter estimates are given in parentheses.)

α β γ δ *R*2 **e′e**

Model 1:*q*/*A* = α + βln*k* + γ*d* + δ(*d*ln*k*) + ε

.4524 .2381 .94355 .00213

(.00903) (.00932)

.4477 .2396 .01900 .99914 .000032

(.00113) (.00117) (.000384)

.4476 .2397 .02746 ‑.08883 .99915 .000032

(.00115) (.00118) (.0119) (.0126)

Model 2: *q*/*A* = α ‑ β(1/*k*) + γ*d* + δ(*d*/*k*) + ε

.9168 .6186 .94915 .001915

(.00891) (.0229)

.9167 .6185 .01961 .99321 .000256

(.00331) (.00849) (.00108)

.9168 .6187 .008651 .02140 .99322 .000255

(.00336) (.00863) (.0354) (.0917)

Model 3: ln(*q*/*A*) = α + βln*k* + γ*d* + δ(*d*ln*k*) + ε

‑.7227 .3516 .94069 .004882

(.0137) (.0141)

‑.7298 .3538 .002881 .99918 .000068

(.00164) (.00169) (.000554)

‑.7300 .3540 .04961 ‑.02182 .99921 .000065

(.00164) (.00148) (.0171) (.0179)

Model 4: ln(*q*/*A*) = α ‑ β(1/*k*) + γ*d* + δ(*d*/*k*) + ε

‑.03219 .9150 .94964 .004146

(.0131) (.0337)

‑.03665 .9148 .02572 .99629 .000305

(.00361) (.00928) (.00118)

‑.03646 .9153 .004290 .05556 .99632 .000303

(.00366) (.00941) (.0386) (.0999)

d. For the four models, the *F* test of the third specification against the first is equivalent to the Chow‑test. The statistics are:

Model 1: *F* = (.002126 ‑ .000032)/2 / (.000032/37) = 1210.6

Model 2: *F* = = 120.43

Model 3: *F* = = 1371.0

Model 4: *F* = = 234.64

The critical value from the F table for 2 and 37 degrees of freedom is 3.26, so all of these are statistically significant. The hypothesis that the same model applies in both subperiods must be rejected.

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? Application 6.4

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According to the full model, the expected number of incidents for a ship of the base type A built in the base period 1960 to 1964, is 3.4. The other 19 predicted values follow from the previous results and are left as an exercise. The relevant test statistics for differences across ship type and year are as follows:





The 5 percent critical values from the *F* table with these degrees of freedom are 3.26 and 3.49, respectively, so we would conclude that the average number of incidents varies significantly across ship types but not across years.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Regression Coefficients** | | | | |
|  | ***Full Model*** | ***Time Effects*** | ***Type Effects*** | ***No Effects*** |
| Constant | 3.4 | 6.0 | 8.25 | 10.85 |
| B | 27.75 | 0 | 27.75 | 0 |
| C | –7.0 | 0 | –7.0 | 0 |
| D | –4.5 | 0 | –4.5 | 0 |
| E | –3.25 | 0 | –3.25 | 0 |
| 65–69 | 7.0 | 7.0 | 0 | 0 |
| 70–74 | 11.4 | 11.4 | 0 | 0 |
| 75–79 | 1.0 | 1.0 | 0 | 0 |
| *R*2 | 0.84823 | 0.0986 | 0.74963 | 0 |
| **e**′**e** | 660.9 | 3925.2 | 1090.2 | 4354.5 |

Chapter 7

Nonlinear, Semiparametric, and Nonparametric Regression Models

**Exercises**

1. We cannot simply take logs of both sides of the equation as the disturbance is additive rather than multiplicative. So, we must treat the model as a nonlinear regression. The linearized equation is

y ≈ 

where α0 and β0 are the expansion point. For given values of α0 and β0, the estimating equation would be

+ ε\*

or  + ε\*.

Estimates of α and β are obtained by applying ordinary least squares to this equation. The process is repeated with the new estimates in the role of α0 and β0. The iteration could be continued until convergence. Starting values are always a problem. If one has no particular values in mind, one candidate would be α0 = and β0 = 0 or β0 = 1 and α0 either **x′y**/**x′x** or/. Alternatively, one could search directly for the α and β to minimize the sum of squares, *S*(α,β) = Σ*i* (*yi* - α*x*β)2 = Σ*i**εi*2. The first order conditions for minimization are

∂*S*(α,β)/∂α = ‑2Σ*i* (*yi* - α*x*β)*x*β = 0 and ∂*S*(α,β)/∂β = ‑2Σ*i* (*yi* - α*x*β)α(ln*x*)xβ = 0.

Methods for solving nonlinear equations such as these are discussed in Appendix E..

2. The proof can be done by mathematical induction. For convenience, denote the *i*th derivative by *f*i. The first derivative appears in Equation (10-34). Just by plugging in *i*=1, it is clear that *f*1 satisfies the relationship. Now, use the chain rule to differentiate *f*1,

*f*2 = (-1/λ2)[*x*λ(ln*x*) - *x*(λ)] + (1/λ)[(ln*x*)*x*λ(ln*x*) - *f*1]

Collect terms to yield *f*2 = (-1/λ)*f*1 + (1/λ)[*x*λ(ln*x*)2 - *f*1] = (1/λ)[*x*λ(ln*x*)2 - 2*f*1].

So, the relationship holds for *i* = 0, 1, and 2. We now assume that it holds for *i* = *K*-1, and show that if so, it also holds for *i* = *K*. This will complete the proof. Thus, assume

*fK*-1 = (1/λ)[*x*λ(ln*x*)*K*-1 - (*K*-1)*fK*-2]

Differentiate this to give *fK* = (-1/λ)*fK*-1 + (1/λ)[(ln*x*)*x*λ(ln*x*)*K*-1 - (*K*-1)*fK*-1].

Collect terms to give *fK* = (1/λ)[*x*λ(ln*x*)*K* - *KfK*-1], which completes the proof for the general case.

Now, we take the limiting value

limλ→0 *fi* = limλ→0 [*x*λ(ln*x*)*i* - *ifi*-1]/λ.

Use L'Hospital's rule once again.

limλ→0 *fi* = limλ→0 *d*{[*x*λ(ln*x*)*i* - *ifi*-1]/*d*λ}/limλ→0 *d*λ/*d*λ.

Then, limλ→0 *fi* = limλ→0 {[*x*λ(ln*x*)*i*+1 - *ifi*]}

Just collect terms, (*i*+1)limλ→0 *fi* = limλ→0 [*x*λ(ln*x*)*i*+1]

or limλ→0 *f*i = limλ→0 [*x*λ(ln*x*)*i*+1]/(*i*+1) = (ln*x*)*i*+1/(*i*+1).

Apps.

2. The search for the minimum sum of squares produced the following results:



1

λ **e′e**

‑.500 .78477

‑.400 .67033

‑.300 .60587

‑.250 .59479

‑.245 .59451

‑.244 .59447

‑.243 .59444

‑.242 .59441

‑.241 .59439

‑.240 .59438

‑.239 .59437

‑.238 .59436

‑.237 .59437

‑.235 .59440

‑.225 .59492

‑.200 .59897

‑.100 .65598

0.000 .78143

.100 .97742

.200 1.24354

The sum of squared residuals is minimized at λ = ‑.238. At this value, the regression results are as follows:

**Parameter Estimate OLS Std.Error Correct Std.Error**

α 2.06092 .07718 .09723

β*k* .178232 .04638 .04378

β*l* .737988 .06996 .12560

λ ‑.238 ‑‑‑‑ .07710

**Estimated Asymptotic Covariance Matrix**

α βk βl λ

α .00945

β*k* .00262 .00192

β*l* .00511 ‑.00199 .01578

λ .00500 .00037 .00825 .00594

The output elasticities for this function evaluated at the sample means are

∂ln*Y*/∂ln*K* = βkKλ = (.178232).175905‑.238 = .2695

∂ln*Y*/∂ln*L* = βlLλ = (.443954).737988‑.238 = .7740.

The estimates found for Zellner and Revankar's model were .254 and .882, respectively, so these are quite similar. For the simple log‑linear model, the corresponding values are .2790 and .927.

3. The Wald test is based on the unrestricted model. The statistic is the square of the usual t‑ratio,

W = (‑.232 / .0771)2 = 9.0546. The critical value from the chi‑squared distribution is 3.84, so the hypothesis that λ = 0 can be rejected. The likelihood ratio statistic is based on both models. The sum of squared residuals for both unrestricted and restricted models is given above. The log‑likelihood is

ln*L* = ‑(*n*/2)[1 + ln(2π) + ln(**e′e**/*n*)], so the likelihood ratio statistic is

*LR* = *n*[ln(**e′e**/*n*)|λ=0 ‑ ln(**e′e**/*n*)| λ=‑.238] = *n*ln[(**e′e|**λ**=**0) / (**e′e**|λ=‑.238)

= 25ln(.78143/.54369) = 6.8406.

Finally, to compute the Lagrange Multiplier statistic, we regress the residuals from the log‑linear regression on a constant, ln*K*, ln*L*, and (1/2)(*b*kln2*K* + *bl*ln2*L*) where the coefficients are those from the log‑linear model (.27898 and .92731). The *R*2 in this regression is .23001, so the Lagrange multiplier statistic is *LM* = *nR*2 = 25(.23001) = 5.7503. All three statistics suggest the same conclusion, the hypothesis should be rejected.

4. Instead of minimizing the sum of squared deviations, we now maximize the concentrated log‑likelihood function, ln*L* = ‑(*n*/2)ln(1+ln(2π)) + (λ ‑ 1)Σ*i* ln*Yi* ‑ (*n*/2)ln(ε**′**ε/*n*).

The search for the maximum of ln*L* produced the results on the next page

The log‑likelihood is maximized at λ = .124. At this value, the regression results are as follows:

**Parameter Estimate OLS Std.Error Correct Std.Error**

α 2.59465 .1283 .7151

βk .378094 .1070 .3228

βl 1.13653 .1117 .4121

λ .124 ‑‑‑‑ .2482

σ2 .036922 ‑‑‑‑ .0179

**Estimated Asymptotic Covariance Matrix**

α βk βl λ σ2

α .5114

βk .2203 .1042

βl .2612 .0951 .1698

λ .1747 .0730 .0953 .0617

σ2 .0104 .0044 .0059 .0038 .00032



2

λ ln*L*

‑.200 ‑13.6284

‑.150 ‑12.8568

‑.100 ‑12.2423

‑.050 ‑11.7764

0.000 ‑11.4476

.050 ‑11.2427

.100 ‑11.1480

.110 ‑11.1410

.120 ‑11.1378

.121 ‑11.1377

.122 ‑11.1376

.123 ‑11.1376

.124 ‑11.1375

.125 ‑11.1376

.130 ‑11.1383

.140 ‑11.1423

.200 ‑11.2344

.300 ‑11.6064

.400 ‑12.8371

The output elasticities for this function evaluated at the sample means, = .175905, = .737988, = 2.870777, are ∂ln*Y*/∂ln*K* = *bk*(*K*/*Y*)λ = .2674

∂ln*Y*/∂ln*L* = *bl*(*L*/*Y*)λ = .9017.

These are quite similar to the estimates given above. The sum of the two output elasticities for the states given in the example in the text are given below for the model estimated with and without transforming the dependent variable. Note that the first of these makes the model look much more similar to the Cobb Douglas model for which this sum is constant.

**State Full Box‑Cox Model lnQ on left hand side**

Florida 1.2840 1.6598

Louisiana 1.2019 1.4239

California 1.1574 1.1176

Maryland 1.1657 1.0261

Ohio 1.1899 .9080

Michigan 1.1604 .8506

Once again, we are interested in testing the hypothesis that λ = 0. The Wald test statistic is

*W* = (.123 / .2482)2 = .2455. We would now not reject the hypothesis that λ = 0. This is a surprising outcome. The likelihood ratio statistic is based on both models. The sum of squared residuals for the restricted model is given above. The sum of the logs of the outputs is 19.29336, so the restricted log‑likelihood is ln*L*0 = (0‑1)(19.29336) ‑ (25/2)[1 + ln(2π) + ln(.781403/25)] = ‑11.44757. The likelihood ratio statistic is ‑2[ ‑11.13758 ‑ (‑11.44757)] = .61998. Once again, the statistic is small. Finally, to compute the Lagrange multiplier statistic, we now use the method described in Example 7.4. The result is *LM* = 1.5621. All of these suggest that the log‑linear model is not a significant restriction on the Box‑Cox model. This rather peculiar outcome would appear to arise because of the rather substantial reduction in the log‑likelihood function which occurs when the dependent variable is transformed along with the right hand side. This is not a contradiction because the model with only the right hand side transformed is not a parametric restriction on the model with both sides transformed. Some further evidence is given in the next exercise.

4. This application actually does not work very well at all.

**Sample ; 1 - 123 $**

**create ; wage=pl/1000$ <== the wage variable needed to be scaled for NLSQ.**

**Create ; k = cost\*sk/pk ; m = cost\*sl/wage $**

**Create ; logk=log(k) ; logm = log(m) ; logkm=.5\*(logk-logm)^2 $**

**Create ; logq = log(q) $**

**Regress; lhs = logq ; Rhs = one,logk,logm,logkm ; Printvc $**

**Wald ; Start = b ; var = varb ; labels = b1,b2,b3,b4**

**; fn1 = gma = exp(b1)**

**; fn2 = delta = b2/(b2+b3)**

**; fn3 = nu = b2+b3**

**; fn4 = ro = b4\*(b2+b3)/(b2\*b3) $**

-----------------------------------------------------------------------------

Ordinary least squares regression ............

LHS=LOGQ Mean = 8.17947

Standard deviation = 1.82990

No. of observations = 123 Degrees of freedom

Regression Sum of Squares = 388.880 3

Residual Sum of Squares = 19.6402 119

Total Sum of Squares = 408.520 122

Standard error of e = .40626

Fit R-squared = .95192 R-bar squared = .95071

Model test F[ 3, 119] = 785.40559 Prob F > F\* = .00000

Diagnostic Log likelihood = -61.70131 Akaike I.C. = -1.76956

Restricted (b=0) = -248.35133

Chi squared [ 3] = 373.30005 Prob C2 > C2\* = .00000

--------+--------------------------------------------------------------------

| Standard Prob. 95% Confidence

LOGQ| Coefficient Error t |t|>T\* Interval

--------+--------------------------------------------------------------------

Constant| 11.4637\*\*\* .38097 30.09 .0000 10.7170 12.2104

LOGK| 1.33526\*\*\* .40682 3.28 .0014 .53790 2.13262

LOGM| -.15835 .39922 -.40 .6923 -.94080 .62410

LOGKM| .16752 .21802 .77 .4438 -.25978 .59483

--------+--------------------------------------------------------------------

Note: \*\*\*, \*\*, \* ==> Significance at 1%, 5%, 10% level.

-----------------------------------------------------------------------------

Cov.[b^]| ONE LOGK LOGM LOGKM

--------+--------------------------------------------------------

ONE| .145138 .151682 -.148244 .0765554

LOGK| .151682 .165506 -.162025 .0866987

LOGM| -.148244 -.162025 .159375 -.0845120

LOGKM| .0765554 .0866987 -.0845120 .0475311

-----------------------------------------------------------------------------

WALD procedure. Estimates and standard errors

for nonlinear functions and joint test of

nonlinear restrictions.

Wald Statistic = 4241.29434

Prob. from Chi-squared[ 4] = .00000

Functions are computed at means of variables

--------+--------------------------------------------------------------------

| Standard Prob. 95% Confidence

WaldFcns| Coefficient Error z |z|>Z\* Interval

--------+--------------------------------------------------------------------

GMA| 95197.9\*\*\* 36267.54 2.62 .0087 24114.9 166281.0

DELTA| 1.13455\*\*\* .33847 3.35 .0008 .47117 1.79793

NU| 1.17691\*\*\* .02883 40.83 .0000 1.12041 1.23341

RO| -.93247 1.47972 -.63 .5286 -3.83268 1.96773

--------+--------------------------------------------------------------------

Note: \*\*\*, \*\*, \* ==> Significance at 1%, 5%, 10% level.

-----------------------------------------------------------------------------

calc ; c10=log(waldfns(1)) ? log gamma

; c20=waldfns(2) ? delta

; c30=waldfns(3) ? nu

; c40=waldfns(4) $ ? rho

nlsq ; lhs = logq

; labels = c1,c2,c3,c4

; fcn = c1 - (c3/c4)\*log( c2/(k^c4) + (1-c2)/(m^c4))

; start = c10,.7,c30,c40 ? <== The estimate of delta was outside (0,1)

; output = 3;pri$

Itr 32 F= .9529D+01 gtHg= .1944D-01 chg.F= .2121D-02 max|db|= .5100D-02

1st derivs. .55004D-02 .20087D+02 -.20581D-01 -.90620D-03

Parameters: .11445D+02 .99989D+00 .11866D+01 -.33864D+01

Itr 33 F= .9528D+01 gtHg= .1164D-01 chg.F= .7744D-03 max|db|= .4248D-02

Itr 33 F= .9528D+01 gtHg= .2009D+02 chg.F= .1247D-06 max|db|= .2009D+02

Line search at iteration 33 does not improve fn. Exiting optimization.

Function= .78300061247D+02, at entry, .95279044765D+01 at exit

-----------------------------------------------------------------------------

User Defined Optimization.........................

Nonlinear least squares regression ............

LHS=LOGQ Mean = 8.17947

Standard deviation = 1.82990

Number of observs. = 123

Model size Parameters = 4

Degrees of freedom = 119

Residuals Sum of squares = 19.0558

Standard error of e = .39361

Fit R-squared = .95335

Adjusted R-squared = .95373

Model test F[ 3, 119] (prob) = 810.7(.0000)

Diagnostic Log likelihood = -59.84348

Restricted(b=0) = -248.35133

Chi-sq [ 3] (prob) = 377.0( .0000)

Info criter. Akaike Info. Criter. = -1.79977

Not using OLS or no constant. Rsqrd & F may be < 0

Model was estimated on Mar 29, 2011 at 06:45:03 PM

--------+--------------------------------------------------------------------

| Standard Prob. 95% Confidence

UserFunc| Coefficient Error z |z|>Z\* Interval

--------+--------------------------------------------------------------------

C1| 11.4447\*\*\* .08192 139.71 .0000 11.2842 11.6053

C2| .99989\*\*\* .00087 1154.83 .0000 .99819 1.00159

C3| 1.18658\*\*\* .02844 41.73 .0000 1.13085 1.24232

C4| -3.38636 4.22912 -.80 .4233 -11.67529 4.90256

--------+--------------------------------------------------------------------

Note: \*\*\*, \*\*, \* ==> Significance at 1%, 5%, 10% level.

-----------------------------------------------------------------------------

Cov.[b^]| C1 C2 C3 C4

--------+--------------------------------------------------------

C1| .00671050 .216691E-04 .00200283 -.114085

C2| .216691E-04 .749667E-06 .315138E-05 -.00364865

C3| .00200283 .315138E-05 .808618E-03 -.0208124

C4| -.114085 -.00364865 -.0208124 17.8855

5. Using the gasoline market data in Appendix Table F2.2, use the partially linear regression method in Section 16.3.3 to fit an equation of the form

ln(*G*/*Pop*) = *β*1ln(*Income*) + *β*2ln*Pnew cars* + *β*3ln*Pused cars* + *g*(ln*Pgasoline*) + *ε*

crea;gp=lg;ip=ly;ncp=lpnc;upp=lpuc;pgp=lpg$

sort;lhs=pgp;rhs=gp,ip,ncp,upp$

crea;dgp=.809\*gp - .5\*gp[-1] - .309\*gp[-2]$

crea;dip=.809\*ip - .5\*ip[-1] - .309\*ip[-2]$

crea;dnc=.809\*ncp -.5\*ncp[-1]-.309\*ncp[-2]$

crea;duc=.809\*upp -.5\*upp[-1]-.309\*upp[-2]$

samp;3-36$

regr;lhs=dgp;rhs=dip,dnc,duc;res=e$

+-----------------------------------------------------------------------+

| Ordinary least squares regression Weighting variable = none |

| Dep. var. = DGP Mean= .9708646870E-02, S.D.= .4738748109E-01 |

| Model size: Observations = 34, Parameters = 3, Deg.Fr.= 31 |

| Residuals: Sum of squares= .1485994289E-01, Std.Dev.= .02189 |

| Fit: R-squared= .799472, Adjusted R-squared = .78653 |

| Model test: F[ 2, 31] = 61.80, Prob value = .00000 |

| Diagnostic: Log-L = 83.2587, Restricted(b=0) Log-L = 55.9431 |

| LogAmemiyaPrCrt.= -7.559, Akaike Info. Crt.= -4.721 |

| Model does not contain ONE. R-squared and F can be negative! |

| Autocorrel: Durbin-Watson Statistic = 1.34659, Rho = .32671 |

+-----------------------------------------------------------------------+

+---------+--------------+----------------+--------+---------+----------+

|Variable | Coefficient | Standard Error |t-ratio |P[|T|>t] | Mean of X|

+---------+--------------+----------------+--------+---------+----------+

DIP .9629902959 .11631885 8.279 .0000 .14504254E-01

DNC -.1010972781 .87755182E-01 -1.152 .2581 .20153536E-01

DUC -.3197058148E-01 .51875022E-01 -.616 .5422 .35656776E-01

--> matr;varpl={1+1/(2\*2)}\*varb$

--> matr;stat(b,varpl)$

+---------------------------------------------------+

|Number of observations in current sample = 34 |

|Number of parameters computed here = 3 |

|Number of degrees of freedom = 31 |

+---------------------------------------------------+

+---------+--------------+----------------+--------+---------+

|Variable | Coefficient | Standard Error |b/St.Er.|P[|Z|>z] |

+---------+--------------+----------------+--------+---------+

B\_1 .9629902959 .13004843 7.405 .0000

B\_2 -.1010972781 .98113277E-01 -1.030 .3028

B\_3 -.3197058148E-01 .57998037E-01 -.551 .5815

6.

+---------------------------------------+

| Nonparametric Regression for G |

| Observations = 36 |

| Points plotted = 36 |

| Bandwidth = .468092 |

| Statistics for abscissa values---- |

| Mean = 2.316611 |

| Standard Deviation = 1.251735 |

| Minimum = .914000 |

| Maximum = 4.109000 |

| ---------------------------------- |

| Kernel Function = Logistic |

| Cross val. M.S.E. = 121.084982 |

| Results matrix = KERNEL |

+---------------------------------------+



Chapter 8

Endogeneity and Instrumental Variable Estimation

**Exercises**

1. There is no need for a separate proof different from the usual for OLS. Formally, however, it follows from the results at (8-4) that

**b** = 

Then,



and



The large sample distribution of this statistic will be the same as the large sample of the statistic with X′X/n replaced with its probablity limit, which is **QXX**. Thus,



To deduce the large sample behavior of this statistic, we can invoke the results from chapter 4. The only change here is the nonzero mean (probability limit) of the vector in brackets. [See (8-3).] Thus, the same proof applies. The consistency, asymptotic normality and asymptotic covariance matrix equal to Asy.Var[**b**] = σε2 (**X′X**)-1

2. A logical solution to this one is simple. For y and x\*,

Cov2(y,x\*)/[Var(y)Var(x\*)] = β2(σ\*2)2/[(β2σ\*2+σε2)(σ\*2)]

Cov2(y,x) /[Var(y)Var(x)] = Cov[βx\*+ε,x\*+u] / [Var(y)Var(x)]

= {Cov[y,x\*] +Cov[y,u]}2 / [Var(y)Var(x)] .

The second term is zero, since y=βx\*+ε which is uncorrelated with u. Thus,

Cov2(y,x) /[Var(y)Var(x)] = Cov[y,x\*] / [Var(y)Var(x)].

The numerator is the same. The denominator is larger, since [Var(y)Var(x)] = Var[y](Var[x\*] + Var[u]),

so the squared correlation must be smaller. If both variables are measured with errors, then we are comparing Cov2(y\*,x\*)/{Var[y\*]Var[x\*]} to Cov2(y,x)/{Var[y]Var[x]}.

The numerator is the covariance of (βx\* + ε + v) with (x\* + u), so the numerator of the fraction is still β2(σ\*2)2. The denominator is still obviously larger, so the same result holds when both variables are measured with error.

3. We work off (8-19), using repeatedly the result **Σ***uu* = (σ*u***j**)(σ*u***j**)′ where **j** has a 1 in the first position and 0 in the remaining K-1. From (8-19),

plim **b** = **β** - [**Q**\* + **Σ***uu*]-1**Σ***uu***β**. The vector is **Σ***uu***β** equals [σ*u*2β1,0,...,0]′. The inverse matrix is

[**Q**\* + **Σ***uu*]-1 = 

This can be simplified since the quadratic form in the denominator just picks off the 1,1 diagonal element. Thus,

[**Q**\* + **Σ**uu]-1 = 

Then

[**Q**\* + **Σ**uu]-1**Σ**uu**β**= β

= **β** - β

= **j** σu2β1 - 

= **j **σu2β1

=**j **σu2β1

=**j **

Finally, **j** equals the first column of  = [q\*11, q\*21,...,q\*k1]. Therefore, the first element, given by (8-20a) is

plim b1 = β1 - ****q\*11 = β1****

For (8-20b),

plim b2 = β2 - ****q\*k1

4. To obtain the result, note first:

plim **b** = **β** + **QXX**-1**γ**

Asy.Var[**b**] = (σ2/n)**QXX**-1

Asy.Var[**b**2sls] = (σ2/n)**QZX**-1**QZZQXZ**-1.

The mean squared error of the OLS estimator is the variance plus the squared bias,

M(**b**|**β**) = (σ2/*n*)**QXX**-1 + **QXX**-1**γγ′QXX**-1

the mean squared error of the 2SLS estimator equals its variance. For OLS to be more precise then 2SLS, we would have to have

(σ2/*n*)**QXX**-1 + **QXX**-1**γγ′QXX**-1 << (σ2/n)**QZX**-1**QZZQXZ**-1.

For convenience, let **δ** = **QXX**-1**γ** so M(b|β) = (σ2/n)**QXX**-1 + **δδ′**. If the mean squared error matrix of the OLS estimator is smaller than that of the 2SLS estimator, then its inverse is larger. Use (A-66) to do the inversion. The result would be

[(σ2/*n*)**QXX**-1 + **δδ′**]-1 >> [(σ2/n)**QZX**-1**QZZQXZ**-1]-1

Now, use A-66

[(σ2/*n*)**QXX**-1 + **δδ′**]-1 = (*n*/σ2) **QXX** - (*n*/σ2) **QXXδδ′**(*n*/σ2) **QXX**

Reinsert **δ** = **QXX**-1**γ** and the right hand side above reduces to

(*n*/σ2) **QXX** - (*n*/σ2)2 **γγ′**

Therefore, if the mean squared error matrix of OLS is smaller, then

(*n*/σ2) **QXX** - (*n*/σ2)2 **γγ′ >>** (n/σ2)**QXZQZZ-1QZX**

Collect the terms, and this implies

(*n*/σ2)[ **QXX** - **QXZQZZ-1QZX**] >> (*n*/σ2)2 **γγ′**

divide both sides by (*n*/σ2),

**QXX** - **QXZQZZ-1QZX** >> **γγ′**

and divide numerator and denominator of the fraction by n/σ2

**QXX** - **QXZQZZ-1QZX** >> **γγ′**

which is the desired result. Is it possible? It is possible, since

**QXX** - **QXZQZZ-1QZX =** plim (1/n)[**X′X** - **X′Z**(**Z′Z**)-1**Z′X**]

= plim (1/n) X**′MZX**

which is a positive definite matrix. SInce **γ** varies independently of **Z** and **X**, certainly there is some configuration of the data and parameters for which this is the case. The result is that it is, indeed, possible for OLS to be more precise, in the mean squared error sense, than 2SLS.

5. The matrices are **X** = [**i**,**x**] and **Z** = [**i**,**z**]. For the OLS estimators, we know from chapter 2 that

*a* =  and *b* = Cov[*x*,*y*]/Var[*x*].

For the IV estimator, (**Z′X**)-1**Z′y**, we obtain the result in detail. Given the forms,



where subscript 1 indicates the mean of the observations for which z equals 1, and n1 is the number of observations. Multiplying the matrix times the vector and cancelling terms produces the solutions



6. To obtain the asymptotic distribution, write the result already in hand as **b** = (**β** + **Q**-1**γ**) + (**X′X**)-1**X′ε** - **Q**-1**ε.** We have established that plim **b** = **β** + **Q**-1**γ**. For convenience, let **θ** ≠ **β** denote **β** + **Q**-1**γ** = plim **b**. Write the preceding in the form **b** - **θ** = (**X′X**/*n*)-1(**X′ε**/*n*) - **Q**-1**γ**. Since plim(**X′X**/*n*) = **Q**, the large sample behavior of the right hand side is the same as that of plim (**b** - **θ**) = **Q**-1plim(**X′ε**/*n*) - **Q**-1**γ**. That is, we may replace (**X′X**/*n*) with **Q** in our derivation. Then, we seek the asymptotic distribution of (**b** - **θ**) which is the same as that of

[**Q**-1plim(**X′ε**/*n*) - **Q**-1**γ**] = **Q**-1. From this point, the derivation is exactly the same as that when **γ** = **0**, so there is no need to redevelop the result. We may proceed directly to the same asymptotic distribution we obtained before. The only difference is that the least squares estimator estimates **θ**, not **β**.

a. The statement of the problem is actually a bit optimistic. GIven the way it is stated, it would imply that the exogenous variables in the “demand” equation would be, in principle, (Ed, Union, Fem) which are also in the supply equation, plus the remainder, (Exp, Exp2, Occ, Ind, South, SMSA, Blk). The problem is that the model as stated would not be identified – the supply equation would, but the demand equation would not be. The way out would be to assume that at least one of (Ed, Union, Fem) does not appear in the demand equation. Since surely education would, that leaves one or both of Union and Fem. We will assume both of them are omitted. So, our equation is

lnWageit = α1 + α2Edit + α3Expit + α4Expit2 + α5Occit +

α6Indit + α7Southit + α8SMSAit + α9Blkit + γ Wksit + uit.

NAMELIST ; X = one,Ed,Exp,Expsq,Occ,Ind,South,SMSA,Blk,Wks $

NAMELIST ; Z = one,Ed,Exp,expsq,Occ,Ind,south,SMSA,Blk,Union,Fem $

Regress ; Lhs = lwage ; Rhs = X $

2SLS ; Lhs = lwage ; Rhs = X ; Inst = Z $

REGRESS ; Lhs = Wks ; Rhs = Z ; cls:b(10)=0,b(11)=0$

+----------------------------------------------------+

| Ordinary least squares regression |

| LHS=LWAGE Mean = 6.676346 |

| Standard deviation = .4615122 |

| WTS=none Number of observs. = 4165 |

| Model size Parameters = 10 |

| Degrees of freedom = 4155 |

| Residuals Sum of squares = 581.2717 |

| Standard error of e = .3740280 |

| Fit R-squared = .3446066 |

| Adjusted R-squared = .3431870 |

| Model test F[ 9, 4155] (prob) = 242.74 (.0000) |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

Constant| 5.13171052 .07238152 70.898 .0000

ED | .06112766 .00277226 22.050 .0000 12.8453782

EXP | .04291665 .00229783 18.677 .0000 19.8537815

EXPSQ | -.00070803 .506204D-04 -13.987 .0000 514.405042

OCC | -.07814434 .01502100 -5.202 .0000 .51116447

IND | .09066812 .01247863 7.266 .0000 .39543818

SOUTH | -.07629062 .01318346 -5.787 .0000 .29027611

SMSA | .13789225 .01278553 10.785 .0000 .65378151

BLK | -.26269494 .02304380 -11.400 .0000 .07226891

WKS | .00484184 .00113470 4.267 .0000 46.8115246

+----------------------------------------------------+

| Two stage least squares regression |

| LHS=LWAGE Mean = 6.676346 |

| Standard deviation = .4615122 |

| WTS=none Number of observs. = 4165 |

| Model size Parameters = 10 |

| Degrees of freedom = 4155 |

| Residuals Sum of squares = 602.3138 |

| Standard error of e = .3807377 |

| Fit R-squared = .3192467 |

| Adjusted R-squared = .3177722 |

| Model test F[ 9, 4155] (prob) = 216.50 (.0000) |

+----------------------------------------------------+

| Instrumental Variables:

|ONE ED EXP EXPSQ OCC IND SOUTH SMSA

|BLK UNION FEM

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

Constant| 4.46105888 .27680953 16.116 .0000

ED | .06167266 .00283031 21.790 .0000 12.8453782

EXP | .04207640 .00236282 17.808 .0000 19.8537815

EXPSQ | -.00068241 .525268D-04 -12.992 .0000 514.405042

OCC | -.07605669 .01531301 -4.967 .0000 .51116447

IND | .08348143 .01302032 6.412 .0000 .39543818

SOUTH | -.08242895 .01364036 -6.043 .0000 .29027611

SMSA | .13244624 .01319402 10.038 .0000 .65378151

BLK | -.25212290 .02383132 -10.579 .0000 .07226891

WKS | .01922950 .00583960 3.293 .0010 46.8115246

This is the test of relevance of the instrumental variables. In the regression of WKS on the full set of exogenous variables, we test the hypothesis that the coefficients on the instruments, UNION and FEM are jointly zero. The results show that the hypothesis is rejected. We conclude that the instruments are relevant.

+----------------------------------------------------+

| Linearly restricted regression |

| Ordinary least squares regression |

| LHS=WKS Mean = 46.81152 |

| Standard deviation = 5.129098 |

| WTS=none Number of observs. = 4165 |

| Model size Parameters = 9 |

| Degrees of freedom = 4156 |

| Residuals Sum of squares = 108653.5 |

| Standard error of e = 5.113097 |

| Fit R-squared = .8138966E-02 |

| Adjusted R-squared = .6229705E-02 |

| Model test F[ 8, 4156] (prob) = 4.26 (.0000) |

| Restrictns. F[ 2, 4154] (prob) = 84.57 (.0000) |

| Not using OLS or no constant. Rsqd & F may be < 0. |

| Note, with restrictions imposed, Rsqd may be < 0. |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

Constant| 46.6129896 .67547781 69.007 .0000

ED | -.03787988 .03789322 -1.000 .3175 12.8453782

EXP | .05840099 .03139904 1.860 .0629 19.8537815

EXPSQ | -.00178055 .00069145 -2.575 .0100 514.405042

OCC | -.14509978 .20533021 -.707 .4798 .51116447

IND | .49950389 .17041135 2.931 .0034 .39543818

SOUTH | .42663864 .18010107 2.369 .0178 .29027611

SMSA | .37851979 .17468415 2.167 .0302 .65378151

BLK | -.73479892 .31481083 -2.334 .0196 .07226891

UNION | .444089D-15 .182255D-08 .000 1.0000 .36398559

FEM | .000000 ......(Fixed Parameter).......

Chapter 9

The Generalized Regression Model and Heteroscedasticity

**Exercises**

1. Write the two estimators as = **β** + (**X′Ω**-1**X**)-1**X′Ω**-1**ε** and **b** = **β** + (**X′X**)-1**X′ε**. Then,

(- **b**) = [(**X′Ω**-1**X**)-1**X′Ω**-1 ‑ (**X′X**)-1**X′**]**ε** has *E*[‑ **b**] = **0** since both estimators are unbiased. Therefore,

Cov[,‑ **b**] = *E*[(- **β**)(- **b**)**′**].

Then,

*E*{(**X′Ω**-1**X**)-1**X′Ω**-1**εε′**[(**X′Ω**-1**X**)-1**X′Ω-**1 ‑ (**X′X**)-1**X′**]**′**}

= (**X′Ω**-1**X**)-1**X′Ω**-1(σ2**Ω**)[**Ω**-1**X**(**X′Ω-**1**X**)-1 ‑ **X**(**X′X**)-1]

= σ2(**X′Ω**-1**X**)-1**X′Ω**-1**ΩΩ**-1**X**(**X′Ω**-1**X**)-1 ‑ (**X′Ω**-1**X**)-1**X′Ω**-1**ΩX**(**X′X**)-1

= (**X′Ω**-1**X**)-1(**X′Ω**-1**X**)(**X′Ω**-1**X**)-1 ‑ (**X′Ω**-1**X**)-1(**X′X**)(**X′X**)-1 = **0**

once the inverse matrices are multiplied.

2 First, (**R** ‑ **q**) = **R**[**β** + (**X′Ω**-1**X**)-1**X′Ω**-1**ε**)] ‑ **q** = **R**(**X′Ω**-1**X**)-1**X′Ω**-1**ε** if **Rβ** ‑ **q** = **0**.

Now, use the inverse square root matrix of **Ω**, **P** = **Ω**-1/2 to obtain the transformed data,

**X**\* = **PX** = **Ω**-1/2**X**, **y**\* = **Py** = **Ω**-1/2**y**, and **ε**\* = **Pε** = **Ω**-1/2**ε**.

Then, *E*[**ε\*ε\*′**] = *E*[**Ω**-1/2**εε′Ω**-2] = **Ω**-1/2(σ2**Ω**)**Ω**-1/2 = σ2**I**,

and, = (**X′Ω**-1**X**)-1**X′Ω**-1**y** = (**X**\***′X**\*)-1**X**\***′y**\*

= the OLS estimator in the regression of **y**\* on **X**\*.

Then, **R** ‑ **q** = **R**(**X**\***′X**\*)-1**X**\***′ε**\*

and the numerator is **ε\*′X**\*(**X**\***′X**\*)-1**R′**[**R**(**X**\***′X**\*)-1**R′**]-1**R**(**X**\***′X**\*)-1**X**\***′ε**\* / *J*. By multiplying it out, we find that the matrix of the quadratic form above is idempotent. Therefore, this is an idempotent quadratic form in a normally distributed random vector. Thus, its distribution is that of σ2 times a chi‑squared variable with degrees of freedom equal to the rank of the matrix. To find the rank of the matrix of the quadratic form, we can find its trace. That is

tr{**X**\*(**X**\***′X**\*)-1**R′**[**R**(**X**\***′X**\*)-1**R′**]-1**R**(**X**\***′X**\*)-1**X**\*}

= tr{(**X**\***′X**\*)-1**R′**[**R**(**X**\***′X**\*)-1**R′**]-1**R**(**X**\***′X**\*)-1**X**\***′X**\*}

= tr{(**X**\***′X**\*)-1**R′**[**R**(**X**\***′X**\*)-1**R′**]-1**R**}

= tr{[**R**(**X**\***′X**\*)-1**R′**][**R**(**X**\***′X**\*)-1**R′**]-1} = tr{**I***J*} = *J*,

which might have been expected. Before proceeding, we should note, we could have deduced this outcome from the form of the matrix. The matrix of the quadratic form is of the form **Q** = **X**\***ABA′X**\***′** where **B** is the nonsingular matrix in the square brackets and **A** = (**X**\***′X**\*)-1**R′**, which is a *K*×*J* matrix which cannot have rank higher than *J*. Therefore, the entire product cannot have rank higher than *J*. Continuing, we now find that the numerator (apart from the scale factor, σ2) is the ratio of a chi‑squared[*J*] variable to its degrees of freedom.

We now turn to the denominator. By multiplying it out, we find that the denominator is

(**y**\* ‑ **X**\*)**′**(**y**\* ‑ **X**\*)/(*n* ‑ *K*). This is exactly the sum of squared residuals in the least squares regression of **y**\* on **X**\*. Since **y**\* = **X**\***β** + **ε**\* and = (**X**\***′X**\*)-1**X**\***′y**\* the denominator is **ε**\***′M**\***ε**\*/(*n* ‑ *K*), the familiar form of the sum of squares. Once again, this is an idempotent quadratic form in a normal vector (and, again, apart from the scale factor, σ2, which now cancels). The rank of the **M** matrix is *n* ‑ *K*, as always, so the denominator is also a chi‑squared variable divided by its degrees of freedom.

It remains only to show that the two chi‑squared variables are independent. We know they are if the two matrices are orthogonal. They are since **M**\***X**\* = **0**. This completes the proof, since all of the requirements for the *F* distribution have been shown.

3. First, we know that the denominator of the *F* statistic converges to σ2. Therefore, the limiting distribution of the *F* statistic is the same as the limiting distribution of the statistic which results when the denominator is replaced by σ2. It is useful to write this modified statistic as

*W* \* = (1/σ2)(**R** ‑ **q**)**′**[**R**(**X**\***′X**\*)-1**R′**]-1(**R**‑ **q**)/*J*.

Now, incorporate the results from the previous problem to write this as

*W* \* = **ε**\***′X**\*(**X**\***′X**\*)-1**R′**[**Rσ**2(**X**\***′X**\*)-1**R′**]-1**R**(**X**\***′X**\*)-1**X**\***′ε**/*J*

Let **ε**0 = **R**(**X**\***′X**\*)-1**X**\***′ε**\*.

Note that this is a *J*×1 vector. By multiplying it out, we find that E[**ε**0**ε**0**′**] = Var[**ε**0] = **R**{σ2(**X**\***′X**\*)-1}**R′**. Therefore, the modified statistic can be written as *W* \* = **ε**0**′**Var[ε0]-1**ε**0/*J*. This is the ‘full rank quadratic form’ discussed in Appendix B. For convenience, let **C** = Var[**ε**0], **T** = **C**-1/2, and **v** = **Tε**0. Then, *W* \* = **v′v**. By construction, **v** = Var[**ε**0]-1/2**ε**0, so *E*[**v**] = **0** and Var[**v**] = **I**. The limiting distribution of **v′v** is chi‑squared *J* if the limiting distribution of **v** is standard normal. All of the conditions for the central limit theorem apply to **v**, so we do have the result we need. This implies that as long as the data are well behaved, the numerator of the *F* statistic will converge to the ratio of a chi‑squared variable to its degrees of freedom.

4. The development is unchanged. As long as the limiting behavior of (1/*n*)**′** = (1/*n*)**X′-1X** is the same as that of (1/*n*)**X**\***′X**\*, the limiting distribution of the test statistic will be the same as if the true **Ω** were used instead of the estimate **.**

5. First, in order to simplify the algebra somewhat without losing any generality, we will scale the columns of **X** so that for each **x***k*, **x***k***′x**k = 1. We do this by beginning with our original data matrix, say, **X**0 and obtaining **X** as **X** = **X**0**D**-1/2, where **D** is a diagonal matrix with diagonal elements **D***kk* = **x***k*0**′x***k*0. By multiplying it out, we find that the GLS slopes based on **X** instead of **X**0 are

= [(**X**0**D**-1/2)**′Ω**-1(**X**0**D**-1/2)]-1[(**X**0**D**-1/2)**′Ω**-1**y**] = **D**1/2[**X′Ω**-1**X**](**D′**)1/2(**D′**)-1/2**X′Ω**-1**y** = **D**1/2 0

with variance Var[] = **D**1/2σ2[**X′Ω**-1**X**]-1(**D′**)1/2 = **D**1/2Var[0](**D′**)1/2. Likewise, the OLS estimator based on **X** instead of **X**0 is **b** = **D**1/2**b**0 and has variance Var[**b**] = **D**1/2Var[**b**0](**D′**)1/2. Since the scaling affects both estimators identically, we may ignore it and simply assume that **X′X** = **I**.

If each column of **X** is a characteristic vector of **Ω**, then, for the *k*th column, **x***k*, **Ωx***k* = λ*k***x***k*. Further, **x***k***′Ωx***k* = λ*k*  and **x***k***′Ωx***j* = 0 for any two different columns of **X**. (We neglect the scaling of **X**, so that **X′X** = **I**, which we would usually assume for a set of characteristic vectors. The implicit scaling of **X** is absorbed in the characteristic roots.) Recall that the characteristic vectors of **Ω**-1 are the same as those of **Ω** while the characteristic roots are the reciprocals. Therefore, **X′ΩX** = **Λ***K*, the diagonal matrix of the *K* characteristic roots which correspond to the columns of **X**. In addition, **X′Ω**-1**X** = **Λ***K*-1, so (**X′Ω**-1**X**)-1 = **Λ***K*, and**X′Ω**-1**y** =**Λ***K*-1**X′y**. Therefore, the GLS estimator is simply = **X′y** with variance Var[] = σ2**Λ***K*. The OLS estimator is **b** = (**X′X**)-1**X′y** = **X′y**. Its variance is Var[**b**] = σ2(**X′X**)-1**X′ΩX**(**X′X**)-1 = σ2**Λ***K*, which means that OLS and GLS are identical in this case.

6. Write **b** = **β** + (**X′X**)-1**X′ε** and = **β** + (**X′Ω**-1**X**)-1**X′Ω**-1**ε**. The covariance matrix is

E[(**b** ‑ **β**)( ‑ **β**)**′**] = *E*[(**X′X**)-1**X′εε′Ω**-1**X**(**X′Ω**-1**X**)-1] = (**X′X**)-1**X**′(σ2**Ω**)**Ω-**1**X**(**X′Ω**-1**X**)-1 = σ2(**X′Ω**-1**X**)-1.

For part (b), **e** = **Mε** as always, so *E*[**ee′**] = σ2**MΩM**. No further simplification is possible for the general case.

For part (c), = **y** ‑ **X** = **y** ‑ **X**[**β** + (**X′**Ω-1**X**)-1**X′**Ω-1**ε**]

= **Xβ** + ε ‑ **X**[β + (**X′**Ω-1**X**)-1**X′**Ω-1**ε**]

= [**I** ‑ **X**(**X′**Ω-1**X**)-1**X′**Ω-1]ε.

Thus, *E*[**′**] = [**I** ‑ **X**(**X′Ω**-1**X**)-1**X′Ω**-1]*E*[**εε′**][**I** ‑ **X**(**X′Ω**-1**X**)-1**X′Ω**-1] **′**

= [**I** ‑ **X**(**X′Ω**-1**X**)-1**X′Ω**-1](σ2**Ω**)[**I** ‑ **X**(**X′Ω**-1**X**)-1**X′Ω**-1] **′**

= [σ2**Ω** ‑ σ2**X**(**X′Ω**-1**X**)-1**X′**][**I** ‑ **X**(**X′Ω**-1**X**)-1**X′Ω**-1] **′**

= [σ2**Ω** ‑ σ2**X**(**X′Ω**-1**X**)-1**X′**][**I** ‑ **Ω**-1**X**(**X′Ω**-1**X**)-1**X′**]

= σ2**Ω**‑ σ2**X**(**X′Ω**-1**X**)-1**X′** ‑ σ2**X**(**X′Ω**-1**X**)-1**X′** + σ2**X**(**X′Ω**-1)**X**)-1**X′Ω**-1**X**(**X′Ω**-1**X**)-1**X′**

= σ2[**Ω** ‑ **X**(**X′Ω**-1**X**)-1**X′**]

The GLS residual vector appears in the preceding part. As always, the OLS residual vector is **e** = **Mε** =

[**I** ‑ **X**(**X′X**)-1**X**′]**ε**. The covariance matrix is

*E*[**e****′**] = *E*[(**I** ‑ **X**(**X′X**)-1**X′**)**εε′**(**I** ‑ **X**(**X′Ω**-1**X**)-1**X′Ω**-1)**′**]

= (**I** ‑ **X**(**X′X**)-1**X′**)(σ2**Ω**)(**I** ‑ Ω-1**X**(**X′Ω**-1**X**)-1**X′**)

= σ2**Ω** ‑ σ2**X**(**X′X**)-1**X′Ω** ‑ σ2**ΩΩ**-1**X**(**X′Ω**-1**X**)-1**X′** + σ2**X**(**X′X**)-1**X′ΩΩ**-1**X**(**X′Ω**-1**X**)-1**X′**

= σ2**Ω** ‑ σ2**X**(**X′X**)-1**X′**

= σ2**MΩ**.

7. The GLS estimator is = (**X′Ω**-1**X**)-1**X′**-1**y** = [Σ*i***x***i***x***i***′**/(**β′x***i*)2]-1[Σ*i***x***iyi*/(**β′x***i*)2]. The log‑likelihood for this model is ln*L* = ‑Σ*i*ln(**β′x***i*) ‑ Σ*iyi*/(**β′x***i*).

The likelihood equations are

∂ln*L*/∂**β** = ‑Σi(1/**β′x***i*)**x***i*  + Σ*i*[*yi*/(**β′x***i*)2]**x***i*  = **0**

or Σ*i*(**x***iyi*/(**β′x***i*)2) = Σ*i***x***i*/(**β′x** *i*).

Now, write Σ*i***x***i*/(**β′x***i*) = Σ*i***x***i***x***i***′β**/(**β′x***i*)2,

so the likelihood equations are equivalent to Σ*i*(**x***iyi*/(**β′x** *i*).2) = Σ*i***x***i***x***i***′β**/(**β′x** *i*).2, or **X′Ω**-1**y** = (**X′Ω**-1**X**)**β**. These are the normal equations for the GLS estimator, so the two estimators are the same. We should note, the solution is only implicit, since **Ω** is a function of **β**. For another more common application, see the discussion of the FIML estimator for simultaneous equations models in Chapter 13.

8.The covariance matrix is

.

The matrix **X** is a column of 1s, so the least squares estimator of *μ* is  Inserting this **Ω** into (10-5), we obtain  The limit of this expression is *ρσ*2, not zero. Although ordinary least squares is unbiased, it is not consistent. For this model, **X**′**ΩX**/*n* = 1 + *ρ*(*n* – 1), which does not converge. Using Theorem 8.2 instead, **X** is a column of 1s, so **X**′**X** = *n*, a scalar, which satisfies condition 1. To find the characteristic roots, multiply out the equation **Ωx** = λ**x** = (1-ρ)**Ix** + ρ**ii′x** = λ**x**. Since **i′x** = Σixi, consider any vector **x** whose elements sum to zero. If so, then it’s obvious that λ = ρ. There are *n*-1 such roots. Finally, suppose that **x** = **i**. Plugging this into the equation produces λ = 1 - ρ + *n*ρ. The characteristic roots of **Ω** are (1 – *ρ*) with multiplicity *n* – 1 and (1 – *ρ* + *nρ*), which violates condition 2.

9. This is a heteroscedastic regression model in which the matrix **X** is a column of ones. The efficient estimator is the GLS estimator, ****= (**X′Ω**-1**X**)-1**X′Ω**-1**y** = [Σ*i*1*yi*/*xi*2] / [Σ*i*12/***x****i*2] = [Σ*i*(*yi*/*xi*2)] / [Σ*i*(1/*xi*2)]. As always, the variance of the estimator is Var[****] = σ2(**X′Ω**-1**X**)-1 = σ2/[Σ*i*(1/*xi*2)]. The ordinary least squares estimator is (**X′X**)-1**X′y** =. The variance ofis σ2(**X′X**)-1(**X′ΩX**)(**X′X**)-1 = (σ2/*n*2)Σ*ixi*2. To show that the variance of the OLS estimator is greater than or equal to that of the GLS estimator, we must show that (σ2/*n*2)Σ*ixi*2 > σ2/Σ*i*(1/*xi*2) or (1/*n*2)(Σ*ixi*2)(Σi(1/*xi*2)) > 1 or Σ*i*Σ*j*(*xi*2/*xj*2) > *n*2. The double sum contains *n* terms equal to one. There remain *n*(*n*‑1)/2 pairs of the form (*xi*2/*xj*2 + *xj*2/*xi*2). If it can be shown that each of these sums is greater than or equal to 2, the result is proved. Just let *z*i = *x*i2. Then, we require *zi*/*zj* + *zj*/*zi* ‑ 2 > 0. But, this is equivalent to (*zi2* + *zj2* ‑ *2zizj*) / *zizj*  > 0 or (*zi* ‑ *zj*)2/*zizj* > 0, which is certainly true if zi and *zj* are positive. They are since *zi* equals *xi*2. This completes the proof.

10. Consider, first, . We saw earlier that Var[] = (σ2/*n*2)Σ*ixi*2 = (σ2/*n*)(1/*n*)Σ*ixi*2. The expected value is *E*[] = *E*[(1/*n*)Σ*iyi*] = α. If the mean square of *x* converges to something finite, then is consistent for α. That is, if plim(1/*n*)Σ*ixi*2 = whereis some finite number, then, plim = α. As such, it follows that *s*2 and *s*\*2 = (1/(*n*‑1))Σ*i*(*yi* ‑ α)2 have the same probability limit. We consider, therefore, plim *s*\*2 = plim(1/(*n*‑1))Σ*i*ε*i*2. The expected value of *s*\*2 is *E*[(1/(*n*‑1)) Σ*i*ε*i*2] = σ2(1/Σ*ixi*2). Once again, nothing more can be said without some assumption about *x*i. Thus, we assume again that the average square of *xi* converges to a finite, positive constant, . Of course, the result is unchanged by division by (*n*‑1) instead of *n*, so limn→∞ *E*[*s*\*2] = σ2. The variance of *s*\*2 is Var[*s*\*2] = Σ*i*Var[ε*i*2]/(*n* ‑ 1)2 . To characterize this, we will require the variances of the squared disturbances, which involves their fourth moments. But, if we assume that every fourth moment is finite, then the preceding is (*n*/(*n*‑1)2) times the average of these fourth moments. If every fourth moment is finite, then the term is dominated by the leading (*n*/(*n*‑1)2) which converges to zero. It follows that plim *s*\*2 = σ2. Therefore, the conventional estimator estimates Asy.Var[]= σ2/*n*.

The appropriate variance of the least squares estimator is Var[]= (σ2/*n*2)Σ*ixi*2, which is, of course, precisely what we have been analyzing above. It follows that the conventional estimator of the variance of the OLS estimator in this model is an appropriate estimator of the true variance of the least squares estimator. This follows from the fact that the regressor in the model, **i**, is unrelated to the source of heteroscedasticity, as discussed in the text.

11. The residuals from the least squares regression are *ei* = *yi* ‑= α + ε*i* ‑ (α +) = ε*i* ‑ . The expected value of the squared residual is

*E*[*ei*2] = *E*[ε*i*2] + *E*[2] ‑ 2*E*[ε*i*] = σi2 + (1/*n*2)*E*[(Σ*i*ε*i*)2] ‑ (2/*n*)*E*[ε*i*(Σ*j*ε*j*)]

= σ*i*2 + (1/*n*2)Σi*E*[ε*i*2] ‑ (2/*n*)*E*[ε*i*2]

since the disturbances are uncorrelated. We can write this as

*E*[*e*i2] = σ2 + σ2γ2*xi*2 + (1/*n*){[(1/*n*)Σ*i*σ*i*2] ‑ [2σ*i*2]}.

And, of course, *e*i2 = *E*[*ei*2] + (*ei*2 ‑ *E*[*ei*2]) = *E*[*ei*2] + *vi*, where *vi* is uncorrelated with *E*[*ei*2] by construction. Now, if we regress *ei*2 on a constant and *xi*2, the estimates of σ2 and (σ2γ2) will be biased in a finite sample because of the left out variable, namely the term multiplied by (1/*n*) in the expression for *E*[*ei*2]. But, if the two terms inside the curled brackets above converge to finite quantities as *n*→∞, then the entire term will vanish, and the omitted variable problem will vanish with it. Surely the second does since it is the variance of ε*i*2, assuming that *xi*2 is finite. To make the first converge, we will require that (1/*n*)Σ*i*[σ2 + σ2γ2*xi*2] = σ2 + σ2γ2(1/*n*)Σ*ixi*2 converge to a finite quantity, or that the mean square of the *x*s converge to a finite quantity. This is a minimal requirement for a heteroscedastic regression, and would surely be met. As such, if *ei*2 is regressed on a constant and *xi*2, we obtain consistent estimators of σ2 and σ2γ2. The estimator of σ is the square root of the ratio of the slope to the constant.

The estimator is not efficient. The expected fourth moment of a normally distributed variable is 3 times the square of the variance. Therefore, in the regression above, the variance of *vi* must be a function of

[σ2(1 + γ2*xi*2)]2. Since the regression is heteroscedastic in a way which is not dependent on the sample size, OLS will not be efficient, but it will be consistent.

12. The sample moments are obtained using, for example, Sxx = **x′x** ‑ *n*2 and so on. For the two samples, we obtain** **  *S*xx *S*yy *S*xy

**Sample 1** 6 6 300 300 200

**Sample 2** 6 6 300 1000 400

The parameter estimates are computed directly using the results of Chapter 6.

InterceptSlope*R*2 *s*2

**Sample 1** 2 2/3 4/9 (1500/9)/48 = 3.472

**Sample 2** -2 4/3 16/30 (4200/9)/48 = 9.722

The pooled moments based on 100 observations are **X′X = , X′y = , y′y =** 4900. The coefficient vector based on these data is [*a*,*b*] = [0,1]. This might have been predicted since the two **X′X** matrices are identical. OLS which ignores the heteroscedasticity would simply average the estimates. The sum of squared residuals would be **e′e** = **y′y** ‑ **b′X′y** = 4900 ‑ 4200 = 700, so the estimate of σ2 is *s*2 = 700/98 = 7.142. Note that the earlier values obtained were 3.472 and 9.722, so the pooled estimate is between the two, once again, as might be expected. The asymptotic covariance matrix of these estimates is *s*2(**X′X**)-1 = .

To test the equality of the variances, we can use the Goldfeld and Quandt test. Under the null hypothesis of equal variances, the ratio *F* = [**e**1**′e**1/(*n*1 ‑ 2)]/[**e**2**′e**2/(*n*2 ‑ 2)] (or vice versa for the subscripts) is the ratio of two independent chi‑squared variables each divided by their respective degrees of freedom. Although it might seem so from the discussion in the text (and the literature) there is nothing in the test which requires that the coefficient vectors be assumed equal across groups. Since for our data, the second sample has the larger residual variance, we refer *F*[48,48] = *s*22/*s*12 = 9.722 / 3.472 = 2.8 to the *F* table. The critical value for 95% significance is 1.61, so the hypothesis of equal variances is rejected.

The method of Example 12.9 can be applied to this groupwise heteroscedastic model. The two step estimator is = [(1/*s*12)**X**1**′X**1 + (1/*s*22)**X**2**′X**2]-1[(1/*s*12)**X**1**′y**1 + (1/*s*22)**X**2**′y**2]. The **X′X** matrices are the same in this problem, so this simplifies to = [(1/*s*12 + 1/*s*22)**X′X**]-1[(1/*s*12)**X**1**′y**1 + (1/*s*22)**X**2**′y**2] . The estimator is, therefore.

?=======================================================

? Application 9.1

?=======================================================

a. The ordinary least squares regression of *Y* on a constant, *X*1, and *X*2 produces the following results:

Sum of squared residuals 1911.9275

*R*2 .03790

Standard error of regression 6.3780

**Variable Coefficient Standard Error t‑ratio**

One .190394 .9144 .208

*X*1 1.13113 .9826 1.151

*X*2 .376825 .4399 .857

**b. Covariance Matrix White’s Corrected Matrix**

.836212 .524589

‑.115451 .96551 .076578 .282366

‑.047133 .051081 .193532 .399218 ‑.091608 1.14447

c. To apply White's test, we first obtain the residuals from the regression of *Y* on a constant, *X*1, and *X*2. Then, we regress the squares of these residuals on a constant, *X*1, *X*2, *X*12, *X*22, and *X*1*X*2. The *R*2 in this regression is .78296, so the chi‑squared statistic is 50×0.78296 = 39.148. The critical value from the table of chi‑squared with 5 degrees of freedom is 11.08, so we would conclude that there is evidence of heteroscedasticity.

d. Lagrange multiplier test.

Regress;Lhs=y;rhs=one,x1,x2 ; Res=e ; het $

create ; lmi=e\*e/(sumsqdev/n) - 1 $

Name ; x=one,x1,x2 $

Calc ; list ; .5\*xss(x,lmi)$

The result was reported with the regression,

| Br./Pagan LM Chi-sq [ 2] (prob) = 72.78 (.0000) |

e. Two step estimator

read;nobs=50;nvar=1;names=y;byva $

-1.42 2.75 2.10 -5.08 1.49 1.00 .16 -1.11 1.66

-.26 -4.87 5.94 2.21 -6.87 .90 1.61 2.11 -3.82

-.62 7.01 26.14 7.39 .79 1.93 1.97 -23.17 -2.52

-1.26 -.15 3.41 -5.45 1.31 1.52 2.04 3.00 6.31

5.51 -15.22 -1.47 -1.48 6.66 1.78 2.62 -5.16 -4.71

-.35 -.48 1.24 .69 1.91

read;nobs=50;nvar=1;names=x1;byva $

-1.65 1.48 .77 .67 .68 .23 -.40 -1.13 .15

-.63 .34 .35 .79 .77 -1.04 .28 .58 -.41

-1.78 1.25 .22 1.25 -.12 .66 1.06 -.66 -1.18

-.80 -1.32 .16 1.06 -.60 .79 .86 2.04 -.51

.02 .33 -1.99 .70 -.17 .33 .48 1.90 -.18

-.18 -1.62 .39 .17 1.02

read;nobs=50;nvar=1;names=x2;byva $

-.67 .70 .32 2.88 -.19 -1.28 -2.72 -.70 -1.55

-.74 -1.87 1.56 .37 -2.07 1.20 .26 -1.34 -2.10

.61 2.32 4.38 2.16 1.51 .30 -.17 7.82 -1.15

1.77 2.92 -1.94 2.09 1.50 -.46 .19 -.39 1.54

1.87 -3.45 -.88 -1.53 1.42 -2.70 1.77 -1.89 -1.85

2.01 1.26 -2.02 1.91 -2.23

Regress;Lhs=y;rhs=one,x1,x2 ; Res=e $

+----------------------------------------------------+

| Ordinary least squares regression |

| Model was estimated May 12, 2007 at 08:33:20PM |

| LHS=Y Mean = .3938000 |

| Standard deviation = 6.368374 |

| WTS=none Number of observs. = 50 |

| Model size Parameters = 3 |

| Degrees of freedom = 47 |

| Residuals Sum of squares = 1911.928 |

| Standard error of e = 6.378033 |

| Fit R-squared = .3790450E-01 |

| Adjusted R-squared = -.3035736E-02 |

| Model test F[ 2, 47] (prob) = .93 (.4033) |

| Diagnostic Log likelihood = -162.0430 |

| Restricted(b=0) = -163.0091 |

| Chi-sq [ 2] (prob) = 1.93 (.3806) |

| Info criter. LogAmemiya Prd. Crt. = 3.763988 |

| Akaike Info. Criter. = 3.763844 |

| Autocorrel Durbin-Watson Stat. = 1.8560359 |

| Rho = cor[e,e(-1)] = .0719820 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |t-ratio |P[|T|>t]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

Constant| .19039401 .91444640 .208 .8360

X1 | 1.13113339 .98260352 1.151 .2555 .10820000

X2 | .37682493 .43992218 .857 .3960 .21500000

Create ; e2 = e\*e $

Create ; loge2 = log(e2) $

Regress ; lhs = loge2 ; Rhs = one,x1,x2 ; keep=vi $

Create ; vi = 1/exp(vi) $

Regress ; Lhs = y ; rhs = one,x1,x2 ; wts = vi $

+----------------------------------------------------+

| Ordinary least squares regression |

| Model was estimated May 12, 2007 at 08:33:20PM |

| LHS=Y Mean = -.5316339 |

| Standard deviation = 4.535703 |

| WTS=VI Number of observs. = 50 |

| Model size Parameters = 3 |

| Degrees of freedom = 47 |

| Residuals Sum of squares = 890.9017 |

| Standard error of e = 4.353775 |

| Fit R-squared = .1162193 |

| Adjusted R-squared = .7861157E-01 |

| Model test F[ 2, 47] (prob) = 3.09 (.0548) |

| Diagnostic Log likelihood = -150.0732 |

| Restricted(b=0) = -153.1619 |

| Chi-sq [ 2] (prob) = 6.18 (.0456) |

| Info criter. LogAmemiya Prd. Crt. = 3.000355 |

| Akaike Info. Criter. = 3.285051 |

| Autocorrel Durbin-Watson Stat. = 1.9978648 |

| Rho = cor[e,e(-1)] = .0010676 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |t-ratio |P[|T|>t]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

Constant| .16662621 .71981411 .231 .8179

X1 | .77648745 .63883379 1.215 .2303 -.51884171

X2 | .84717700 .36328984 2.332 .0240 -.34867101

?=======================================================

Application 9.2

?=======================================================

The procedure is shown in Section 14.9.2.a.

create ; lc=log(c);lq=log(q);lq2=lq\*lq;lpf=log(pf)$

namelist ; z = one,lf $

regress ; lhs=lc;rhs=one,lq,lq2,lpf ; res = e $

calc ; s2 = e'e/n ; ls2 = log(s2)+1.2704 $

matrix ; d = [ls2/0] $

Calc ; change = 1 $

Proc

create ; wti=1/exp(d'z) $

regr ; quietly ; lhs = lc ; rhs = one,lq,lq2,lpf ; wts = wti ; res = e$

create ; e2=e^2/exp(d'z)- 1 $

regress ; quietly ; lhs = e2 ; rhs = one,lf $

matrix ; d = d + b $

calc ; list ; change = b'b $

Endproc $

Execute ; while change > .00001 $

regr ; lhs = lc ; rhs = one,lq,lq2,lpf ; wts = wti ; res = e$

matrix ; zz = 2\* <z'z> $

display ; parameters = d ; covariance = zz ; labels = z $

calc ; list ; sigmasq = exp(d(1)) $

**[CALC] CHANGE = 4.0934839**

**[CALC] CHANGE = 8.9498852**

**[CALC] CHANGE = 9.5669462**

**[CALC] CHANGE = 3.9298595**

**[CALC] CHANGE = 1.0081074**

**[CALC] CHANGE = .2295960**

**[CALC] CHANGE = .0503866**

**[CALC] CHANGE = .0109164**

**[CALC] CHANGE = .0023538**

**[CALC] CHANGE = .0005065**

**[CALC] CHANGE = .0001089**

**[CALC] CHANGE = .0000234**

**[CALC] CHANGE = .0000050**

**-> regr ; lhs = lc ; rhs = one,lq,lq2,lpf ; wts = wti ; res = e$**

**-----------------------------------------------------------------------------**

**Ordinary least squares regression ............**

**LHS=LC Mean = 12.99902**

**Standard deviation = 1.19008**

**WTS=WTI Number of observs. = 90**

**Model size Parameters = 4**

**Degrees of freedom = 86**

**Residuals Sum of squares = 1.28516**

**Standard error of e = .12224**

**Fit R-squared = .98980**

**Adjusted R-squared = .98945**

**Model test F[ 3, 86] (prob) = 2783.0(.0000)**

**Diagnostic Log likelihood = 57.31224**

**Restricted(b=0) = -149.04878**

**Chi-sq [ 3] (prob) = 412.7( .0000)**

**Info criter. Akaike Info. Criter. = -4.02259**

**Model was estimated on Mar 29, 2011 at 08:49:26 PM**

**--------+--------------------------------------------------------------------**

**| Standard Prob. 95% Confidence**

**LC| Coefficient Error t |t|>T\* Interval**

**--------+--------------------------------------------------------------------**

**Constant| 9.26108\*\*\* .21476 43.12 .0000 8.84016 9.68200**

**LQ| .91932\*\*\* .03304 27.83 .0000 .85456 .98407**

**LQ2| .02328\*\* .01124 2.07 .0413 .00125 .04531**

**LPF| .40267\*\*\* .01668 24.14 .0000 .36997 .43536**

**--------+--------------------------------------------------------------------**

**Note: \*\*\*, \*\*, \* ==> Significance at 1%, 5%, 10% level.**

**-----------------------------------------------------------------------------**

**--> matrix ; zz = 2\* <z'z> $**

**--> display ; parameters = d ; covariance = zz ; labels = z $**

**-----------------------------------------------------------------------------**

**User Specified Model**

**--------+--------------------------------------------------------------------**

**| Standard Prob. 95% Confidence**

**LC| Coefficient Error z |z|>Z\* Interval**

**--------+--------------------------------------------------------------------**

**Constant| -9.59229\*\*\* 1.59839 -6.00 .0000 -12.72508 -6.45951**

**LF| 9.77913\*\*\* 2.83949 3.44 .0006 4.21383 15.34443**

**--------+--------------------------------------------------------------------**

**Note: \*\*\*, \*\*, \* ==> Significance at 1%, 5%, 10% level.**

**-----------------------------------------------------------------------------**

**Chapter 10**

⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

**Systems of Regression Equations**

1. The model can be written as . Therefore, the OLS estimator is

*m* = (**i′i** + **i′i**)-1(**i′y**1 + **i′y**2) = (*n*+ *n*) / (*n* + *n*) = (**+**)/2 = 1.5.

The sampling variance would be Var[*m*] = (1/2)2{Var[**] + Var[**] + 2Cov[(**1, **)]}.

We would estimate the parts with Est.Var[**] = *s*11/*n* = ((150 ‑ 100(1)2)/99)/100 = .0051

Est.Var[**] = *s*22/*n* = ((550 ‑ 100(2)2)/99)/100 = .0152

Est.Cov[**, **] = *s*12/*n* = ((260 ‑ 100(1)(2))/99)/100 = .0061

Combining terms, Est.Var[*m*] = .0079.

The GLS estimator would be

[(σ11 + σ12)**i′y**1 + (σ22 + σ12)**i′y**2]/[(σ11 + σ12)**i′i** + (σ22 + σ12)**i′i**] = *w*+ (1‑*w*) **

where *w* = (σ11 + σ12) / (σ11 + σ22 + 2σ12). Denoting , **Σ**-1 = .

The weight simplifies a bit as the determinant appears in both the denominator and the numerator. Thus,

*w* = (σ22 ‑ σ12) / (σ11 + σ22 ‑ 2σ12). For our sample data, the two step estimator would be based on the variances computed above and *s*11 = .5051, *s*22 = 1.5152, *s*12 = .6061. Then, *w* = 1.1250. The FGLS estimate is 1.125(1) + (1 ‑ 1.125)(2) = .875. The sampling variance of this estimator is

*w*2Var[**] + (1 ‑ *w*)2Var[**] + 2*w*(1 ‑ *w*)Cov[**, **] = .0050 as compared to .0079 for the OLS estimator.

2. The model is **y** =  = **Xβ** + **ε** = , σ2**Ω** = .

The generalized least squares estimator is

 = 

= 

where  *s*xx = **x′x**/*n*, *s*x1 = **x′y**1/*n*, *s*x2 = **x′y**2/*n*

and σij = the *ij*th element of the 2×2 Σ-1.

To obtain the explicit form, note, first, that all terms σij are of the form σji/(σ11σ22 - σ212) But, the denominator in these ratios will be cancelled as it appears in both the inverse matrix and in the vector. Therefore, in terms of the original parameters, (after cancelling *n*), we obtain

 =  = .

The two elements are = [σ11*s*xx(σ22- σ12) - σ12 (σ12*s*x1 -σ11*s*x2)]/[σ11σ22*s*xx - (σ12)2]

= [σ12(σ22 - σ12) - σ22(σ12*s*x1 - σ11*s*x2)]/[σ11σ22*s*xx - (σ12)2]

The asymptotic covariance matrix is

[**X′Ω**-1**X**]-1 = 

The OLS estimator is **b** = (**X′X**)-1**X′y** = . The sampling variance is

(**X′X**)-1**X′ΩX**(**X′X**)-1 = . The *n*s are carried outside the product and reduce to (1/*n*). This leaves Var[**b**] = .

Using the results above, the OLS coefficients are *b*1 == 150/50 = 3 and b2 = **x′y**2/**x′x** = 50/100 = 1/2.

The estimators of the disturbance (co‑)variances are

*s*11 = Σ*i* (*yi1* -)2/*n* = (500 ‑ 50(3)2)/50 = 1

*s*22 = Σ*i* (*yi2* - *b*2*x*i)2/*n* = (90 ‑ (1/2)50)/50 = 1.3

*s*12 = Σ*i* (*yi1* -)(*yi2* - *b*2*x*i)2/*n* = [**y**1′**y**2 - *n*- *b*2**x′y**1 + *nb*2]/*n*

= (40 ‑ 50(3)(1) ‑ (1/2)60 + 50(1/2)(3)(2)/50 = .2

Therefore, we estimate the asymptotic covariance matrix of the OLS estimates as

Est.Var[**b**] = .

To compute the FGLS estimates, we use our results from part a. The necessary statistics for the computation are *s*11 = 1, *s*22 = 1.3,  *s*11 = .2, *s*xx = 100/50 = 2, = 100/50 = 2,

= 150/50 = 3, = 50/50 = 1

*s*x1 = 60/50 = 1.2, *s*x2 = 50/50 = 1

Then, = {1(2)[1.3(3) ‑ .2(1)] ‑ .2(2)[.2(1.2) ‑ 1(1)]}/{1(1.3) ‑ [.2(2)]2} = 3.157

= {2(2)[1.3(3) ‑ .2(1)] ‑ 1.3[.2(1.2) ‑ 1(1)]}/{1(1.3) ‑ [.2(2)]2} = 1.011

The estimate of the asymptotic covariance matrix is

(1/50)[1(1.3) ‑ (.2)2]/{1(1.3)2 ‑ [.2(2)]2}. Notice that the estimated variance of the FGLS estimator of the parameter of the first equation is larger. The result for the *true* GLS estimator based on known values of the disturbance variances and covariance does not guarantee that the *estimated* variances will be smaller in a finite sample. However, the estimated variance of the second parameter is considerably smaller than that for the OLS estimate.

Finally, to test the hypothesis that β2 = 1 we use the *z*‑statistic (asymptotically distributed as standard normal),  *z* = (1.011 ‑ 1) / (.007945)2 = .123. The hypothesis cannot be rejected.

3. The ordinary least squares estimates of the parameters are

*b*1 = **x**1**′y**1/**x**1**′x**1 = 4/5 = .8 and *b*2 = **x**2**′y**2/**x**2**′x**2 = 6/10 = .6

Then, the variances and covariance of the disturbances are

*s*11 = (**y**1**′y**1 ‑ *b*1**x**1**′y**1)/*n* = (20 ‑ .8(4))/20 = .84

*s*22 = (**y**2**′y**2 ‑ *b*2**x**2**′y**2)/*n* = (10 ‑ .6(6))/20 = .32

*s*12 = (**y**1**′y**2 ‑ *b*2**x**2**′y**1 - *b*1**x**1**′y**2 + *b*1*b*2**x**1**′x**2 )/*n* = (6 ‑ .6(3) ‑ .8(3) + .8(.6)(2))/20 = .246

We will require **S**-1 = . Then, the FGLS estimator is

. Inserting the values given in the problem produces the FGLS estimates, = .505335,  = .541741 with estimated asymptotic covariance matrix equal to the inverse matrix shown above, Est.Var=. To test the hypothesis, we use the *t* statistic, *t* = (.505335 ‑ .541741)/[.132565 + .0252505 ‑ 2(.0077645)]2 = ‑.0965 which is quite small. We would not reject the hypothesis.

To compute the maximum likelihood estimates, we would begin with the OLS estimates of σ11, σ22, and σ12. Then, we iterate between the following calculations

(1) Compute the 2×2 matrix, **S**-1

(2) Compute the 2×2 matrix [**X′**(**S**-1⊗**I**)**X**] =

[**X′**(**S**-1⊗**I**)**y**] =

(3) Compute the coefficient vector = [**X′**(**S**-1⊗**I**)**X**]-1[**X′**(**S**-1⊗**I**)**y**]

Compare this estimate to the previous one. If they are similar enough, exit the iterations.

(4) Recompute **S** using *s*ij = **y**i**′y**j - **x**i**′y**j - **x**j**′y**i + **x**i**′x**j, i,j = 1,2.

(5) Go back to step (1) and continue.

Our iterations produce the two slope estimates

1: .505335 .541741

2: .601889 .564998

3: .614884 .566875

4: .616559 .567186

5: .616775 .567227

6: .616803 .567232

7: .616807 .567232 converged.

At convergence, we find the estimate of the asymptotic covariance matrix of the estimates as

[**X**′(**S**-1⊗**I**)**X**]-1 =  and **S** = .

To use the likelihood ratio method to test the hypothesis, we will require the restricted maximum likelihood estimate. Under the hypothesis,the model is the one in Section 15.2.2. The restricted estimate is given in (15‑12) and the equations which follow. To obtain them, we make a small modification in our algorithm above. We replace step (3) with

(3') = [*s*11**x**1**′y**1 + *s*22**x**2**′y**2 + *s*12(**x**1**′y**2 + **x**2**′y**1)]/[*s*11**x**1**′x**1 + *s*22**x**2**′x**2 + 2*s*12**x**1**′x**2].

Step 4 is then computed using this common estimate for both and . The iterations produce

1: .5372671

2: .5703837

3: .5725274

4: .5726687

5: .5726780

6: .5726786 converged.

At this estimate, the estimate of **Σ** is  The likelihood ratio statistic is given in (15-56). Using our unconstrained and constrained estimates, we find |**W***u*| = .2471714 and |**W**r| = .2473338. The statistic is λ = 20(ln.2473338 ‑ ln.2471714) = .0131. This is far below the critical value of 3.84, so once again, we do not reject the hypothesis.

4. The GLS estimator is

****= 

The matrix to be inverted equals [**Σ**-1 ⊗**X′X**]-1. But, [**Σ-**1⊗**X′X**]-1 = **Σ**⊗(**X′X**)-1. (See (2‑76).) Therefore,

****=

We now make the replacements **X′y**1 = (**X′X**)**b**1 and **X′y**2 = (**X′X**)**b**2. After multiplying out the product, we find that

****= 

The four scalar terms in the matrix product are the corresponding elements of **ΣΣ**-1 = **I**. Therefore, ****= .

5. The algebraic result is a little tedious, but straightforward. The GLS estimator which is computed is

.

It helps at this point to make some simplifying substitutions. The elements in the inverse matrix, σ*ij*, are all equal to elements of the original matrix divided by the determinant. But, the determinant appears in the leading matrix, which is inverted and in the trailing vector (which is not). Therefore, the determinant will cancel out. Making the substitutions, . Now, we are concerned with probability limits. We divide every element of the matrix to be inverted by *n*, then because of the inversion, divide the vector on the right by *n* as well. Suppose, for simplicity, that

limn→∞**x**i**′x**j/*n* = *qij*, i,j = 1,2,3. Then, *plim*

Then, we will use *plim* (1/*n*)**x**1**′y**1 = β1*q*11 + *plim* (1/*n*)**x**1Nε1 = β1*q*11

*plim* (1/*n*)**x**1**′y**2 = β2*q*12 + β3*q*13

*plim* (1/*n*)**x**2**′y**1 = β1*q*12

*plim* (1/*n*)**x**2**′y**2 = β2*q*22 + β3*q*23.

Therefore, after multiplying out all the terms,

*plim*.

The inverse matrix is , so with Δ = (σ11F22*q*11*q*22 - (F12*q*12)2)

plim. Taking the first coefficient separately and collecting terms,

plim= β1[σ11σ22*q*11*q*22-(σ12*q*12)2]/Δ + β2[σ11*q*22σ12*q*12 + σ12*q*12σ11*q*22]/Δ + β3[σ11*q*22σ12*q*13 + σ12*q*12σ11*q*23]/Δ

The first term in brackets equals Δ while the second equals 0. That leaves

plim = β1 - β3[σ11σ12(*q*22*q*13 - *q*12*q*23)]/Δ which is not equal to β1. There are two special cases worthy of note, though. The right hand side does equal β1 if either (1) σ12 = 0; the regressions are actually unrelated,

or (2) *q*12 = *q*13 = 0; the regressors in the two equations are uncorrelated. The second of these is similar to our finding for omitted variables in the classical regression model.

6. The model is . The GLS estimator of the full coefficient vector, θ, is . Let *qxx* equal **x′x**/*n*, *qx*1 = **x′y**1/*n* and, *qx*2 = **x′y**2/*n*. The *n*s in the inverse and in the vector cancel. Also, as suggested, we assume that = 0. As in the previous exercise, we replace elements of the inverse with elements from the original matrix and cancel the determinant which multiplies the matrix (after inversion) and divides the vector. Thus,  The inverse of the matrix is straightforward. Proceeding directly, we obtain 

It remains only to multiply the matrices and collect terms. The result is

=, =,= [(q*x*1/q*xx*) - (σ12σ22)(q*x*2/q*xx*)] = *b*1 - γ*b*2.

7. Once again, nothing is lost by assuming that = 0. Now, the OLS estimators are

*a*1 =, *a*2 =, *a*3 =, *b* = **x′y**1/**x′x**.

The vector of residuals is *ei*1 = *yi*1 ‑ ‑ *bxi*

*ei*2 = *yi*2 ‑ 

*ei*3 = *yi*3 - 

Now, if *yi*2 + *yi*3 = 1 at every observation, then (1/*n*)Σi(*yi*2 + *yi*3) = + = 1 as well. Therefore, by just adding the two equations, we see that *ei*2 + *ei*3 = 0 for every observation. Let **e***i* be the 3×1 vector of residuals. Then, **e**i**′c** = 0, where **c** = [0,1,1]**′**. The sample covariance matrix of the residuals is

**S** = [(1/*n*)Σ*i***e***i***e***i***′**]. Then, **Sc** = [(1/*n*)Σ*i***e***i***e***i***′**]**c** = [(1/*n*)Σ*i***e***i***e***i***′c**] = [(1/*n*)Σ*i* **e***i*×0] = **0,** which means, by definition, that **S** is singular.

We can proceed simply by dropping the third equation. The adding up condition implies that α3 = 1 ‑ α2. So, we can treat the first two equations as a seemingly unrelated regression model and estimate a3 using the estimate of α2.

8. (a) Since nothing is excluded from either equation and there are no other restrictions, neither equation passes the order condition for identification.

(1) We use (13‑12) and the equations which follow it. For the first equation, [**A**3**′**,**A**5**′**] = β22, a scalar which has rank *M*‑1 = 1 unless β22 = 0. For the second, [**A**3**′**,**A**5**′**] = β31. Thus, both equations are identified.

(2) This restriction does not restrict the first equation, so it remains unidentified. The second equation is now identified, as [**A**3**′**,**A**5**′**] = [β11,β21] has rank 1 if either of the two ceofficients are nonzero.

(3) If γ1 equals 0, the model becomes partially recursive. The first equation becomes a regression which can be estimated by ordinary least squares. However, the second equation continues to fail the order condition. To see the problem, consider that even with the restriction, any linear combination of the two equations has the same variables as the original second eqation.

(4) We know from above that if β32 = 0, the second equation is identifiable. If it is, then γ2 is identified. We may treat it as known. As such, γ1 is known. By regressing **y**1 ‑ γ1**y**2 on the **x**s, we would obtain estimates of the remaining parameters, so these restrictions identify the model. It is instructive to analyze this from the standpoint of false structures as done in the text. A false structure which incorporates the known restrictions would be ×. If the false structure is to obey the restrictions, then *f*11 ‑ γ*f*21 = 1, *f*22 ‑ γ*f*12 = 1, *f*21 ‑ γ*f*11 = *f*12 ‑ γ*f*22, β31 *f*12 = 0. It follows then that *f*12 = 0 so *f*11 = 1. Then, *f*21 ‑ γ*f*11 = ‑γ or *f*21 = (*f*11 ‑ 1)γ so that *f*11 ‑ γ2(*f*11 ‑ 1) = 1. This can only hold for all values of γ if *f*11 = 1 and, then, *f*21 = 0. Therefore, **F** = **I** which establishes identification.

(5) If β31 = 0, the first equation is identified by the usual rank and order conditions. Consider, then, the off‑diagonal element of **Σ** = **Γ′ΩΓ**. **Ω** is identified since it is the reduced form covariance matrix. The off‑diagonal element is σ12 = ω11 + ω22 ‑ (γ1 + γ2)ω12 = 0. Since γ1 is zero, γ2 = ω12/(ω11 + ω22). With γ2 known, the remaining parameters are estimable by least squares regression of (**y**2 ‑ γ2**y**1) on the **x**s. Therefore, the restrictions identify the model.

(6) Since this is only a single restriction, it will not likely identify the entire model. Consider again the false structure. The restrictions implied by the theory are *f*11 ‑ γ2*f*21 = 1, *f*22 ‑ γ1*f*12 = 1, β21*f*11 + β22*f*21 = β21*f*12 + β22*f*22. The three restrictions on four unknown elements of **F** do not serve to pin down any of them. This restriction does not even partially identify the model.

(7) The last four restrictions remove *x*2 and *x*3 from the model. The remaining model is not identified by the usual rank and order conditions. From part (5), we see that the first restriction implies σ12 = ω11 + ω22 ‑ (γ1 + γ2)ω12 = 0. But, with neither γ1 nor γ2 specified, this does not identify either parameter.

(8) The first equation is identified by the conventional rank and order conditions. The second equation fails the order condition. But, the restriction σ12 = 0 provides the necessary additional information needed to identify the model. For simplicity, write the model with the restrictions imposed as

*y*1 = γ1*y*2 + ε1 and *y*2 = γ2*y*1 + β*x* + ε2.

The reduced form is *y*1 = π1*x* + *v*1 and y2 = π2*x* + *v*2

where π1 = γ1β/Δ and π2 = β/Δ with Δ = (1 - γ1γ2), and *v*1 = (ε1 + γ1ε2)/Δ and *v*2 = (ε2 + γ2ε1)/Δ. The reduced form variances and covariances are ω11 = (γ12σ22 + σ11)/Δ2, ω22 = (γ22σ11 + σ22)/Δ2, ω12 = (γ1σ22 + γ2σ11)/Δ2.

All reduced form parameters are estimable directly by using least squares, so the reduced form is identified in all cases. Now, γ1 = π1/π2. σ11 is the residual variance in the euqation (y1 - γ1y2) = ε1, so σ11 must be estimable (identified) if γ1 is. Now, with a bit of manipulation, we find that γ1ω12 - ω11 = -σ11/Δ. Therefore, with σ11 and γ1 "known" (identified), the only remaining unknown is γ2, which is therefore identified. With γ1 and γ2 in hand, β may be deduced from π2. With γ2 and β in hand, σ22 is the residual variance in the equation (*y*2 - β*x* - γ2*y*1) = ε2, which is directly estimable, therefore, identified.

9. Obtain the reduced form for the model in Exercise 1 under each of the assumptions made in parts (a) and (b1), (b6), and (b9).

(1). The model is *y*1 = γ1*y*2 + β11*x*1 + β21*x*2 + β31*x*3 + ε1

*y*2 = γ2*y*1 + β12*x*1 + β22*x*2 + β32*x*3 + ε2.

Therefore, **Γ** =  and **B** = and **Σ** is unrestricted. The reduced form is

**Π**=and

**Ω** = (**Γ**-1)**′Σ**(**Γ**-1) = 

(6) The model is *y*1 = β11*x*1 + β21*x*2 + β31*x*3 + ε1

*y*2 = γ2*y*1 + β12*x*1 + β22*x*2 + β32*x*3 + ε2

The first equation is already a reduced form. Substituting it into the second provides the second reduced form. The coefficient matrix is **P**=, **Γ**-1 = so **Ω** = (**Γ**-1)**′Σ**(**Γ**-1) =

(9) The model is

*y*1 = γ1*y*2 + ε1

*y*2 = γ2*y*1 + β12*x*1 + ε2

Then, **Π** = -**BΓ**-1 = [β12γ1/(1-γ1γ2) β12/(1-γ1γ2)] and **Ω** = .

10. The relevant submatrices are **X′X =** , **X′y**1 = , **X′y**2 = , **y1′y1** = 20, **y**2**′y**2 = 10, **y**1**′y**2 = 6, **X′Z**1 = , **X′Z**2 =  **Z**1**′Z**1 = , **Z**2**′Z**2 =,

**Z**1**′Z**2 = , **Z**1**′y**1 = , **Z**1**′y**2 =, **Z**2**′y**1 = , **Z**2**′y**2 = .

The two OLS coefficient vectors are

**d**1 = (**X′X**)-1**X′y**1 = [.439024,.536585] **′**

**d**2 = (**X′X**)-1**X′y**2 = [.193016,.384127,.19746] **′**.

The two stage least squares estimators are

****= [**Z**1**′X**(**X′X**)-1**X′Z**1]-1[**Z**1**′X**(**X′X**)-1**X′y**1] = [.368816,.578711] **′**.

****= [**Z**2**′X**(**X′X**)-1**X′Z**2]-1[**Z**2**′X**(**X′X**)-1**X′y**2] = [.484375,.367188,.109375] **′**.

= (**y**1**′y**1 ‑ 2**y**1**′Z** + **′Z**1**′Z**1****) / 25 = .610397, = .268384.

The estimated asymptotic covariance matrices are

Est.Var[****] = [**Z**1′**X**(**X′X**)-1**X′Z**1]-1 = 

Est.Var[Est.Var[****]] = .

The three stage least squares estimate is



= [.368817,.578708,.4706,.306363,.168294]**′** .

The estimated standard errors are the square roots of the diagonal elements of the inverse matrix,

[.4637,.4466,.3626,.1716,.1628], compared to the 2SLS values, [.4637,.4466,.3639,.2174,.2081].

To compute the limited information maximum likelihood estimator, we require the matrix of sums of squares and cross products of residuals of the regressions of **y**1 and **y**2 on **x**1 and on **x**1, **x**2, and **x**3. These are

**W**0 = **Y′Y** ‑ **Y**′**x**1(**x**1**′x**1)-1**x**1**′Y** = , **W**1 = **Y′Y** ‑ **Y**′**X**(**X**′**X**)-1**X**′**Y** = 

The two characteristic roots of (**W**1)-1**W**0 are 1.53157 and 1.00837. We carry the smaller one into the *k*‑class computation [see, for example, Theil (1971) or Judge, et al (1985)];

**= **

Finally, the two estimates of the reduced form are

(OLS) **P** = 

and (2SLS) .

**11.** For the model  *y*1 = γ1*y*2 + β11*x*1 + β21*x*2 + ε1

*y*2 = γ2*y*1 + β32*x*3 + β42*x*4 + ε2

show that there are two restrictions on the reduced form coefficients. Describe a procedure for estimating the model while incorporating the restrictions.

The structure is [*y*1 *y*2] 

or **y′ Γ** + **x′B** = ε**′.** The reduced form coefficient matrix is

**Π** = -**BΓ**-1 =  =  The two restrictions are π12/π11 = π22/π21 and π31/π32 = π41/π42. If we write the reduced form as

*y*1 = π11*x*1 + π21*x*2 + π31*x*3 + π41*x*4 + *v*1

*y*2 = π12*x*1 + π22*x*2 + π32*x*3 + π42*x*4 + *v*2.

We could treat the system as a nonlinear seemingly unrelated regressions model. One possible way to handle the restrictions is to eliminate two parameters directly by making the substitutions

π12 = π11π22/π21 and π31 = π32π41/π42.

The pair of equations would be

*y*1 = π11*x*1 + π21*x*2 + (π32π41/π42)*x*3 + π41*x*4 + *v*1

*y*2 = (π11π22/π21)*x*1 + π22*x*2 + π32*x*3 + π42*x*4 + *v*2.

This nonlinear system could now be estimated by nonlinear GLS. The function to be minimized would be

Σ *vi*12σ11 + *vi*22σ22 + 2*vi*1*vi*2σ12 = *n*tr(**Σ**-1**W**).

Needless to say, this would be quite involved.

12. Prove plim **Y***j***′ε**/*T* = **ωj** ‑ **Ω***jj***γ**j.

Consistent with the partitioning **y′** = [*yj* **Y***j***′** **Y***i*\***′**], partition **Ω** into

**ω***jj*   **ω***j***′ ω**\**j***′**

**Ω** = **ω***j*  **Ω***jj* **Ω***j***′**

**ω\****j*  **Ω\****j* **Ω***j*\*

and, as in the equation preceding (13-8), partition the *j*th column of **Γ** as **Γ***j* = . Since the full set of reduced form disturbances is **V** = **EΓ**-1, it follows that **E** = **VΓ**. In particular, the *j*th column of **E** is **ε***j* = **VΓ***j*. In the reduced form, now referring to (15-8),  **Y***j*  = **XΠ***j*  + **V***j*, where **Π***j* is the *Mj* columns of **Π** corresponding to the included endogenous variables and **V***j* is the *T*×*Mj* matrix of their reduced form disturbances. Since **X** is uncorrelated with all columns of **E**, we have

plim **Y***j***′**ε*j*/*T* = plim **V***j***′ Γ***j* /*T* = [**ω***j* **Ω***jj* **Ω***j\**]  = **ω***j* - **Ω***jj***γ***j* as required.

13. Prove that an underidentified equation cannot be estimated by two stage least squares.

If the equation fails the order condition, then the number of excluded exogenous variables is less than the number of included endogenous. The matrix of instrumental variables to be used for two stage least squares is of the form = [**XA**,**X***j*], where **XA** is *Mj* linear combination of all *K* columns in **X** and **X***j* is *Kj* columns of **X**. In total, *K* = *Kj*\* + *Kj*. If the equation fails the order condition, then *Kj*\* < *Mj*, so is *Mj* + *Kj* columns which are linear combinations of *K* = *Kj*\* + *Kj* < *Mj* + *Kj*. Therefore,  cannot have full column rank. In order to compute the two stage least squares estimator, we require (**′**)-1, which cannot be computed.

**Applications**

1. By adding the share equations vertically, we find the restrictions

β1 + β2 + β3 = 1

δ11 + δ12 + δ13 = 0

δ12 + δ22 + δ23 = 0

δ13 + δ23 + δ33 = 0

γ*y*1 + γ*y*2 + γ*y*3 = 0.

Note that the adding up condition also implies ε1 + ε2 + ε3 = 0.

We will eliminate the third share equation. The restrictions imply

β3 = 1 ‑ β1 ‑ β2

δ13 = ‑ δ11 ‑ δ12

δ23 = ‑ δ12 ‑ δ22

δ33 = ‑ δ13 ‑ δ23 = δ11 + δ22 + 2δ12

γ*y*3 = ‑ γ*y*1 ‑ γ*y*2.

By inserting these in the three share equations, we find

*S*1 = β1 + δ11ln*p*1 + δ12ln*p*2 ‑ δ11ln*p*3 ‑ δ12ln*p*3 + γ*y*1ln*Y* + ε1

= β1 + δ11ln(*p*1/*p*3) + δ12ln(*p*2/*p*3) + γ*y*1ln*Y* + ε1

*S*2 = β2 + δ12ln*p*1 + δ22ln*p*2 ‑ δ12ln*p*3 ‑ δ22ln*p*3 + γ*y*2ln*Y* + ε2

= β2 + δ12ln(*p*1/*p*3) + δ22ln(*p*2/*p*3) + γy2ln*Y* + ε2

*S*3 = 1 - β1 - β2 - δ11ln*p*1 - δ12ln*p*1 - δ12ln*p*2 - δ22ln*p*2 + δ11ln*p*3 + δ12ln*p*3 + δ12ln*p*3

+ δ22ln*p*3 - γ*y*1ln*p*3 - γ*y*2ln*p*3 - ε1 - ε2

= 1 - *S*1 - *S*2

For the cost function, making the substitutions for β3, δ13, δ23, δ33, and γy3 produces

ln*C* = α + β1(ln*p*1 ‑ ln*p*3) + β2(ln*p*2 ‑ ln*p*3)

+ δ11((ln2*p*1)/2 ‑ ln*p*1ln*p*3 + (ln2*p*3)/2)

+ δ22((ln2*p*2)/2 ‑ ln*p*2ln*p*3 + (ln2*p*3)/2) + δ12(ln*p*1ln*p*2 ‑ ln*p*1ln*p*3 ‑ ln*p*2ln*p*3 + (ln2*p*3))

+ γy1ln*Y*(ln*p*1 ‑ ln*p*3) + γy2ln*Y*(ln*p*2 ‑ ln*p*3) + βyln*Y* + βyy(ln2*Y*)/2 + εc

= α + β1ln(*p*1/*p*3) + β2ln(*p*2/*p*3)

+ δ11(ln2(*p*1/*p*3))/2 + δ22(ln2(*p*2/*p*3))/2 + δ12ln(*p*1/*p*3)ln(*p*2/*p*3)

+ γy1ln*Y*ln(*p*1/*p*3) + γy2ln*Y*ln(*p*2/*p*3) + βyln*Y* + βyy(ln2*Y*)/2 + εc

The system of three equations (cost and two shares) can be estimated as discussed in the text. Invariance is achieved by using a maximum likelihood estimator. The five parameters eliminated by the restrictions can be estimated after the others are obtained just by using the restrictions. The restrictions are linear, so the standard errors are also striaghtforward to obtain.

The least squares estimates are shown below. Estimated standard errors appear in parentheses.

**Variable Cost Function Capital Share Labor Share**

One 51.32 (45.91) ‑.0174 (.4697) .2172 (.2408)

ln(pk/pf) ‑21.74 (20.14) .2380 (.1045) .0033 (.0534)

ln(pl/pf) 32.39 (21.81) .0065 (.1059) .0168 (.0542)

ln2(pk/pf)/2 4.596 (4.604) ‑.0007 (.0098) ‑.0117 (.0050)

ln2(pl/pf)/2 8.216 (5.159)

ln(pk/pf)ln(pl/pf) ‑6.238 (4.684)

lnY 1.674 (.9297)

ln2Y/2 ,006997 (.0313)

lnYln(pk/pf) ‑.3223 (.2652)

lnYln(pl/pf) .08631 (.1981)

The estimates do not even come close to satisfying the cross equation restrictions. The parameters in the cost function are extremely large, owing primarily to rather severe multicollinearity among the price terms.

The results of estimation of the system by direct maximum likelihood are shown. The convergence criterion is the value of Belsley (discussed near the end of Section 5.5). The value α shown below is **g′H**-1**g** where **g** is the gradient and **H** is the Hessian of the log‑likelihood.

Iteration 0, F=46.76391, ln\***S**\*= ‑7.514268, α= 2.054399

Iteration 1, F=136.7448, ln\***S**\*= ‑16.51236, α= .5796486

Iteration 2, F=146.9803, ln\***S**\*= ‑17.53591, α= .02179947

Iteration 3, F=147.2268, ln\***S**\*= ‑17.56055, α= .0004222

Residual covariance matrix

Cost Capital Labor

Cost .0145572

Capital .000304768 .00303853

Labor ‑.000317554 ‑.000887258 .000798128

Coefficient Estimate Std. Error

α ‑6.41878 .6637

β*k* ‑.0546555 .2422

β*l* .250976 .2138

δ*kk* .245259 .06904

δ*ll* .0245770 .04788

δ*kl* ‑.00403448 .04779

β*y* .572452 .1340

β*yy* .0456587 .01908

γ*yk* ‑.00124236 .008409

γ*yl* ‑.0116921 .004442

β*f* .8036795

δ*kf* ‑.2412245

δ*lf* ‑.0205425

δ*ff* .261767

γ*yf* .0129345

The means of the variables are:  = 3531.8, *k* = 169.35,  = 2.039,  = 26.41. The three factor shares computed at these means are *Sk* = .4182, *Sl* = .0865, *Sf* = .4953. (The sample means are .411, .0954, and .4936.) The matrix of elasticities computed according to (15-72) is

*k l f*

.01115  *k*

**Σ** = .8885 ‑7.2756  *l*

‑.1646 .5206 .04819  *f*

(Two of the three diagonals have the `wrong' sign. This may be due to the very small sample size. The cross elasticities however do conform to what one might expect, the primary one being the evident substitution between capital and fuel.

To test the hypothesis that γ*yi* = 0, we reestimate the model without the interaction terms between ln*Y* and the prices in the cost function and without ln*Y* in the factor share equations. The iterations for this restricted model are shown below.

Iter.= 0, F=46.76391, log|**S**|= ‑7.514268, α= 1.912223

Iter.= 1, F=123.7521, log|**S**|= ‑15.21308, α= .5888180

Iter.= 2, F=136.3410, log|**S**|= ‑16.47198, α= .2771995

Iter.= 3, F=141.3491, log|**S**|= ‑16.97279, α= .08024513

Iter.= 4, F=142.5591, log|**S**|= ‑17.09379, α= .01636212

Converged achieved

Since we are interested only in the test statistic, we have not listed the parameter estimates. The test statistic given in (17‑26) is λ = T(ln|**S***r*| ‑ ln|**S***u*|) = 20(‑17.09379 ‑ (‑17.56055)) = 9.3352. There are two restrictions since only two of the three parameters are free. The critical value from the chi‑squared table is 5.99, so we would reject the hypothesis.

?===========================================

? Application 10.2

?===========================================

? a. Separate regressions and aggregation test.

? This saves the residuals to be used later.

CALC ; SS1=0 $

MATRIX ; EOLS = Init(20,10,0) $

PROCEDURE $

Include ; new ; Firm = company $

REGRESS ; Lhs = I ; Rhs = F,C,one ; Res = e$

CALC ; SS1=SS1 + Sumsqdev $

MATRIX ; EOLS(\*,company) = e $

ENDPROC $

EXECUTE ; Company=1,10 $

SAMPLE ; 1-200 $

+----------------------------------------------------+

| Residuals Sum of squares = 143205.9 |

| Standard error of e = 91.78167 |

| Fit R-squared = .9213540 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |t-ratio |P[|T|>t]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F | .11928083 .02583417 4.617 .0002 4333.84500

C | .37144481 .03707282 10.019 .0000 648.435000

Constant| -149.782453 105.842125 -1.415 .1751

+----------------------------------------------------+

| Residuals Sum of squares = 158093.3 |

| Standard error of e = 96.43445 |

| Fit R-squared = .4708624 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |t-ratio |P[|T|>t]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F | .17485602 .07419805 2.357 .0307 1971.82500

C | .38964189 .14236688 2.737 .0140 294.855000

Constant| -49.1983219 148.075365 -.332 .7438

+----------------------------------------------------+

| Residuals Sum of squares = 13216.59 |

| Standard error of e = 27.88272 |

| Fit R-squared = .7053067 |

| Adjusted R-squared = .6706369 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |t-ratio |P[|T|>t]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F | .02655119 .01556610 1.706 .1063 1941.32500

C | .15169387 .02570408 5.902 .0000 400.160000

Constant| -9.95630645 31.3742491 -.317 .7548

+----------------------------------------------------+

| Residuals Sum of squares = 2997.444 |

| Standard error of e = 13.27856 |

| Fit R-squared = .9135784 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |t-ratio |P[|T|>t]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F | .07794782 .01997330 3.903 .0011 693.210000

C | .31571819 .02881317 10.957 .0000 121.245000

Constant| -6.18996051 13.5064781 -.458 .6525

+----------------------------------------------------+

| Residuals Sum of squares = 1396.836 |

| Standard error of e = 9.064592 |

| Fit R-squared = .6804076 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |t-ratio |P[|T|>t]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F | .16237770 .05703645 2.847 .0111 231.470000

C | .00310174 .02196531 .141 .8894 486.765000

Constant| 22.7071160 6.87207605 3.304 .0042

+----------------------------------------------------+

| Residuals Sum of squares = 1110.533 |

| Standard error of e = 8.082418 |

| Fit R-squared = .9521422 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |t-ratio |P[|T|>t]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F | .13145484 .03117234 4.217 .0006 419.865000

C | .08537427 .10030597 .851 .4065 104.285000

Constant| -8.68554338 4.54516804 -1.911 .0730

+----------------------------------------------------+

| Residuals Sum of squares = 1507.403 |

| Standard error of e = 9.416516 |

| Fit R-squared = .7635009 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |t-ratio |P[|T|>t]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F | .08752720 .06562593 1.334 .1999 149.790000

C | .12378141 .01706483 7.254 .0000 314.945000

Constant| -4.49953436 11.2893942 -.399 .6952

+----------------------------------------------------+

| Residuals Sum of squares = 1773.234 |

| Standard error of e = 10.21312 |

| Fit R-squared = .7444461 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |t-ratio |P[|T|>t]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F | .05289413 .01570650 3.368 .0037 670.910000

C | .09240649 .05609897 1.647 .1179 85.6400000

Constant| -.50939018 8.01528894 -.064 .9501

+----------------------------------------------------+

| Residuals Sum of squares = 1407.360 |

| Standard error of e = 9.098674 |

| Fit R-squared = .6655145 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |t-ratio |P[|T|>t]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F | .07538794 .03395227 2.220 .0403 333.650000

C | .08210356 .02799168 2.933 .0093 297.900000

Constant| -7.72283708 9.35933952 -.825 .4207

+----------------------------------------------------+

| Residuals Sum of squares = 20.02673 |

| Standard error of e = 1.085377 |

| Fit R-squared = .6431578 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |t-ratio |P[|T|>t]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F | .00457343 .02716079 .168 .8683 70.9210000

C | .43736919 .07958891 5.495 .0000 5.94150000

Constant| .16151857 2.06556414 .078 .9386

+----------------------------------------------------+

| Ordinary least squares regression |

| LHS=I Mean = 145.9582 |

| Standard deviation = 216.8753 |

| WTS=none Number of observs. = 200 |

| Model size Parameters = 3 |

| Degrees of freedom = 197 |

| Residuals Sum of squares = 1755850. |

| Standard error of e = 94.40840 |

| Fit R-squared = .8124080 |

| Adjusted R-squared = .8105035 |

| Model test F[ 2, 197] (prob) = 426.58 (.0000) |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |t-ratio |P[|T|>t]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F | .11556216 .00583571 19.803 .0000 1081.68110

C | .23067849 .02547580 9.055 .0000 276.017150

Constant| -42.7143694 9.51167603 -4.491 .0000

? b. Aggregation test

REGRESS ; LHS = I ; RHS = F,C,one $

CALC ; SS0=Sumsqdev $

CALC ; List ; Fstat = ((SS0 - SS1)/(9\*3)) / (SS0/(n-10\*3))

; FC = Ftb(.95,27,170) $

+------------------------------------+

| Listed Calculator Results |

+------------------------------------+

FSTAT = 5.131854

FC = 1.551534

? c. SUR model

NAMELIST ; X1=F1,C1,one $

NAMELIST ; X2=F2,C2,one $

NAMELIST ; X3=F3,C3,one $

NAMELIST ; X4=F4,C4,one $

NAMELIST ; X5=F5,C5,one $

NAMELIST ; X6=F6,C6,one $

NAMELIST ; X7=F7,C7,one $

NAMELIST ; X8=F8,C8,one $

NAMELIST ; X9=F9,C9,one $

NAMELIST ; X10=F10,C10,one $

NAMELIST ; Y=I1,I2,I3,I4,I5,I6,I7,I8,I9,I10 $

SAMPLE ; 1 - 20 $

SURE ; Lhs = Y ; Eq1=X1;Eq2=X2;Eq3=X3;Eq4=X4;Eq5=X6;Eq6=X6

; Eq7=X7;Eq8=X8;Eq9=X9;Eq10=X10

; Maxit=0 ; OLS $

Criterion function for GLS is log-likelihood.

Iteration 0, GLS = -737.6463

Iteration 1, GLS = -730.1070

+----------------------------------------------------+

| Estimates for equation: I1 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F1 | .12472490 .01490044 8.371 .0000 4333.84500

C1 | .37951869 .02912686 13.030 .0000 648.435000

Constant| -178.611571 65.7890483 -2.715 .0066

+----------------------------------------------------+

| Estimates for equation: I2 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F2 | .16828512 .04057787 4.147 .0000 1971.82500

C2 | .33587688 .10299836 3.261 .0011 294.855000

Constant| -20.3887867 83.2537952 -.245 .8065

+----------------------------------------------------+

| Estimates for equation: I3 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F3 | .03425481 .00925706 3.700 .0002 1941.32500

C3 | .12538119 .02040101 6.146 .0000 400.160000

Constant| -14.3822597 20.6146424 -.698 .4854

+----------------------------------------------------+

| Estimates for equation: I4 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F4 | .06760969 .01597735 4.232 .0000 693.210000

C4 | .30752805 .02536245 12.125 .0000 121.245000

Constant| 1.96954637 11.0026359 .179 .8579

+----------------------------------------------------+

| Estimates for equation: I5 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F6 | .00635232 .02903793 .219 .8268 419.865000

C6 | .12737505 .09456013 1.347 .1780 104.285000

Constant| 45.8520779 4.86959707 9.416 .0000

+----------------------------------------------------+

| Estimates for equation: I6 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F6 | .12891587 .01798607 7.168 .0000 419.865000

C6 | .06768693 .06029084 1.123 .2616 104.285000

Constant| -5.77499083 3.44886478 -1.674 .0940

+----------------------------------------------------+

| Estimates for equation: I7 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F7 | .09106397 .04535783 2.008 .0447 149.790000

C7 | .12913287 .01446995 8.924 .0000 314.945000

Constant| -6.71472214 8.72476796 -.770 .4415

+----------------------------------------------------+

| Estimates for equation: I8 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F8 | .05179274 .00835658 6.198 .0000 670.910000

C8 | .04729955 .03473521 1.362 .1733 85.6400000

Constant| 4.09249729 5.09237714 .804 .4216

+----------------------------------------------------+

| Estimates for equation: I9 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F9 | .07275469 .02111017 3.446 .0006 333.650000

C9 | .06640816 .02194422 3.026 .0025 297.900000

Constant| -2.16859331 7.30885683 -.297 .7667

+----------------------------------------------------+

| Estimates for equation: I10 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F10 | -.01695668 .01550963 -1.093 .2743 70.9210000

C10 | .37466423 .05739586 6.528 .0000 5.94150000

Constant| 2.06101718 1.16003699 1.777 .0756?

c. Aggregation test according to (10-15)

MATRIX ; Z=Init(3,3,0) ; J=Iden(3); L=-1\*J $

MATRIX ; R=[j,z,z,z,z,z,z,z,z,l /

z,j,z,z,z,z,z,z,z,l /

z,z,j,z,z,z,z,z,z,l /

z,z,z,j,z,z,z,z,z,l /

z,z,z,z,j,z,z,z,z,l /

z,z,z,z,z,j,z,z,z,l /

z,z,z,z,z,z,j,z,z,l /

z,z,z,z,z,z,z,j,z,l /

z,z,z,z,z,z,z,z,j,l ]

; d = R\*b ; Vd = R\*Varb\*R'

; list ; AggF = 1/27 \* d'<vd>d $

Matrix AGGF has 1 rows and 1 columns.

1

+--------------

1| 98.53777

CALC ; List ; Ftb(.95,27,(200-10\*3)) $

+------------------------------------+

| Listed Calculator Results |

+------------------------------------+

Result = 1.551534

? d. Using separate OLS regressions, compute LM statistic

? OLS residuals were saved in matrix EOLS earlier.

MATRIX ; VEOLS = 1/20\*EOLS'EOLS

; VI = Diag(VEOLS) ; SDI = ISQR(VI)

; ROLS = SDI\*VEOLS\*SDI

; RR = ROLS' \*ROLS $

CALC ; List ; LMStat = (20/2)\*(Trc(RR)-10)

; Ctb(.95, (9\*10/2))$

+------------------------------------+

| Listed Calculator Results |

+------------------------------------+

LMSTAT = 97.617948

Result = 61.656233

? Constrained Sur model with one coefficient vector.

? This is the unconstrained model in (10-19)-(10-21)

SAMPLE ; 1 - 200 $

REGRESS; Lhs = I ; Rhs = F,C,one $

+----------------------------------------------------+

| Ordinary least squares regression |

| LHS=I Mean = 145.9582 |

| Standard deviation = 216.8753 |

| WTS=none Number of observs. = 200 |

| Model size Parameters = 3 |

| Degrees of freedom = 197 |

| Residuals Sum of squares = 1755850. |

| Standard error of e = 94.40840 |

| Fit R-squared = .8124080 |

| Adjusted R-squared = .8105035 |

| Model test F[ 2, 197] (prob) = 426.58 (.0000) |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |t-ratio |P[|T|>t]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F | .11556216 .00583571 19.803 .0000 1081.68110

C | .23067849 .02547580 9.055 .0000 276.017150

Constant| -42.7143694 9.51167603 -4.491 .0000

TSCS ; Lhs = I ; Rhs = F,C,one ; Pds=20 ; Model=S2,R0 $

+--------------------------------------------------+

| Groupwise Regression Models |

| Estimator = 2 Step GLS |

| Groupwise Het. and Correlated (S2) |

| Nonautocorrelated disturbances (R0) |

| Test statistics against the correlation |

| Deg.Fr. = 45 C\*(.95) = 61.66 C\*(.99) = 69.96 |

| Test statistics against the correlation |

| Likelihood ratio statistic = 320.2052 |

| Log-likelihood function = -853.084972 |

+--------------------------------------------------+

+--------+--------------+----------------+--------+--------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]|

+--------+--------------+----------------+--------+--------+

F | .10806238 .00241169 44.808 .0000

C | .15079551 .00386063 39.060 .0000

Constant| -20.1588844 .79950153 -25.214 .0000

CREATE ; WI = (SDI(firm,firm))^2 $

REGRESS; Lhs = I ; Rhs = F,C,one ; Wts = WI $

+----------------------------------------------------+

| Ordinary least squares regression |

| LHS=I Mean = 6.993136 |

| Standard deviation = 18.01824 |

| WTS=WI Number of observs. = 200 |

| Model size Parameters = 3 |

| Degrees of freedom = 197 |

| Residuals Sum of squares = 11690.82 |

| Standard error of e = 7.703521 |

| Fit R-squared = .8190465 |

| Adjusted R-squared = .8172094 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |t-ratio |P[|T|>t]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

F | .07847124 .00459121 17.092 .0000 96.8424912

C | .09896094 .00761314 12.999 .0000 23.8374846

Constant| -2.96519441 .66964256 -4.428 .0000

?=========================================================

? Application 10.3 - Simultaneous Equations

?=========================================================

? Read the data

? For convenience, rename the variables so they correspond

? to the example in the text.

sample ; 1 - 204 $

create ; ct=realcons$

create ; it=realinvs$

create ; gt=realgovt$

create ; rt=tbilrate $

? Impose (artifically) the adding up condition on total demand.

create ; yt=ct+it+gt $

create ; ct1=ct[-1] $

create ; yt1 = yt[-1] $

create ; dyt = yt - yt1 $

sample ; 2-204 $

names ; xt = one,gt,rt,ct1,yt1$

? Estimate equations by 2sls and save coefficients with

? the names used in the example.

2sls ; lhs = ct ; rhs=one,yt,ct1 ; inst = xt $

+----------------------------------------------------+

| Two stage least squares regression |

| LHS=CT Mean = 3008.995 |

| Standard deviation = 1456.900 |

| WTS=none Number of observs. = 203 |

| Model size Parameters = 3 |

| Degrees of freedom = 200 |

| Residuals Sum of squares = 75713.32 |

| Standard error of e = 19.45679 |

| Fit R-squared = .9998208 |

| Adjusted R-squared = .9998190 |

| Model test F[ 2, 200] (prob) =\*\*\*\*\*\*\* (.0000) |

+----------------------------------------------------+

| Instrumental Variables:

|ONE GT RT CT1 YT1

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

Constant| -13.8657181 5.31536302 -2.609 .0091

YT | .05843862 .01790473 3.264 .0011 4663.67389

CT1 | .92200662 .02657199 34.698 .0000 2982.97438

calc ; a0=b(1) ; a1=b(2) ; a2=b(3) $

2sls ; lhs = it ; rhs=one,rt,dyt ; inst = xt $

+----------------------------------------------------+

| Two stage least squares regression |

| LHS=IT Mean = 654.5296 |

| Standard deviation = 391.3705 |

| WTS=none Number of observs. = 203 |

| Model size Parameters = 3 |

| Degrees of freedom = 200 |

| Residuals Sum of squares = .7744227E+08 |

| Standard error of e = 622.2631 |

| Fit R-squared = -1.540485 |

| Adjusted R-squared = -1.565889 |

+----------------------------------------------------+

| Instrumental Variables:

|ONE GT RT CT1 YT1

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

Constant| -300.699429 125.980850 -2.387 .0170

RT | 56.5192542 15.4643912 3.655 .0003 5.24965517

DYT | 16.5359646 2.02509785 8.166 .0000 39.8236453

calc ; b0=b(1) ; b1=b(2) ; b2=b(3) $

?

? Create the coefficients of the reduced form. We only need the parts

? for the dynamics. These are in the second half of the example.

calc ; a=1-a1-b2 $

?

? Construct the matrix that governs the dynamics of the system. Note that

? the I equation is static. It is a function of y(t-1) and c(t-1) but not

? of I(t-1). This is the DELTA(1) submatrix in (13-42). The dominant

? root is the largest rood of DELTA(1).

calc ; list ; C11=(1-b2)/a ; C12=-a1\*b2/a ; C21=a2/a ; C22=-b2/a $

matrix ; C = [c11,c12 / c21,c22] $

+------------------------------------+

| Listed Calculator Results |

+------------------------------------+

C11 = .996253

C12 = .061967

C21 = -.059124

C22 = 1.060378

Matrix ; list ; roots = cxrt(c)$

Calc ; list ; domroot = sqr(roots(1,1)^2 + roots(1,2)^2)$

**--> Matrix ; list ; roots = cxrt(c)$**

Matrix ROOTS has 2 rows and 2 columns.

1 2

+----------------------------

1| 1.02832 -.05134

2| 1.02832 .05134

**--> Calc ; list ; domroot = sqr(roots(1,1)^2 + roots(1,2)^2)$**

+------------------------------------+

| Listed Calculator Results |

+------------------------------------+

DOMROOT = 1.029596

? The largest root is larger than on in absolute value. The system is unstable.

3sls ; lhs = ct,it ; eq1=one,yt,ct1 ; eq2=one,rt,dyt ; inst=xt ; maxit=0 $

+----------------------------------------------------+

| Estimates for equation: CT |

| InstVar/GLS least squares regression |

| LHS=CT Mean = 3008.995 |

| Residuals Sum of squares = 73370.06 |

| Standard error of e = 19.15334 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

Constant| -17.4780776 4.55837624 -3.834 .0001

YT | .07312129 .01415744 5.165 .0000 4663.67389

CT1 | .90026227 .02103720 42.794 .0000 2982.97438

+----------------------------------------------------+

| Estimates for equation: IT |

| InstVar/GLS least squares regression |

| LHS=IT Mean = 654.5296 |

| Residuals Sum of squares = .9735005E+08 |

| Standard error of e = 697.6749 |

+----------------------------------------------------+

+--------+--------------+----------------+--------+--------+----------+

|Variable| Coefficient | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|

+--------+--------------+----------------+--------+--------+----------+

Constant| -236.744328 122.661644 -1.930 .0536

RT | 30.5417941 12.9861014 2.352 .0187 5.24965517

DYT | 18.3544221 1.93633720 9.479 .0000 39.8236453