Chapter 1

Introduction

There are no exercises or applications in Chapter 1.

Chapter 2

The Classical Multiple Linear Regression Model

There are no exercises or applications in Chapter 2.

Chapter 3

Least Squares

Exercises

1. Let $\mathbf{X} = \begin{bmatrix} 1 & x_1 \\ ... & ... \\ 1 & x_n \end{bmatrix}$.

(a) The normal equations are given by (3-12), $\mathbf{X'e} = \mathbf{0}$ (we drop the minus sign), hence for each of the columns of **X**, $\mathbf{x_k}$, we know that $\mathbf{x_k}'$e = 0. This implies that $\Sigma_{i=1}^{n} e_i = 0$ and $\Sigma_{i=1}^{n} x_i e_i = 0$.

(b) Use $\Sigma_{i=1}^{n} e_i$ to conclude from the first normal equation that $a = \bar{y} - b\bar{x}$.

(c) We know that $\Sigma_{i=1}^{n} e_i = 0$ and $\Sigma_{i=1}^{n} x_i e_i = 0$. It follows then that $\Sigma_{i=1}^{n}(x_i - \bar{x})e_i = 0$ because $\Sigma_{i=1}^{n}\bar{x}e_i = \bar{x}\Sigma_{i=1}^{n}e_i = 0$. Substitute e_i to obtain

$\Sigma_{i=1}^{n}(x_i - \bar{x})(y_i - a - bx_i) = 0$ or $\Sigma_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y} - b(x_i - \bar{x})) = 0$

Then, $\Sigma_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = b\Sigma_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x}))$ so $b = \dfrac{\Sigma_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\Sigma_{i=1}^{n}(x_i - \bar{x})^2}$.

(d) The first derivative vector of e′e is -2**X′e**. (The normal equations.) The second derivative matrix is ∂^2(**e′e**)/∂**b**∂**b′** = 2**X′X**. We need to show that this matrix is positive definite. The diagonal elements are 2n and $2\Sigma_{i=1}^{n}x_i^2$ which are clearly both positive. The determinant is (2n)($2\Sigma_{i=1}^{n}x_i^2$)-($2\Sigma_{i=1}^{n}x_i$)^2
= $4n\Sigma_{i=1}^{n}x_i^2$ -4($n\bar{x}$)^2 = $4n[(\Sigma_{i=1}^{n}x_i^2) - n\bar{x}^2] = 4n[(\Sigma_{i=1}^{n}(x_i - \bar{x})^2]$. Note that a much simpler proof appears after (3-6).

2. Write **c** as **b** + (**c** - **b**). Then, the sum of squared residuals based on **c** is

(**y** - **Xc**)′(**y** - **Xc**) = [**y** - **X**(**b** + (**c** - **b**))] ′[**y** - **X**(**b** + (**c** - **b**))] = [(**y** - **Xb**) + **X**(**c** - **b**)] ′[(**y** - **Xb**) + **X**(**c** - **b**)]
= (**y** - **Xb**) ′(**y** - **Xb**) + (**c** - **b**) ′**X′X**(**c** - **b**) + 2(**c** - **b**) ′**X′**(**y** - **Xb**).

But, the third term is zero, as 2(**c** - **b**) ′**X′**(**y** - **Xb**) = 2(**c** - **b**)**X′e** = **0**. Therefore,

(**y** - **Xc**) ′(**y** - **Xc**) = **e′e** + (**c** - **b**) ′**X′X**(**c** - **b**)

or (**y** - **Xc**) ′(**y** - **Xc**) - **e′e** = (**c** - **b**) ′**X′X**(**c** - **b**).

The right hand side can be written as **d′d** where **d** = **X**(**c** - **b**), so it is necessarily positive. This confirms what we knew at the outset, least squares is least squares.

3. The residual vector in the regression of **y** on **X** is $\mathbf{M_X y}$ = [**I** - **X**(**X′X**)$^{-1}$**X′**]**y**. The residual vector in the regression of **y** on **Z** is

$\mathbf{M_Z y}$ = [**I** - **Z**(**Z′Z**)$^{-1}$**Z′**]**y**
= [**I** - **XP**((**XP**)′(**XP**))$^{-1}$(**XP**)′)**y**
= [**I** - **XPP**$^{-1}$(**X′X**)$^{-1}$(**P′**)$^{-1}$**P′X′**)**y**
= $\mathbf{M_X y}$

Since the residual vectors are identical, the fits must be as well. Changing the units of measurement of the regressors is equivalent to postmultiplying by a diagonal **P** matrix whose kth diagonal element is the scale factor to be applied to the kth variable (1 if it is to be unchanged). It follows from the result above that this will not change the fit of the regression.

4. In the regression of **y** on **i** and **X**, the coefficients on **X** are **b** = (**X′M^{0}X**)$^{-1}$**X′M**^{0}**y**. **M**^{0} = **I** - **i**(**i′i**)$^{-1}$**i′** is the matrix which transforms observations into deviations from their column means. Since **M**^{0} is idempotent and symmetric we may also write the preceding as [(**X′M**^{0}**′**)(**M**^{0}**X**)]$^{-1}$(**X′M**^{0}**′**)(**M**^{0}**y**) which implies that the

regression of $\mathbf{M}^0\mathbf{y}$ on $\mathbf{M}^0\mathbf{X}$ produces the least squares slopes. If only **X** is transformed to deviations, we would compute $[(\mathbf{X}'\mathbf{M}^{0\prime})(\mathbf{M}^0\mathbf{X})]^{-1}(\mathbf{X}'\mathbf{M}^{0\prime})\mathbf{y}$ but, of course, this is identical. However, if only **y** is transformed, the result is $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{M}^0\mathbf{y}$ which is likely to be quite different.

5. What is the result of the matrix product $\mathbf{M}_1\mathbf{M}$ where $\mathbf{M}_1$ is defined in (3-19) and **M** is defined in (3-14)?

$$\mathbf{M}_1\mathbf{M} = (\mathbf{I} - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1')(\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}') = \mathbf{M} - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{M}$$

There is no need to multiply out the second term. Each column of $\mathbf{M}\mathbf{X}_1$ is the vector of residuals in the regression of the corresponding column of $\mathbf{X}_1$ on all of the columns in **X**. Since that **x** is one of the columns in **X**, this regression provides a perfect fit, so the residuals are zero. Thus, $\mathbf{M}\mathbf{X}_1$ is a matrix of zeroes which implies that $\mathbf{M}_1\mathbf{M} = \mathbf{M}$.

6. The original X matrix has n rows. We add an additional row, x_s'. The new y vector likewise has an additional element. Thus, $\mathbf{X}_{n,s} = \begin{bmatrix}\mathbf{X}_n \\ \mathbf{x}_s'\end{bmatrix}$ and $\mathbf{y}_{n,s} = \begin{bmatrix}\mathbf{y}_n \\ y_s\end{bmatrix}$. The new coefficient vector is

$\mathbf{b}_{n,s} = (\mathbf{X}_{n,s}'\mathbf{X}_{n,s})^{-1}(\mathbf{X}_{n,s}'\mathbf{y}_{n,s})$. The matrix is $\mathbf{X}_{n,s}'\mathbf{X}_{n,s} = \mathbf{X}_n'\mathbf{X}_n + \mathbf{x}_s\mathbf{x}_s'$. To invert this, use (A -66);

$(\mathbf{X}_{n,s}'\mathbf{X}_{n,s})^{-1} = (\mathbf{X}_n'\mathbf{X}_n)^{-1} - \dfrac{1}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}$. The vector is

$(\mathbf{X}_{n,s}'\mathbf{y}_{n,s}) = (\mathbf{X}_n'\mathbf{y}_n) + \mathbf{x}_s y_s$. Multiply out the four terms to get

$(\mathbf{X}_{n,s}'\mathbf{X}_{n,s})^{-1}(\mathbf{X}_{n,s}'\mathbf{y}_{n,s}) =$

$$\mathbf{b}_n - \frac{1}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s\mathbf{x}_s'\mathbf{b}_n + (\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s y_s - \frac{1}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s y_s$$

=

$$\mathbf{b}_n + (\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s y_s - \frac{\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s y_s - \frac{1}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s\mathbf{x}_s'\mathbf{b}_n$$

$$\mathbf{b}_n + \left[1 - \frac{\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}\right](\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s y_s - \frac{1}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s\mathbf{x}_s'\mathbf{b}_n$$

$$\mathbf{b}_n + \frac{1}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s y_s - \frac{1}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s\mathbf{x}_s'\mathbf{b}_n$$

$$\mathbf{b}_n + \frac{1}{1+\mathbf{x}_s'(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s}(\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{x}_s(y_s - \mathbf{x}_s'\mathbf{b}_n)$$

7. Define the data matrix as follows: $\mathbf{X} = \begin{bmatrix}\mathbf{i} & \mathbf{x} & \mathbf{0} \\ 1 & 0 & 1\end{bmatrix} = \begin{bmatrix}\mathbf{X}_1, & \mathbf{0} \\ & 1\end{bmatrix} = \begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_2\end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix}\mathbf{y}_o \\ y_m\end{bmatrix}$. (The subscripts on the parts of **y** refer to the "observed" and "missing" rows of **X**. We will use Frish-Waugh to obtain the first two columns of the least squares coefficient vector. $\mathbf{b}_1 = (\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1)^{-1}(\mathbf{X}_1'\mathbf{M}_2\mathbf{y})$. Multiplying it out, we find that $\mathbf{M}_2$ = an identity matrix save for the last diagonal element that is equal to 0.

$\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1 = \mathbf{X}_1'\mathbf{X}_1 - \mathbf{X}_1'\begin{bmatrix}\mathbf{0} & \mathbf{0} \\ \mathbf{0}' & 1\end{bmatrix}\mathbf{X}_1$. This just drops the last observation. $\mathbf{X}_1'\mathbf{M}_2\mathbf{y}$ is computed likewise. Thus, the coeffients on the first two columns are the same as if y_0 had been linearly regressed on $\mathbf{X}_1$. The denomonator of R^2 is different for the two cases (drop the observation or keep it with zero fill and the dummy variable). For the first strategy, the mean of the *n*-1 observations should be different from the mean of the full n unless the last observation happens to equal the mean of the first *n*-1.

For the second strategy, replacing the missing value with the mean of the other *n*-1 observations, we can deduce the new slope vector logically. Using Frisch-Waugh, we can replace the column of *x*'s with deviations from the means, which then turns the last observation to zero. Thus, once again, the coefficient on the *x* equals what it is using the earlier strategy. The constant term will be the same as well.

8. For convenience, reorder the variables so that $\mathbf{X} = [\mathbf{i}, \mathbf{P}_d, \mathbf{P}_n, \mathbf{P}_s, \mathbf{Y}]$. The three dependent variables are $\mathbf{E}_d$, $\mathbf{E}_n$, and $\mathbf{E}_s$, and $\mathbf{Y} = \mathbf{E}_d + \mathbf{E}_n + \mathbf{E}_s$. The coefficient vectors are

$$\mathbf{b}_d = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{E}_d,$$
$$\mathbf{b}_n = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{E}_n, \text{ and}$$
$$\mathbf{b}_s = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{E}_s.$$

The sum of the three vectors is

$$\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\mathbf{E}_d + \mathbf{E}_n + \mathbf{E}_s] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}.$$

Now, $\mathbf{Y}$ is the last column of $\mathbf{X}$, so the preceding sum is the vector of least squares coefficients in the regression of the last column of $\mathbf{X}$ on all of the columns of $\mathbf{X}$, including the last. Of course, we get a perfect fit. In addition, $\mathbf{X}'[\mathbf{E}_d + \mathbf{E}_n + \mathbf{E}_s]$ is the last column of $\mathbf{X}'\mathbf{X}$, so the matrix product is equal to the last column of an identity matrix. Thus, the sum of the coefficients on all variables except income is 0, while that on income is 1.

9. Let $\bar{R}_K^2$ denote the adjusted R^2 in the full regression on K variables including $\mathbf{x}_k$, and let $\bar{R}_1^2$ denote the adjusted R^2 in the short regression on K-1 variables when $\mathbf{x}_k$ is omitted. Let R_K^2 and R_1^2 denote their unadjusted counterparts. Then,

$$R_K^2 = 1 - \mathbf{e}'\mathbf{e}/\mathbf{y}'\mathbf{M}^0\mathbf{y}$$
$$R_1^2 = 1 - \mathbf{e}_1'\mathbf{e}_1/\mathbf{y}'\mathbf{M}^0\mathbf{y}$$

where $\mathbf{e}'\mathbf{e}$ is the sum of squared residuals in the full regression, $\mathbf{e}_1'\mathbf{e}_1$ is the (larger) sum of squared residuals in the regression which omits $\mathbf{x}_k$, and $\mathbf{y}'\mathbf{M}^0\mathbf{y} = \Sigma_i (y_i - \bar{y})^2$

Then, $$\bar{R}_K^2 = 1 - [(n-1)/(n-K)](1 - R_K^2)$$

and $$\bar{R}_1^2 = 1 - [(n-1)/(n-(K-1))](1 - R_1^2).$$

The difference is the change in the adjusted R^2 when $\mathbf{x}_k$ is added to the regression,

$$\bar{R}_K^2 - \bar{R}_1^2 = [(n-1)/(n-K+1)][\mathbf{e}_1'\mathbf{e}_1/\mathbf{y}'\mathbf{M}^0\mathbf{y}] - [(n-1)/(n-K)][\mathbf{e}'\mathbf{e}/\mathbf{y}'\mathbf{M}^0\mathbf{y}].$$

The difference is positive if and only if the ratio is greater than 1. After cancelling terms, we require for the adjusted R^2 to increase that $\mathbf{e}_1'\mathbf{e}_1/(n-K+1)]/[(n-K)/\mathbf{e}'\mathbf{e}] > 1$. From the previous problem, we have that $\mathbf{e}_1'\mathbf{e}_1 = \mathbf{e}'\mathbf{e} + b_K^2(\mathbf{x}_k'\mathbf{M}_1\mathbf{x}_k)$, where $\mathbf{M}_1$ is defined above and b_k is the least squares coefficient in the full regression of $\mathbf{y}$ on $\mathbf{X}_1$ and $\mathbf{x}_k$. Making the substitution, we require $[(\mathbf{e}'\mathbf{e} + b_K^2(\mathbf{x}_k'\mathbf{M}_1\mathbf{x}_k))(n-K)]/[(n-K)\mathbf{e}'\mathbf{e} + \mathbf{e}'\mathbf{e}] > 1$. Since $\mathbf{e}'\mathbf{e} = (n-K)s^2$, this simplifies to $[\mathbf{e}'\mathbf{e} + b_K^2(\mathbf{x}_k'\mathbf{M}_1\mathbf{x}_k)]/[\mathbf{e}'\mathbf{e} + s^2] > 1$. Since all terms are positive, the fraction is greater than one if and only $b_K^2(\mathbf{x}_k'\mathbf{M}_1\mathbf{x}_k) > s^2$ or $b_K^2(\mathbf{x}_k'\mathbf{M}_1\mathbf{x}_k/s^2) > 1$. The denominator is the estimated variance of b_k, so the result is proved.

10. This R^2 must be lower. The sum of squares associated with the coefficient vector which omits the constant term must be higher than the one which includes it. We can write the coefficient vector in the regression without a constant as $\mathbf{c} = (0,\mathbf{b}^*)$ where $\mathbf{b}^* = (\mathbf{W}'\mathbf{W})^{-1}\mathbf{W}'\mathbf{y}$, with $\mathbf{W}$ being the other K-1 columns of $\mathbf{X}$. Then, the result of the previous exercise applies directly.

11. We use the notation 'Var[.]' and 'Cov[.]' to indicate the sample variances and covariances. Our information is Var[N] = 1, Var[D] = 1, Var[Y] = 1.

Since $C = N + D$, Var[C] = Var[N] + Var[D] + 2Cov[N,D] = 2(1 + Cov[N,D]).

From the regressions, we have

Cov[C,Y]/Var[Y] = Cov[C,Y] = .8.

But, Cov[C,Y] = Cov[N,Y] + Cov[D,Y].

Also, Cov[C,N]/Var[N] = Cov[C,N] = .5,

but, Cov[C,N] = Var[N] + Cov[N,D] = 1 + Cov[N,D], so Cov[N,D] = -.5,

so that Var[C] = 2(1 + -.5) = 1.

And, Cov[D,Y]/Var[Y] = Cov[D,Y] = .4.

Since Cov[C,Y] = .8 = Cov[N,Y] + Cov[D,Y], Cov[N,Y] = .4.

Finally, Cov[C,D] = Cov[N,D] + Var[D] = -.5 + 1 = .5.

Now, in the regression of C on D, the sum of squared residuals is $(n-1)\{\text{Var}[C] - (\text{Cov}[C,D]/\text{Var}[D])^2\text{Var}[D]\}$

based on the general regression result $\Sigma e^2 = \Sigma(y_i - \bar{y})^2 - b^2\Sigma(x_i - \bar{x})^2$. All of the necessary figures were obtained above. Inserting these and n-1 = 20 produces a sum of squared residuals of 15.

12. The relevant submatrices to be used in the calculations are

	Investment	Constant	GNP	Interest
Investment	*	3.0500	3.9926	23.521
Constant		15	19.310	111.79
GNP			25.218	148.98
Interest				943.86

The inverse of the lower right 3×3 block is $(\mathbf{X}'\mathbf{X})^{-1}$,

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} 7.5874 & & \\ -7.41859 & 7.84078 & \\ .27313 & -.598953 & .06254637 \end{bmatrix}$$

The coefficient vector is $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (-.0727985, .235622, -.00364866)'$. The total sum of squares is $\mathbf{y}'\mathbf{y} = .63652$, so we can obtain $\mathbf{e}'\mathbf{e} = \mathbf{y}'\mathbf{y} - \mathbf{b}'\mathbf{X}'\mathbf{y}$. $\mathbf{X}'\mathbf{y}$ is given in the top row of the matrix. Making the substitution, we obtain $\mathbf{e}'\mathbf{e} = .63652 - .63291 = .00361$. To compute R^2, we require $\Sigma_i (x_i - \bar{y})^2 = .63652 - 15(3.05/15)^2 = .01635333$, so $R^2 = 1 - .00361/.0163533 = .77925$.

13. The results cannot be correct. Since log S/N = log S/Y + log Y/N by simple, exact algebra, the same result must apply to the least squares regression results. That means that the second equation estimated must equal the first one plus log Y/N. Looking at the equations, that means that all of the coefficients would have to be identical save for the second, which would have to equal its counterpart in the first equation, plus 1. Therefore, the results cannot be correct. In an exchange between Leff and Arthur Goldberger that appeared later in the same journal, Leff argued that the difference was simple rounding error. You can see that the results in the second equation resemble those in the first, but not enough so that the explanation is credible. Further discussion about the data themselves appeared in subsequent idscussion. [See Goldberger (1973) and Leff (1973).]

14. A proof of Theorem 3.1 provides a general statement of the observation made after (3-8). The counterpart for a multiple regression to the normal equations preceding (3-7) is

$$\begin{aligned}
&b_1 n + b_2\Sigma_i x_{i2} + b_3\Sigma_i x_{i3} + \ldots + b_K\Sigma_i x_{iK} = \Sigma_i y_i \\
&b_1\Sigma_i x_{i2} + b_2\Sigma_i x_{i2}^2 + b_3\Sigma_i x_{i2}x_{i3} + \ldots + b_K\Sigma_i x_{i2}x_{iK} = \Sigma_i x_{i2}y_i \\
&\ldots \\
&b_1\Sigma_i x_{iK} + b_2\Sigma_i x_{iK}x_{i2} + b_3\Sigma_i x_{iK}x_{i3} + \ldots + b_K\Sigma_i x_{iK}^2 = \Sigma_i x_{iK}y_i .
\end{aligned}$$

As before, divide the first equation by n, and manipulate to obtain the solution for the constant term, $b_1 = \bar{y} - b_2\bar{x}_2 - \ldots - b_K\bar{x}_K$. Substitute this into the equations above, and rearrange once again to obtain the equations for the slopes,

$$\begin{aligned}
&b_2\Sigma_i(x_{i2}-\bar{x}_2)^2 + b_3\Sigma_i(x_{i2}-\bar{x}_2)(x_{i3}-\bar{x}_3) + \ldots + b_K\Sigma_i(x_{i2}-\bar{x}_2)(x_{iK}-\bar{x}_K) = \Sigma_i(x_{i2}-\bar{x}_2)(y_i-\bar{y}) \\
&b_2\Sigma_i(x_{i3}-\bar{x}_3)(x_{i2}-\bar{x}_2) + b_3\Sigma_i(x_{i3}-\bar{x}_3)^2 + \ldots + b_K\Sigma_i(x_{i3}-\bar{x}_3)(x_{iK}-\bar{x}_K) = \Sigma_i(x_{i3}-\bar{x}_3)(y_i-\bar{y}) \\
&\ldots \\
&b_2\Sigma_i(x_{iK}-\bar{x}_K)(x_{i2}-\bar{x}_2) + b_3\Sigma_i(x_{iK}-\bar{x}_K)(x_{i3}-\bar{x}_3) + \ldots + b_K\Sigma_i(x_{iK}-\bar{x}_K)^2 = \Sigma_i(x_{iK}-\bar{x}_K)(y_i-\bar{y}).
\end{aligned}$$

If the variables are uncorrelated, then all cross product terms of the form $\Sigma_i(x_{ij}-\bar{x}_j)(x_{ik}-\bar{x}_k)$ will equal zero. This leaves the solution,

$$\begin{aligned}
&b_2\Sigma_i(x_{i2}-\bar{x}_2)^2 = \Sigma_i(x_{i2}-\bar{x}_2)(y_i-\bar{y}) \\
&b_3\Sigma_i(x_{i3}-\bar{x}_3)^2 = \Sigma_i(x_{i3}-\bar{x}_3)(y_i-\bar{y}) \\
&\ldots \\
&b_K\Sigma_i(x_{iK}-\bar{x}_K)^2 = \Sigma_i(x_{iK}-\bar{x}_K)(y_i-\bar{y}),
\end{aligned}$$

which can be solved one equation at a time for

$$b_k = \left[\Sigma_i(x_{ik}-\bar{x}_k)(y_i-\bar{y})\right] \Big/ \left[\Sigma_i(x_{ik}-\bar{x}_k)^2\right], \; k = 2,\ldots,K.$$

Each of these is the slope coefficient in the simple of y on the respective variable.

Application

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?=======================================================================
? Chapter 3 Application 1
?=======================================================================
Read $
(Data appear in the text.)
Namelist ; X1 = one,educ,exp,ability$
Namelist ; X2 = mothered,fathered,sibs$
?=======================================================================
? a.
?=======================================================================
Regress  ; Lhs = wage ; Rhs = x1$
+----------------------------------------------------+
| Ordinary    least squares regression               |
| LHS=WAGE     Mean                 =   2.059333      |
|              Standard deviation   =   .2583869      |
| WTS=none     Number of observs.   =         15      |
| Model size   Parameters           =          4      |
|              Degrees of freedom   =         11      |
| Residuals    Sum of squares       =   .7633163      |
|              Standard error of e  =   .2634244      |
| Fit          R-squared            =   .1833511      |
|              Adjusted R-squared   =  -.3937136E-01 |
| Model test   F[  3,    11] (prob) =    .82 (.5080) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    1.66364000       .61855318     2.690   .0210
 EDUC    |     .01453897       .04902149      .297   .7723   12.8666667
 EXP     |     .07103002       .04803415     1.479   .1673   2.80000000
 ABILITY |     .02661537       .09911731      .269   .7933    .36600000
?=======================================================================
? b.
?=======================================================================
Regress  ; Lhs = wage ; Rhs = x1,x2$
+----------------------------------------------------+
| Ordinary    least squares regression               |
| LHS=WAGE     Mean                 =   2.059333      |
|              Standard deviation   =   .2583869      |
| WTS=none     Number of observs.   =         15      |
| Model size   Parameters           =          7      |
|              Degrees of freedom   =          8      |
| Residuals    Sum of squares       =   .4522662      |
|              Standard error of e  =   .2377673      |
| Fit          R-squared            =   .5161341      |
|              Adjusted R-squared   =   .1532347      |
| Model test   F[  6,     8] (prob) =   1.42 (.3140) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|     .04899633       .94880761      .052   .9601
 EDUC    |     .02582213       .04468592      .578   .5793   12.8666667
 EXP     |     .10339125       .04734541     2.184   .0605   2.80000000
 ABILITY |     .03074355       .12120133      .254   .8062    .36600000
 MOTHERED|     .10163069       .07017502     1.448   .1856   12.0666667
 FATHERED|     .00164437       .04464910      .037   .9715   12.6666667
 SIBS    |     .05916922       .06901801      .857   .4162   2.20000000
?=======================================================================
? c.
?=======================================================================
```

```
Regress  ; Lhs = mothered ; Rhs = x1 ; Res = meds $
Regress  ; Lhs = fathered ; Rhs = x1 ; Res = feds $
Regress  ; Lhs = sibs     ; Rhs = x1 ; Res = sibss $
Namelist ; X2S = meds,feds,sibss $
Matrix   ; list ; Mean(X2S) $
Matrix Result   has  3 rows and  1 columns.
               1
        +--------------
       1| -.1184238D-14
       2|  .1657933D-14
       3| -.5921189D-16
The means are (essentially) zero.  The sums must be zero, as these new variables
are orthogonal to the columns of X1. The first column in X1 is a column of ones,
so this means that these residuals must sum to zero.
?=========================================================================
? d.
?=========================================================================
Namelist ; X = X1,X2 $
Matrix   ; i = init(n,1,1) $
Matrix   ; M0 = iden(n) - 1/n*i*i' $
Matrix   ; b12 = <X'X>*X'wage$
Calc     ; list ; ym0y =(N-1)*var(wage) $
Matrix   ; list ; cod = 1/ym0y * b12'*X'*M0*X*b12 $
Matrix COD      has  1 rows and  1 columns.
               1
        +--------------
       1|     .51613
Matrix   ; e = wage - X*b12 $
Calc     ; list ; cod = 1 - 1/ym0y * e'e $
+------------------------------------+
 COD     =       .516134
The R squared is the same using either method of computation.
Calc     ; list ; RsqAd = 1 - (n-1)/(n-col(x))*(1-cod)$
+------------------------------------+
 RSQAD   =       .153235
? Now drop the constant
Namelist ; X0 = educ,exp,ability,X2 $
Matrix   ; i = init(n,1,1) $
Matrix   ; M0 = iden(n) - 1/n*i*i' $
Matrix   ; b120 = <X0'X0>*X0'wage$
Matrix   ; list ; cod = 1/ym0y * b120'*X0'*M0*X0*b120 $
Matrix COD      has  1 rows and  1 columns.
               1
        +--------------
       1|     .52953
Matrix   ; e0 = wage - X0*b120 $
Calc     ; list ; cod = 1 - 1/ym0y * e0'e0 $
+------------------------------------+
| Listed Calculator Results          |
+------------------------------------+
 COD     =       .515973
The R squared now changes depending on how it is computed.  It also goes up,
completely artificially.
?=========================================================================
? e.
?=========================================================================
The R squared for the full regression appears immediately below.
? f.
Regress ; Lhs = wage ; Rhs = X1,X2 $
+----------------------------------------------------+
| Ordinary    least squares regression               |
| WTS=none     Number of observs.   =         15     |
| Model size   Parameters           =          7     |
|              Degrees of freedom   =          8     |
| Fit          R-squared            =   .5161341     |
+----------------------------------------------------+
+--------+-------------+---------------+--------+--------+---------+
```

```
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|     .04899633       .94880761      .052   .9601
 EDUC    |     .02582213       .04468592      .578   .5793   12.8666667
 EXP     |     .10339125       .04734541     2.184   .0605   2.80000000
 ABILITY |     .03074355       .12120133      .254   .8062    .36600000
 MOTHERED|     .10163069       .07017502     1.448   .1856   12.0666667
 FATHERED|     .00164437       .04464910      .037   .9715   12.6666667
 SIBS    |     .05916922       .06901801      .857   .4162   2.20000000
Regress ; Lhs = wage ; Rhs = X1,X2S $
+----------------------------------------------------+
| Ordinary    least squares regression               |
| WTS=none     Number of observs.   =         15     |
| Model size   Parameters           =          7     |
|              Degrees of freedom   =          8     |
| Fit          R-squared            =   .5161341     |
|              Adjusted R-squared   =   .1532347     |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    1.66364000       .55830716     2.980   .0176
 EDUC    |     .01453897       .04424689      .329   .7509   12.8666667
 EXP     |     .07103002       .04335571     1.638   .1400   2.80000000
 ABILITY |     .02661537       .08946345      .297   .7737    .36600000
 MEDS    |     .10163069       .07017502     1.448   .1856 -.118424D-14
 FEDS    |     .00164437       .04464910      .037   .9715  .165793D-14
 SIBSS   |     .05916922       .06901801      .857   .4162 -.592119D-16
```

In the first set of results, the first coefficient vector is
$\mathbf{b}_1 = (\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{M}_2\mathbf{y}$ and
$b_2 = (\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1}\mathbf{X}_2'M_1\mathbf{y}$
In the second regression, the second set of regressors is M_1X_2, so
$\mathbf{b}_1 = (\mathbf{X}_1'\mathbf{M}_{12}\,\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{M}_{12}\mathbf{y}$ where $\mathbf{M}_{12} = \mathbf{I} - (\mathbf{M}_1\mathbf{X}_2)[(M_1\mathbf{X}_2)'(\mathbf{M}_1\mathbf{X}_2)]^{-1}(\mathbf{M}_1\mathbf{X}_2)'$
Thus, because the "M" matrix is different, the coefficient vector is different. The second set of coefficients in the second regression is
$\mathbf{b}_2 = [(\mathbf{M}_1\mathbf{X}_2)'\mathbf{M}_1(\mathbf{M}_1\mathbf{X}_2)]^{-1}\,(\mathbf{M}_1\mathbf{X}_2)\mathbf{M}_1\mathbf{y} = (\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{M}_1\mathbf{y}$ because $\mathbf{M}_1$ is idempotent.

Chapter 4

Statistical Properties of the Least Squares Estimator

Exercises

1. Consider the optimization problem of minimizing the variance of the weighted estimator. If the estimate is to be unbiased, it must be of the form $c_1\hat{\theta}_1 + c_2\hat{\theta}_2$ where c_1 and c_2 sum to 1. Thus, $c_2 = 1 - c_1$. The function to minimize is $\text{Min}_c L_* = c_1^2 v_1 + (1 - c_1)^2 v_2$. The necessary condition is $\partial L_*/\partial c_1 = 2c_1v_1 - 2(1 - c_1)v_2 = 0$ which implies $c_1 = v_2 / (v_1 + v_2)$. A more intuitively appealing form is obtained by dividing numerator and denominator by v_1v_2 to obtain $c_1 = (1/v_1) / [1/v_1 + 1/v_2]$. Thus, the weight is proportional to the inverse of the variance. The estimator with the smaller variance gets the larger weight.

2. First, $\hat{\beta} = \mathbf{c'y} = \mathbf{c'x} + \mathbf{c'}\boldsymbol{\varepsilon}$. So $E[\hat{\beta}] = \beta\mathbf{c'x}$ and $\text{Var}[\hat{\beta}] = \sigma^2\mathbf{c'c}$. Therefore,

$\text{MSE}[\hat{\beta}] = \beta^2[\mathbf{c'x} - 1]^2 + \sigma^2\mathbf{c'c}$. To minimize this, we set $\partial\text{MSE}[\hat{\beta}]/\partial\mathbf{c} = 2\beta^2[\mathbf{c'x} - 1]\mathbf{x} + 2\sigma^2\mathbf{c} = \mathbf{0}$.

Collecting terms, $\beta^2(\mathbf{c'x} - 1)\mathbf{x} = -\sigma^2\mathbf{c}$

Premultiply by $\mathbf{x'}$ to obtain $\beta^2(\mathbf{c'x} - 1)\mathbf{x'x} = -\sigma^2\mathbf{x'c}$

or $\mathbf{c'x} = \beta^2\mathbf{x'x} / (\sigma^2 + \beta^2\mathbf{x'x})$.

Then, $\mathbf{c} = [(-\beta^2/\sigma^2)(\mathbf{c'x} - 1)]\mathbf{x}$,

so $\mathbf{c} = [1/(\sigma^2/\beta^2 + \mathbf{x'x})]\mathbf{x}$.

Then, $\hat{\beta} = \mathbf{c'y} = \mathbf{x'y} / (\sigma^2/\beta^2 + \mathbf{x'x})$.

The expected value of this estimator is

$$E[\hat{\beta}] = \beta\mathbf{x'x} / (\sigma^2/\beta^2 + \mathbf{x'x})$$

so

$$E[\hat{\beta}] - \beta = \beta(-\sigma^2/\beta^2) / (\sigma^2/\beta^2 + \mathbf{x'x})$$
$$= -(\sigma^2/\beta) / (\sigma^2/\beta^2 + \mathbf{x'x})$$

while its variance is $\text{Var}[\mathbf{x'}(\mathbf{x}\beta + \boldsymbol{\varepsilon}) / (\sigma^2/\beta^2 + \mathbf{x'x})] = \sigma^2\mathbf{x'x} / (\sigma^2/\beta^2 + \mathbf{x'x})^2$

The mean squared error is the variance plus the squared bias,

$$\text{MSE}[\hat{\beta}] = [\sigma^4/\beta^2 + \sigma^2\mathbf{x'x}]/[\sigma^2/\beta^2 + \mathbf{x'x}]^2.$$

The ordinary least squares estimator is, as always, unbiased, and has variance and mean squared error

$$\text{MSE}(b) = \sigma^2/\mathbf{x'x}.$$

The ratio is taken by dividing each term in the numerator

$$\frac{MSE\left[\hat{\beta}\right]}{\text{MSE}(b)} = \frac{(\sigma^4/\beta^2)/(\sigma^2/\mathbf{x'x}) + \sigma^2\mathbf{x'x}/(\sigma^2/\mathbf{x'x})}{\left(\sigma^2/\beta^2 + \mathbf{x'x}\right)^2}$$

$$= [\sigma^2\mathbf{x'x}/\beta^2 + (\mathbf{x'x})^2]/(\sigma^2/\beta^2 + \mathbf{x'x})^2$$
$$= \mathbf{x'x}[\sigma^2/\beta^2 + \mathbf{x'x}]/(\sigma^2/\beta^2 + \mathbf{x'x})^2$$
$$= \mathbf{x'x}/(\sigma^2/\beta^2 + \mathbf{x'x})$$

Now, multiply numerator and denominator by β^2/σ^2 to obtain

$$\text{MSE}[\hat{\beta}]/\text{MSE}[b] = \beta^2\mathbf{x'x}/\sigma^2/[1 + \beta^2\mathbf{x'x}/\sigma^2] = \tau^2/[1 + \tau^2]$$

As $\tau\rightarrow\infty$, the ratio goes to one. This would follow from the result that the biased estimator and the unbiased estimator are converging to the same thing, either as σ^2 goes to zero, in which case the MMSE estimator is the same as OLS, or as $\mathbf{x'x}$ grows, in which case both estimators are consistent.

3. The OLS estimator fit without a constant term is $b = \mathbf{x}'\mathbf{y} / \mathbf{x}'\mathbf{x}$. Assuming that the constant term is, in fact, zero, the variance of this estimator is $\text{Var}[b] = \sigma^2/\mathbf{x}'\mathbf{x}$. If a constant term is included in the regression, then,

$$b' = \Sigma_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) / \Sigma_{i=1}^{n}(x_i - \bar{x})^2$$

The appropriate variance is $\sigma^2/\Sigma_{i=1}^{n}(x_i - \bar{x})^2$ as always. The ratio of these two is

$$\text{Var}[b]/\text{Var}[b'] = [\sigma^2/\mathbf{x}'\mathbf{x}] / [\sigma^2/\Sigma_{i=1}^{n}(x_i - \bar{x})^2]$$

But, $\quad \Sigma_{i=1}^{n}(x_i - \bar{x})^2 = \mathbf{x}'\mathbf{x} + n\bar{x}^2$

so the ratio is $\quad \text{Var}[b]/\text{Var}[b'] = [\mathbf{x}'\mathbf{x} + n\bar{x}^2]/\mathbf{x}'\mathbf{x} = 1 - n\bar{x}^2/\mathbf{x}'\mathbf{x} = 1 - \{n\bar{x}^2/[S_{xx} + n\bar{x}^2]\} \le 1$

It follows that fitting the constant term when it is unnecessary inflates the variance of the least squares estimator if the mean of the regressor is not zero.

4. We could write the regression as $y_i = (\alpha + \lambda) + \beta x_i + (\varepsilon_i - \lambda) = \alpha^* + \beta x_i + \varepsilon_i^*$. Then, we know that $E[\varepsilon_i^*] = 0$, and that it is independent of x_i. Therefore, the second form of the model satisfies all of our assumptions for the classical regression. Ordinary least squares will give unbiased estimators of α^* and β. As long as λ is not zero, the constant term will differ from α.

5. Let the constant term be written as $a = \Sigma_i d_i y_i = \Sigma_i d_i(\alpha + \beta x_i + \varepsilon_i) = \alpha\Sigma_i d_i + \beta\Sigma_i d_i x_i + \Sigma_i d_i\varepsilon_i$. In order for a to be unbiased for all samples of x_i, we must have $\Sigma_i d_i = 1$ and $\Sigma_i d_i x_i = 0$. Consider, then, minimizing the variance of a subject to these two constraints. The Lagrangean is

$$L_* = \text{Var}[a] + \lambda_1(\Sigma_i d_i - 1) + \lambda_2\Sigma_i d_i x_i \quad \text{where} \quad \text{Var}[a] = \Sigma_i \sigma^2 d_i^2.$$

Now, we minimize this with respect to d_i, λ_1, and λ_2. The $(n+2)$ necessary conditions are

$$\partial L_*/\partial d_i = 2\sigma^2 d_i + \lambda_1 + \lambda_2 x_i, \quad \partial L_*/\partial\lambda_1 = \Sigma_i d_i - 1, \quad \partial L_*/\partial\lambda_2 = \Sigma_i d_i x_i$$

The first equation implies that $\quad d_i = [-1/(2\sigma^2)](\lambda_1 + \lambda_2 x_i)$.

Therefore, $\quad \Sigma_i d_i = 1 = [-1/(2\sigma^2)][n\lambda_1 + (\Sigma_i x_i)\lambda_2]$

and $\quad \Sigma_i d_i x_i = 0 = [-1/(2\sigma^2)][(\Sigma_i x_i)\lambda_1 + (\Sigma_i x_i^2)\lambda_2]$.

We can solve these two equations for λ_1 and λ_2 by first multiplying both equations by $-2\sigma^2$ then writing the resulting equations as $\begin{bmatrix} n & \Sigma_i x_i \\ \Sigma_i x_i & \Sigma_i x_i^2 \end{bmatrix}\begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} = \begin{bmatrix} -2\sigma^2 \\ 0 \end{bmatrix}$. The solution is $\begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} = \begin{bmatrix} n & \Sigma_i x_i \\ \Sigma_i x_i & \Sigma_i x_i^2 \end{bmatrix}^{-1}\begin{bmatrix} -2\sigma^2 \\ 0 \end{bmatrix}$.

Note, first, that $\Sigma_i x_i = n\bar{x}$. Thus, the determinant of the matrix is $n\Sigma_i x_i^2 - (n\bar{x})^2 = n(\Sigma_i x_i^2 - n\bar{x}^2) = nS_{xx}$ where $S_{xx}\Sigma_{i=1}^{n}(x_i - \bar{x})^2$. The solution is, therefore, $\begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} = \frac{1}{nS_{xx}}\begin{bmatrix} \Sigma_i x_i^2 & -n\bar{x} \\ -n\bar{x} & 0 \end{bmatrix}\begin{bmatrix} -2\sigma^2 \\ 0 \end{bmatrix}$

or $\quad \lambda_1 = (-2\sigma^2)(\Sigma_i x_i^2/n)/S_{xx}$

$\quad \lambda_2 = (2\sigma 2\bar{x})/S_{xx}$

Then, $\quad d_i = [\Sigma_i x_i^2/n - \bar{x}x_i]/S_{xx}$

This simplifies if we write $\Sigma x_i^2 = S_{xx} + n\bar{x}^2$, so $\Sigma_i x_i^2/n = S_{xx}/n + \bar{x}^2$. Then,

$d_i = 1/n + \bar{x}(\bar{x} - x_i)/S_{xx}$, or, in a more familiar form, $d_i = 1/n - \bar{x}(x_i - \bar{x})/S_{xx}$.

This makes the intercept term $\Sigma_i d_i y_i = (1/n)\Sigma_i y_i - \bar{x}\,\Sigma_{i=1}^{n}(x_i - \bar{x})y_i/S_{xx} = \bar{y} - b\bar{x}$ which was to be shown.

6. Let $q = E[Q]$. Then, $\quad q = \alpha + \beta P$, or $P = (-\alpha/\beta) + (1/\beta)q$.

Using a well known result, for a linear demand curve, marginal revenue is $MR = (-\alpha/\beta) + (2/\beta)q$. The profit maximizing output is that at which marginal revenue equals marginal cost, or 10. Equating MR to 10 and solving for q produces $q = \alpha/2 + 5\beta$, so we require a confidence interval for this combination of the parameters.

The least squares regression results are $\hat{Q} = 20.7691 - .840583$. The estimated covariance matrix of the coefficients is $\begin{bmatrix} 7.96124 & -0.624559 \\ -0.624559 & 0.0564361 \end{bmatrix}$. The estimate of q is 6.1816. The estimate of the variance of $\hat{q}$ is $(1/4)7.96124 + 25(.056436) + 5(-.0624559)$ or 0.278415, so the estimated standard error is 0.5276.

The 95% cutoff value for a t distribution with 13 degrees of freedom is 2.161, so the confidence interval is 6.1816 - 2.161(.5276) to 6.1816 + 2.161(.5276) or 5.041 to 7.322.

7. a. The sample means are (1/100) times the elements in the first column of **X'X**. The sample covariance matrix for the three regressors is obtained as $(1/99)[(\mathbf{X}'\mathbf{X})_{ij} - 100\,\bar{x}_i\bar{x}_j]$.

$$\text{Sample Var}[\mathbf{x}] = \begin{bmatrix} 1.0127 & 0.069899 & 0.555489 \\ 0.069899 & 0.755960 & 0.417778 \\ 0.555489 & 0.417778 & 0.496969 \end{bmatrix}$$ The simple correlation matrix is

$$\begin{bmatrix} 1 & .07971 & .78043 \\ .07971 & 1 & .68167 \\ .78043 & .68167 & 1 \end{bmatrix}$$

b. The vector of slopes is $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ = [-.4022, 6.123, 5.910, -7.525]′.

c. For the three short regressions, the coefficient vectors are

(1) one, x_1, and x_2: [-.223, 2.28, 2.11]′
(2) one, x_1, and x_3 [-.0696, .229, 4.025]′
(3) one, x_2, and x_3: [-.0627, -.0918, 4.358]′

d. The magnification factors are

for x_1: $[(1/(99(1.01727)) / 1.129]^2$ = .094
for x_2: $[(1/99(.75596)) / 1.11]^2$ = .109
for x_3: $[(1/99(.496969))/ 4.292]^2$ = .068.

e. The problem variable appears to be x_3 since it has the lowest magnification factor. In fact, all three are highly intercorrelated. Although the simple correlations are not excessively high, the three multiple correlations are .9912 for x_1 on x_2 and x_3, .9881 for x_2 on x_1 and x_3, and .9912 for x_3 on x_1 and x_2.

8. We consider two regressions. In the first, **y** is regressed on K variables, **X**. The variance of the least squares estimator, $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$, $\text{Var}[\mathbf{b}] = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$. In the second, **y** is regressed on **X** and an additional variable, **z**. Using results for the partitioned regression, the coefficients on **X** when **y** is regressed on **X** and **z** are $\mathbf{b}_{.z} = (\mathbf{X}'\mathbf{M}_z\mathbf{X})^{-1}\mathbf{X}'\mathbf{M}_z\mathbf{y}$ where $\mathbf{M}_z = \mathbf{I} - \mathbf{z}(\mathbf{z}'\mathbf{z})^{-1}\mathbf{z}'$. The true variance of $\mathbf{b}_{.z}$ is the upper left $K\times K$ matrix in $\text{Var}[\mathbf{b},c] = s^2\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{z} \\ \mathbf{z}'\mathbf{X} & \mathbf{z}'\mathbf{X} \end{bmatrix}^{-1}$. But, we have already found this above. The submatrix is $\text{Var}[\mathbf{b}_{.z}] = s^2(\mathbf{X}'\mathbf{M}_z\mathbf{X})^{-1}$. We can show that the second matrix is larger than the first by showing that its inverse is smaller. (See (A-120).) Thus, as regards the true variance matrices $(\text{Var}[\mathbf{b}])^{-1} - (\text{Var}[\mathbf{b}_{.z}])^{-1} = (1/\sigma^2)\mathbf{z}(\mathbf{z}'\mathbf{z})^{-1}\mathbf{z}'$ which is a nonnegative definite matrix. Therefore $\text{Var}[\mathbf{b}]^{-1}$ is larger than $\text{Var}[\mathbf{b}_{.z}]^{-1}$, which implies that Var[**b**] is smaller.

Although the true variance of **b** is smaller than the true variance of $\mathbf{b}_{.z}$, it does not follow that the estimated variance will be. The estimated variances are based on s^2, not the true σ^2. The residual variance estimator based on the short regression is $s^2 = \mathbf{e}'\mathbf{e}/(n - K)$ while that based on the regression which includes **z** is $s_z^2 = \mathbf{e}_{.z}'\mathbf{e}_{.z}/(n - K - 1)$. The numerator of the second is definitely smaller than the numerator of the first, but so is the denominator. It is uncertain which way the comparison will go. The result is derived in the previous problem. We can conclude, therefore, that if t ratio on c in the regression which includes **z** is larger than one in absolute value, then s_z^2 will be smaller than s^2. Thus, in the comparison, $\text{Est.Var}[\mathbf{b}] = s^2(\mathbf{X}'\mathbf{X})^{-1}$ is based on a smaller matrix, but a larger scale factor than $\text{Est.Var}[\mathbf{b}_{.z}] = s_z^2(\mathbf{X}'\mathbf{M}_z\mathbf{X})^{-1}$. Consequently, it is uncertain whether the estimated standard errors in the short regression will be smaller than those in the long one. Note that it is not sufficient merely for the result of the previous problem to hold, since the relative sizes of the matrices also play a role. But, to take a polar case, suppose **z** and **X** were uncorrelated. Then, $\mathbf{XNM}_z\mathbf{X}$ equals **XNX**. Then, the estimated variance of $\mathbf{b}_{.z}$ would be less than that of **b** without **z** even though the true variance is the same (assuming the premise of the previous problem holds). Now, relax this assumption while holding the t ratio on c constant. The matrix in $\text{Var}[\mathbf{b}_{.z}]$ is now larger, but the leading scalar is now smaller. Which way the product will go is uncertain.

9. The F ratio is computed as $[\mathbf{b}'\mathbf{X}'\mathbf{X}\mathbf{b}/K]/[\mathbf{e}'\mathbf{e}/(n - K)]$. We substitute $\mathbf{e} = \mathbf{M}\varepsilon$, and

$\mathbf{b} = \beta + (\mathbf{X'X})^{-1}\mathbf{X'}\varepsilon = (\mathbf{X'X})^{-1}\mathbf{X'}\varepsilon$. Then, $F = [\varepsilon'\mathbf{X}(\mathbf{X'X})^{-1}\mathbf{X'X}(\mathbf{X'X})^{-1}\mathbf{X'}\varepsilon/K]/[\varepsilon'\mathbf{M}\varepsilon/(n - K)] =$ $[\varepsilon'(\mathbf{I} - \mathbf{M})\varepsilon/K]/[\varepsilon'\mathbf{M}\varepsilon/(n - K)]$.

The exact expectation of F can be found as follows: $F = [(n\text{-}K)/K][\varepsilon'(\mathbf{I} - \mathbf{M})\varepsilon]/[\varepsilon'\mathbf{M}\varepsilon]$. So, its exact expected value is $(n\text{-}K)/K$ times the expected value of the ratio. To find that, we note, first, that $\mathbf{M}\varepsilon$ and $(\mathbf{I} - \mathbf{M})\varepsilon$ are independent because $\mathbf{M}(\mathbf{I} - \mathbf{M}) = \mathbf{0}$. Thus, $E\{[\varepsilon'(\mathbf{I} - \mathbf{M})\varepsilon]/[\varepsilon'\mathbf{M}\varepsilon]\} = E[\varepsilon'(\mathbf{I} - \mathbf{M})\varepsilon]\times E\{1/[\varepsilon'\mathbf{M}\varepsilon]\}$. The first of these was obtained above, $E[\varepsilon'(\mathbf{I} - \mathbf{M})\varepsilon] = K\sigma^2$. The second is the expected value of the reciprocal of a chi-squared variable. The exact result for the reciprocal of a chi-squared variable is $E[1/\chi^2(n\text{-}K)] = 1/(n - K - 2)$. Combining terms, the exact expectation is $E[F] = (n - K) / (n - K - 2)$. Notice that the mean does not involve the numerator degrees of freedom.

10. We write $\mathbf{b} = \beta + (\mathbf{X'X})^{-1}\mathbf{X'}\varepsilon$, so $\mathbf{b'b} = \beta'\beta + \varepsilon'\mathbf{X}(\mathbf{X'X})^{-1}(\mathbf{X'X})^{-1}\mathbf{X'}\varepsilon + 2\beta'(\mathbf{X'X})^{-1}\mathrm{X}'\varepsilon$. The expected value of the last term is zero, and the first is nonstochastic. To find the expectation of the second term, use the trace, and permute $\varepsilon'\mathbf{X}$ inside the trace operator. Thus,

$$
\begin{aligned}
\mathrm{E}[\beta'\beta] &= \beta'\beta + E[\varepsilon'\mathbf{X}(\mathbf{X'X})^{-1}(\mathbf{X'X})^{-1}\mathbf{X'}\varepsilon] \\
&= \beta'\beta + E[tr\{\varepsilon'\mathbf{X}(\mathbf{X'X})^{-1}(\mathbf{X'X})^{-1}\mathbf{X'}\varepsilon\}] \\
&= \beta'\beta + E[tr\{(\mathbf{X'X})^{-1}\mathbf{X'}\varepsilon\varepsilon'\mathbf{X}(\mathbf{X'X})^{-1}\}] \\
&= \beta'\beta + tr[E\{(\mathbf{X'X})^{-1}\mathbf{X'}\varepsilon\varepsilon'\mathbf{X}(\mathbf{X'X})^{-1}\}] \\
&= \beta'\beta + tr[(\mathbf{X'X})^{-1}\mathbf{X'}E[\varepsilon\varepsilon']\mathbf{X}(\mathbf{X'X})^{-1}] \\
&= \beta'\beta + tr[(\mathbf{X'X})^{-1}\mathbf{X'}(\sigma^2\mathbf{I})\mathbf{X}(\mathbf{X'X})^{-1}] \\
&= \beta'\beta + \sigma^2 tr[(\mathbf{X'X})^{-1}\mathbf{X'X}(\mathbf{X'X})^{-1}] \\
&= \beta'\beta + \sigma^2 tr[(\mathbf{X'X})^{-1}] \\
&= \beta'\beta + \sigma^2\Sigma_k(1/\lambda_k)
\end{aligned}
$$

The trace of the inverse equals the sum of the characteristic roots of the inverse, which are the reciprocals of the characteristic roots of $\mathbf{X'X}$.

11. The F ratio is computed as $[\mathbf{b'X'Xb}/K]/[\mathbf{e'e}/(n - K)]$. We substitute $\mathbf{e} = \mathbf{M},$ and $\mathbf{b} = \beta + (\mathbf{X'X})^{-1}\mathbf{X'}\varepsilon = (\mathbf{X'X})^{-1}\mathbf{X'}\varepsilon$. Then, $F = [\varepsilon'\mathbf{X}(\mathbf{X'X})^{-1}\mathbf{X'X}(\mathbf{X'X})^{-1}\mathbf{X'}\varepsilon/K]/[\varepsilon'\mathbf{M}\varepsilon/(n - K)] =$ $[\varepsilon'(\mathbf{I} - \mathbf{M})\varepsilon/K]/[\varepsilon'\mathbf{M}\varepsilon/(n - K)]$. The denominator converges to σ^2 as we have seen before. The numerator is an idempotent quadratic form in a normal vector. The trace of $(\mathbf{I} - \mathbf{M})$ is K regardless of the sample size, so the numerator is always distributed as σ^2 times a chi-squared variable with K degrees of freedom. Therefore, the numerator of F does not converge to a constant, it converges to σ^2/K times a chi-squared variable with K degrees of freedom. Since the denominator of F converges to a constant, σ^2, the statistic converges to a random variable, $(1/K)$ times a chi-squared variable with K degrees of freedom.

12. We can write e_i as $e_i = y_i - \mathbf{b'x}_i = (\beta'\mathbf{x}_i + \varepsilon_i) - \mathbf{b'x}_i = \varepsilon_i + (\mathbf{b} - \beta)'\mathbf{x}_i$
We know that plim $\mathbf{b} = \beta$, and $\mathbf{x}_i$ is unchanged as n increases, so as $n\rightarrow\infty$, e_i is arbitrarily close to ε_i.

13. The estimator is $\bar{y} = (1/n)\Sigma_i y_i = (1/n)\Sigma_i(\mu + \varepsilon_i) = \mu + (1/n)\Sigma_i\varepsilon_i$. Then, $E[\bar{y}] = \mu + (1/n)\Sigma_i E[\varepsilon_i] = \mu$ and $\mathrm{Var}[\bar{y}] = (1/n^2)\Sigma_i\Sigma_j \mathrm{Cov}[\varepsilon_i,\varepsilon_j] = \sigma^2/n$. Since the mean equals μ and the variance vanishes as $n\rightarrow\infty$, $\bar{y}$ is mean square consistent. In addition, since $\bar{y}$ is a linear combination of normally distributed variables, $\bar{y}$ has a normal distribution with the mean and variance given above in every sample. Suppose that ε_i were not normally distributed. Then, $\sqrt{n}(\bar{y} - \mu) = (1/\sqrt{n})(\Sigma_i\varepsilon_i)$ satisfies the requirements for the central limit theorem. Thus, the asymptotic normal distribution applies whether or not the disturbances have a normal distribution.

For the alternative estimator, $\hat{\mu} = \Sigma_i w_i y_i$, so $E[\hat{\mu}] = \Sigma_i w_i E[y_i] = \Sigma_i w_i\mu = \mu\Sigma_i w_i = \mu$ and $\mathrm{Var}[\hat{\mu}] = \Sigma_i w_i^2\sigma^2 = \sigma^2\Sigma_i w_i^2$. The sum of squares of the weights is $\Sigma_i w_i^2 = \Sigma_i i^2/[\Sigma_i i]^2 = [n(n+1)(2n+1)/6]/[n(n+1)/2]^2 =$ $[2(n^2 + 3n/2 + 1/2)]/[1.5n(n^2 + 2n + 1)]$. As $n\rightarrow\infty$, the fraction will be dominated by the term $(1/n)$ and will tend to zero. This establishes the consistency of this estimator. The last expression also provides the asymptotic variance. The large sample variance can be found as $\mathrm{Asy.Var}[\hat{\mu}] = (1/n)\lim_{n\rightarrow\infty}\mathrm{Var}[\sqrt{n}(\hat{\mu} - \mu)]$. For the estimator above, we can use $\mathrm{Asy.Var}[\hat{\mu}] = (1/n)\lim_{n\rightarrow\infty} n\mathrm{Var}[\hat{\mu} - \mu] = (1/n)\lim_{n\rightarrow\infty}\sigma^2[2(n^2 +$

$3n/2 + 1/2)]/[1.5(n^2 + 2n + 1)] = 1.3333\sigma^2$. Notice that this is unambiguously larger than the variance of the sample mean, which is the ordinary least squares estimator.

14. To obtain the asymptotic distribution, write the result already in hand as $\mathbf{b} = (\beta + \mathbf{Q}^{-1}\gamma) + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\varepsilon - \mathbf{Q}^{-1}\varepsilon$. We have established that plim $\mathbf{b} = \beta + \mathbf{Q}^{-1}\gamma$. For convenience, let $\theta \neq \beta$ denote $\beta + \mathbf{Q}^{-1}\gamma$ = plim $\mathbf{b}$. Write the preceding in the form $\mathbf{b} - \theta = (\mathbf{X}'\mathbf{X}/n)^{-1}(\mathbf{X}'\varepsilon/n) - \mathbf{Q}^{-1}\gamma$. Since plim$(\mathbf{X}'\mathbf{X}/n) = \mathbf{Q}$, the large sample behavior of the right hand side is the same as that of plim $(\mathbf{b} - \theta) = \mathbf{Q}^{-1}\text{plim}(\mathbf{X}'\varepsilon/n) - \mathbf{Q}^{-1}\gamma$. That is, we may replace $(\mathbf{X}'\mathbf{X}/n)$ with $\mathbf{Q}$ in our derivation. Then, we seek the asymptotic distribution of $\sqrt{n}\,(\mathbf{b} - \theta)$ which is the same as that of $\sqrt{n}\,[\mathbf{Q}^{-1}\text{plim}(\mathbf{X}'\varepsilon/n) - \mathbf{Q}^{-1}\gamma] = \mathbf{Q}^{-1}\sqrt{n}\;(1/n)\Sigma_{i=1}^{n}\left(\mathbf{x}_i\varepsilon_i\; - \;\gamma\right)$. From this point, the derivation is exactly the same as that when $\gamma = \mathbf{0}$, so there is no need to redevelop the result. We may proceed directly to the same asymptotic distribution we obtained before. The only difference is that the least squares estimator estimates θ, not β.

15. a. To solve this, we will use an extension of Exercise 6 in Chapter 3 (adding one row of data), and the necessary matrix result, (A-66b) in which B will be $\mathbf{X}_m$ and $\mathbf{C}$ will be $\mathbf{I}$. Bypassing the matrix algebra, which will be essentially identical to the earlier exercise, we have

$$\mathbf{b}_{c,m} = \mathbf{b}_c + [\mathbf{I} + \mathbf{X}_{\text{m}}(\mathbf{X}_{\text{c}}'\mathbf{X}_{\text{c}})^{-1}\mathbf{X}_{\text{m}}]^{-1}(\mathbf{X}_{\text{c}}'\mathbf{X}_{\text{c}})^{-1}\mathbf{X}_{\text{m}}'(\mathbf{y}_{\text{m}} - \mathbf{X}_{\text{m}}\mathbf{b}_{\text{c}})$$

But, in this case, $\mathbf{y}_{\text{m}}$ is precisely $\mathbf{X}_{\text{m}}\mathbf{b}_{\text{c}}$, so the ending vector is zero. Thus, the coefficient vector is the same. b. The model applies to the first n_{c} observations, so $\mathbf{b}_{\text{c}}$ is the least squares estimator for those observations. Yes, it is unbiased.

c. The residuals at the second step are $\mathbf{e}_{\text{c}}$ and $(\mathbf{X}_{\text{m}}\mathbf{b}_{\text{c}} - \mathbf{X}_{\text{m}}\mathbf{b}_{\text{c}}) = (\mathbf{e}_{\text{c}}', \mathbf{0}')'$. Thus, the sum of squares is the same at both steps.

d. The numerator of s^2 is the same in both cases, however, for the second one, the degrees of freedom is larger. The first is unbiased, so the second one must be biased downward.

Applications

```
?=======================================================================
? Chapter 4 Application 1
?=======================================================================
Read $
Year   GasExp Pop    Gasp   Income PNC    PUC    PPT    PD     PN     PS
1953   7.4    159565 16.668 8883   47.2   26.7   16.8   37.7   29.7   19.4
...
2004   224.5  293951 123.901 27113 133.9  133.3  209.1  114.8  172.2  222.8

Sample ; 1 - 52 $
Create ; G = 1000000*gasexp/(gasp*pop)$
Create ; t = year - 1952 $
Namelist ; X = one,income, gasp,pnc,puc,ppt,pd,pn,ps,t$
?=======================================================================
? a.  Basic regression
?=======================================================================
Regress ; Lhs = g ; Rhs = X $
+----------------------------------------------------+
| Ordinary    least squares regression               |
| LHS=G        Mean                 =   4.935619     |
|              Standard deviation   =   1.059105     |
| WTS=none     Number of observs.   =         52     |
| Model size   Parameters           =         10     |
|              Degrees of freedom   =         42     |
| Residuals    Sum of squares       =   .4985489     |
|              Standard error of e  =   .1089505     |
| Fit          R-squared            =   .9912852     |
|              Adjusted R-squared   =   .9894177     |
| Model test   F[  9,    42] (prob) = 530.82 (.0000) |
+----------------------------------------------------+
```

```
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    1.10587817       .56937860     1.942   .0588
 INCOME  |     .00021575     .517619D-04     4.168   .0001   16805.0577
 GASP    |    -.01108386       .00397812    -2.786   .0080   51.3429615
 PNC     |     .00057735       .01284414      .045   .9644   87.5673077
 PUC     |    -.00587463       .00487032    -1.206   .2345   77.8000000
 PPT     |     .00690726       .00483613     1.428   .1606   89.3903846
 PD      |     .00122888       .01188175      .103   .9181   78.2692308
 PN      |     .01269051       .01259799     1.007   .3195   83.5980769
 PS      |    -.02802781       .00799625    -3.505   .0011   89.7769231
 T       |     .07250369       .01418280     5.112   .0000   26.5000000
?=======================================================================
? b.  Hypothesis that b(NC) = b(UC) $
?=======================================================================
Calc ; list ; (b(4)-b(5))/sqr(varb(4,4)+varb(5,5)-2*varb(4,5)) $
+------------------------------------+
| Listed Calculator Results          |
+------------------------------------+
 Result  =       .494883
?=======================================================================
? c.  Elasticities.  In each case, elasticity = b*xbar/ybar
?=======================================================================
Calc ; g2004   = g(52)$
Calc ; i2004   = income(52)$
Calc ; pg2004  = gasp(52)$
Calc ; ppt2004 = ppt(52)$
Calc ; list ; ei = b(2)*i2004/g2004
            ; ep = b(3)*pg2004/g2004
            ; eppt = b(6)*ppt2004/g2004$
+------------------------------------+
| Listed Calculator Results          |
+------------------------------------+
 EI      =       .948988
 EP      =      -.222792
 EPPT    =       .234311
?=======================================================================
? d. Log regression
?=======================================================================
Create ; logg = log(g) ; logpg = log(gasp) ; logi = log(income)
       ; logpnc=log(pnc) ; logpuc = log(puc) ; logppt = log(ppt)
       ; logpd = log(pd) ; logpn = log(pn) ; logps = log(ps) $
Namelist ; LogX = one,logi,logpg,logpnc,logpuc,logppt,logpd,logpn,logps,t$
Regress ; lhs = logg ; rhs = logx $
+----------------------------------------------------+
| Ordinary    least squares regression               |
| LHS=LOGG     Mean                 =   1.570475     |
|              Standard deviation   =   .2388115     |
| WTS=none     Number of observs.   =         52     |
| Model size   Parameters           =         10     |
|              Degrees of freedom   =         42     |
| Residuals    Sum of squares       =   .3812817E-01 |
|              Standard error of e  =   .3012994E-01 |
| Fit          R-squared            =   .9868911     |
|              Adjusted R-squared   =   .9840821     |
| Model test   F[  9,    42] (prob) = 351.33 (.0000) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|   -7.28719016      2.52056245    -2.891   .0061
 LOGI    |     .99299135       .25037574     3.966   .0003   9.67214751
 LOGPG   |     .06051812       .05401018     1.120   .2689   3.72930296
 LOGPNC  |    -.15471632       .26696298     -.580   .5653   4.38036654
 LOGPUC  |    -.48909058       .08519952    -5.741   .0000   4.10544881
 LOGPPT  |     .01926966       .13644891      .141   .8884   4.14194132
```

```
 LOGPD   |    1.73205775       .25988611     6.665   .0000   4.23906603
 LOGPN   |    -.72953933       .26506853    -2.752   .0087   4.23689080
 LOGPS   |    -.86798166       .35291106    -2.459   .0181   4.17535768
 T       |     .03797198       .00751371     5.054   .0000   26.5000000
?=======================================================================
? e.  Correlations of Price Variables
?=======================================================================
Namelist ; Prices = pnc,puc,ppt,pd,pn,ps$
Matrix   ; list ; xcor(prices) $
Correlation Matrix for Listed Variables

              PNC      PUC      PPT       PD       PN       PS
     PNC  1.00000   .99387   .98074   .99327   .98853   .97849
     PUC   .99387  1.00000   .98242   .98783   .98220   .97685
     PPT   .98074   .98242  1.00000   .95847   .98986   .99751
      PD   .99327   .98783   .95847  1.00000   .97734   .95633
      PN   .98853   .98220   .98986   .97734  1.00000   .99358
      PS   .97849   .97685   .99751   .95633   .99358  1.00000
?=======================================================================
? f.  Renormalizations of price variables
?=======================================================================
/*
In the linear case, the coefficients would be divided by the same
scale factor, so that x*b would be unchanged, where x is a variable
and b is the coefficient.  In the loglinear case, since log(k*x)=
log(k)+log(x), the renomalization would simply affect the constant
term.  The price coefficients woulde be unchanged.
*/
?=======================================================================
? g.  Oaxaca decomposition
?=======================================================================
Dates ; 1953 $
Period ; 1953-1973 $
Matrix ; xb0 = Mean(logx)$
Regress ; lhs = logg ; rhs = logx $
Matrix ; b0 = b ; v0 = varb $
Calc ; yb0 = ybar $
Period ; 1974-2004 $
Matrix ; xb1 = mean(logx) $
Regress ; lhs = logg ; rhs = logx $
Matrix ; b1 = b ; v1 = varb $
Calc ; yb1 = ybar $
? Now the decomposition
Calc ; list ; dybar = yb1 - yb0 $ Total
Calc ; list ; dy_dx = b1'xb1 - b1'xb0 $ Change due to change in x
Calc ; list ; dy_db = b1'xb0 - b0'xb0 $
Matrix ; vdb = v1+v0 ; vdb = xb0'[vdb]xb0 $
Calc ; sdb = sqr(vdb)
     ; list ; lower = dy_db - 1.96*sqr(vdb)
            ; upper = dy_db + 1.96*sqr(vdb) $
+------------------------------------+
| Listed Calculator Results          |
+------------------------------------+
 DYBAR   =       .395377
 DY_DX   =       .122745
 DY_DB   =       .272631
 LOWER   =       .184844
 UPPER   =       .360419
```

```
?======================================================================
? Chapter 4 Application 2
?======================================================================
Create ; lc = log(cost/pf) ; lpl=log(pl/pf) ; lpk=log(pk/pf)$
Create ; lq = log(q) ; lqq = .5*lq*lq $
Namelist ; x = one,lq,lqq,lpk,lpl $
? a.  Cost function
Regress; lhs = lc ; rhs = x ; printvc $
+-----------------------------------------------------+
| Ordinary    least squares regression                |
| LHS=LC       Mean                 =  -.3195570      |
|              Standard deviation   =   1.542364      |
| WTS=none     Number of observs.   =        158      |
| Model size   Parameters           =          5      |
|              Degrees of freedom   =        153      |
| Residuals    Sum of squares       =   2.904896      |
|              Standard error of e  =   .1377906      |
| Fit          R-squared            =   .9922222      |
|              Adjusted R-squared   =   .9920189      |
| Model test   F[  4,   153] (prob) =4879.59 (.0000) |
+-----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|   -6.81816332       .25243920   -27.009   .0000
 LQ      |     .40274543       .03148312    12.792   .0000   8.26548908
 LQQ     |     .06089514       .00432530    14.079   .0000   35.7912728
 LPK     |     .16203385       .04040556     4.010   .0001    .85978893
 LPL     |     .15244470       .04659735     3.272   .0013   5.58162250
               1             2              3             4             5
        +----------------------------------------------------------------------
       1|     .06373      -.00238       .00031       .00399      -.01047
       2|    -.00238       .00099      -.00013       .00010      -.00020
       3|     .00031      -.00013    .1870819D-04 -.1493338D-04  .2453652D-04
       4|     .00399       .00010   -.1493338D-04     .00163      -.00102
       5|    -.01047      -.00020    .2453652D-04    -.00102       .00217
?======================================================================
? b.  capital price coefficient
?======================================================================
Wald ; fn1 = 1 - b_lpk - b_lpl $
+------------------------------------------------+
| WALD procedure. Estimates and standard errors |
| for nonlinear functions and joint test of     |
| nonlinear restrictions.                        |
| Wald Statistic             =    266.36109     |
| Prob. from Chi-squared[ 1] =       .00000     |
+------------------------------------------------+
+--------+--------------+----------------+--------+--------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]|
+--------+--------------+----------------+--------+--------+
 Fncn(1) |     .68552145       .04200352    16.321   .0000
?======================================================================
? c.  efficient scale
?======================================================================
Wald ; fn1 = exp((1-b_lq)/b_lqq) $
+------------------------------------------------+
| WALD procedure. Estimates and standard errors |
| for nonlinear functions and joint test of     |
| nonlinear restrictions.                        |
| Wald Statistic             =     21.74979     |
| Prob. from Chi-squared[ 1] =       .00000     |
+------------------------------------------------+
+--------+--------------+----------------+--------+--------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]|
+--------+--------------+----------------+--------+--------+
 Fncn(1) |    18177.1045      3897.59890     4.664   .0000
Calc ; qstar = waldfns(1) ; vqstar = varwald(1,1)
```

```
     ; list ; lower = qstar - 1.96*sqr(vqstar)
            ; upper = qstar + 1.96*sqr(vqstar) $
?========================================================================
? d.  Raw data
?========================================================================
+------------------------------------+
| Listed Calculator Results          |
+------------------------------------+
 LOWER   =  10537.810653
 UPPER   =  25816.398344
Create ; output = q $
Sort ; lhs = output $
/*
```

The estimated efficient scale is 18177. There are 25 firms in the sample that have output larger than this. As noted in the problem, many of the largest firms in the sample are aggregates of smaller ones, so it is difficult to draw a conclusion here. However, some of the largest firms (Southern, American Electric power) are singly counted, and are much larger than this scale. The important point is that much of the output in the sample is produced by firms that are smaller than this efficient scale. There are unexploited economies of scale in this industry.

```
*/
```

Chapter 5

Inference and Prediction

Exercises

1. The estimated covariance matrix for the least squares estimator is

$$s^2(\mathbf{X'X})^{-1} = \frac{20}{3900}\begin{bmatrix} 3900/29 & 0 & 0 \\ 0 & 80 & -10 \\ 0 & -10 & 80 \end{bmatrix} = \begin{bmatrix} .69 & 0 & 0 \\ 0 & .40 & -.051 \\ 0 & -.051 & .256 \end{bmatrix}$$ where $s^2 = 520/(29-3) = 20$. Then,

the test may be based on $t = (.4 + .9 - 1)/[.410 + .256 - 2(.051)]^{1/2} = .399$. This is smaller than the critical value of 2.056, so we would not reject the hypothesis.

2. In order to compute the regression, we must recover the original sums of squares and cross products for y. These are $\mathbf{X'y} = \mathbf{X'Xb} = [116, 29, 76]'$. The total sum of squares is found using $R^2 = 1 - \mathbf{e'e}/\mathbf{y'M}^0\mathbf{y}$, so $\mathbf{y'M}^0\mathbf{y} = 520 / (52/60) = 600$. The means are $\bar{x}_1 = 0$, $\bar{x}_2 = 0$, $\bar{y} = 4$, so, $\mathbf{y'y} = 600 + 29(4^2) = 1064$. The slope in the regression of $\mathbf{y}$ on $\mathbf{x}_2$ alone is $b_2 = 76/80$, so the regression sum of squares is $b_2^{\,2}(80) = 72.2$, and the residual sum of squares is $600 - 72.2 = 527.8$. The test based on the residual sum of squares is $F = [(527.8 - 520)/1]/[520/26] = .390$. In the regression of the previous problem, the t-ratio for testing the same hypothesis would be $t = .4/(.410)^{1/2} = .624$ which is the square root of .39.

3. For the current problem, $\mathbf{R} = [\mathbf{0},\mathbf{I}]$ where $\mathbf{I}$ is the last K_2 columns. Therefore, $\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{R}$N is the lower right $K_2 \times K_2$ block of $(\mathbf{X'X})^{-1}$. As we have seen before, this is $(\mathbf{X}_2'\mathbf{M}_1\mathbf{X}2)^{-1}$. Also, $(\mathbf{X'X})^{-1}\mathbf{R'}$ is the last K_2 columns of $(\mathbf{X'X})^{-1}$. These are $(\mathbf{X'X})^{-1}\mathbf{R'} = \begin{bmatrix} -(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1} \\ (\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1} \end{bmatrix}$ Finally, since $\mathbf{q} = \mathbf{0}$, $\mathbf{Rb} - \mathbf{q} = (\mathbf{0b}_1 + \mathbf{Ib}_2) - \mathbf{0} = \mathbf{b}_2$. Therefore, the constrained estimator is

$$\mathbf{b}_* = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} - \begin{bmatrix} -(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1} \\ (\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1} \end{bmatrix}(\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)\mathbf{b}_2,$$ where $\mathbf{b}_1$ and $\mathbf{b}_2$ are the multiple regression coefficients in the regression of $\mathbf{y}$ on both $\mathbf{X}_1$ and $\mathbf{X}_2$. Collecting terms, this produces $\mathbf{b}_* = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} - \begin{bmatrix} -(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2\mathbf{b}_2 \\ \mathbf{b}_2 \end{bmatrix}$. But, we have from Section 6.3.4 that $\mathbf{b}_1 = (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{y} - (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2\mathbf{b}_2$ so the preceding reduces to $\mathbf{b}_* = \begin{bmatrix} (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{y} \\ \mathbf{0} \end{bmatrix}$ which was to be shown.

If, instead, the restriction is $\beta_2 = \beta_2^0$ then the preceding is changed by replacing $\mathbf{R}\beta - \mathbf{q} = \mathbf{0}$ with $\mathbf{R}\beta - \beta_2^0 = \mathbf{0}$. Thus, $\mathbf{Rb} - \mathbf{q} = \mathbf{b}_2 - \beta_2^0$. Then, the constrained estimator is

$$\mathbf{b}_* = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} - \begin{bmatrix} -(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2(\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1} \\ (\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1} \end{bmatrix}(\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)(\mathbf{b}_2 - \beta_2^0)$$

or

$$\mathbf{b}_* = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} + \begin{bmatrix} (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2(\mathbf{b}_2 - \beta_2^0) \\ (\beta_2^0 - \mathbf{b}_2) \end{bmatrix}$$

Using the result of the previous paragraph, we can rewrite the first part as

$$\mathbf{b}_{1*} = (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{y} - (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2\beta_2^{\mathbf{0}} = (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'(\mathbf{y} - \mathbf{X}_2\beta_2^0)$$

which was to be shown.

4. By factoring the result in (5-14), we obtain $\mathbf{b}_* = [\mathbf{I} - \mathbf{CR}]\mathbf{b} + \mathbf{w}$ where $\mathbf{C} = (\mathbf{X'X})^{-1}\mathbf{R}'[\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{R}']^{-1}$ and $\mathbf{w} = \mathbf{Cq}$. The covariance matrix of the least squares estimator is

$$\text{Var}[\mathbf{b}_*] = [\mathbf{I} - \mathbf{CR}]\sigma^2(\mathbf{X'X})^{-1}[\mathbf{I} - \mathbf{CR}]'$$
$$= \sigma^2(\mathbf{X'X})^{-1} + \sigma^2\mathbf{CR}(\mathbf{X'X})^{-1}\mathbf{R'C'} - \sigma^2\mathbf{CR}(\mathbf{X'X})^{-1} - \sigma^2(\mathbf{X'X})^{-1}\mathbf{R'C'}.$$

By multiplying it out, we find that $\mathbf{CR}(\mathbf{X'X})^{-1} = (\mathbf{X'X})^{-1}\mathbf{R}'(\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{R}')^{-1}\mathbf{R}(\mathbf{X'X})^{-1} = \mathbf{CR}(\mathbf{X'X})^{-1}\mathbf{R'C'}$
so $\text{Var}[\mathbf{b}_*] = \sigma^2(\mathbf{X'X})^{-1} - \sigma^2\mathbf{CR}(\mathbf{X'X})^{-1}\mathbf{R'C'} = \sigma^2(\mathbf{X'X})^{-1} - \sigma^2(\mathbf{X'X})^{-1}\mathbf{R}'[\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{R}']^{-1}\mathbf{R}(\mathbf{X'X})^{-1}$
This may also be written as $\text{Var}[\mathbf{b}_*] = \sigma^2(\mathbf{X'X})^{-1}\{\mathbf{I} - \mathbf{R}'(\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{R}')^{-1}\mathbf{R}(\mathbf{X'X})^{-1}\}$

$$= \sigma^2(\mathbf{X'X})^{-1}\{[\sigma^2(\mathbf{X'X})^{-1}]^{-1} - \mathbf{R}'[\mathbf{R}\sigma^2(\mathbf{X'X})^{-1}\mathbf{R}']^{-1}\mathbf{R}\}\sigma^2(\mathbf{X'X})^{-1}$$

Since $\text{Var}[\mathbf{Rb}] = \mathbf{R}\sigma^2(\mathbf{X'X})^{-1}\mathbf{R}'$ this is the answer we seek.

5. The variance of the restricted least squares estimator is given in the second equation in the previous exercise. We know that this matrix is positive definite, since it is derived in the form $\mathbf{B}'\sigma^2(\mathbf{X'X})^{-1}\mathbf{B}'$, and $\sigma^2(\mathbf{X'X})^{-1}$ is positive definite. Therefore, it remains to show only that the matrix subtracted from Var[$\mathbf{b}$] to obtain Var[$\mathbf{b}_*$] is positive definite. Consider, then, a quadratic form in Var[$\mathbf{b}_*$]

$$\mathbf{z}'\text{Var}[\mathbf{b}_*]\mathbf{z} = \mathbf{z}'\text{Var}[\mathbf{b}]\mathbf{z} - \sigma^2\mathbf{z}'(\mathbf{X'X})^{-1}(\mathbf{R}'[\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{R}']^{-1}\mathbf{R})(\mathbf{X'X})^{-1}\mathbf{z}$$
$$= \mathbf{z}'\text{Var}[\mathbf{b}]\mathbf{z} - \mathbf{w}'[\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{R}']^{-1}\mathbf{w} \quad \text{where } \mathbf{w} = \sigma\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{z}.$$

It remains to show, therefore, that the inverse matrix in brackets is positive definite. This is obvious since its inverse is positive definite. This shows that every quadratic form in Var[$\mathbf{b}_*$] is less than a quadratic form in Var[$\mathbf{b}$] in the same vector.

6. The result follows immediately from the result which precedes (5-19). Since the sum of squared residuals must be at least as large, the coefficient of determination, $COD = 1 - \text{sum of squares} / \Sigma_i (y_i - \bar{y})^2$,
must be no larger.

7. For convenience, let $\mathbf{F} = [\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{R}']^{-1}$. Then, $\lambda = \mathbf{F}(\mathbf{Rb} - \mathbf{q})$ and the variance of the vector of Lagrange multipliers is $\text{Var}[\lambda] = \mathbf{FR}\sigma^2(\mathbf{X'X})^{-1}\mathbf{R'F} = \sigma^2\mathbf{F}$. The estimated variance is obtained by replacing σ^2 with s^2. Therefore, the chi-squared statistic is

$$\chi^2 = (\mathbf{Rb} - \mathbf{q})'\mathbf{F}'(s^2\mathbf{F})^{-1}\mathbf{F}(\mathbf{Rb} - \mathbf{q}) = (\mathbf{Rb} - \mathbf{q})'[(1/s^2)\mathbf{F}](\mathbf{Rb} - \mathbf{q})$$
$$= (\mathbf{Rb} - \mathbf{q})'[\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{R}']^{-1}(\mathbf{Rb} - \mathbf{q})/[\mathbf{e'e}/(n - K)]$$

This is exactly J times the F statistic defined in (5-19) and (5-20). Finally, J times the F statistic in (5-20) equals the expression given above.

8. We use (5-19) to find the new sum of squares. The change in the sum of squares is

$$\mathbf{e}_*'\mathbf{e}_* - \mathbf{e'e} = (\mathbf{Rb} - \mathbf{q})'[\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{R}']^{-1}(\mathbf{Rb} - \mathbf{q})$$

For this problem, $(\mathbf{Rb} - \mathbf{q}) = b_2 + b_3 - 1 = .3$. The matrix inside the brackets is the sum of the 4 elements in the lower right block of $(\mathbf{X'X})^{-1}$. These are given in Exercise 1, multiplied by $s^2 = 20$. Therefore, the required sum is $[\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{R}'] = (1/20)(.410 + .256 - 2(.051)) = .028$. Then, the change in the sum of squares is $.3^2 / .028 = 3.215$. Thus, $\mathbf{e'e} = 520$, $\mathbf{e}_*'\mathbf{e}_* = 523.215$, and the chi-squared statistic is $26[523.215/520 - 1] = .16$. This is quite small, and would not lead to rejection of the hypothesis. Note that for a single restriction, the Lagrange multiplier statistic is equal to the F statistic which equals, in turn, the square of the t statistic used to test the restriction. Thus, we could have obtained this quantity by squaring the .399 found in the first problem (apart from some rounding error).

9. First, use (5-19) to write $\mathbf{e}_*'\mathbf{e}_* = \mathbf{e'e} + (\mathbf{Rb} - \mathbf{q})'[\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{R}']^{-1}(\mathbf{Rb} - \mathbf{q})$. Now, the result that $E[\mathbf{e'e}] = (n - K)\sigma^2$ obtained in Chapter 6 must hold here, so $E[\mathbf{e}_*'\mathbf{e}_*] = (n - K)\sigma^2 + E[(\mathbf{Rb} - \mathbf{q})'[\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{R}']^{-1}(\mathbf{Rb} - \mathbf{q})]$.
Now, $\mathbf{b} = \beta + (\mathbf{X'X})^{-1}\mathbf{X}'\varepsilon$, so $\mathbf{Rb} - \mathbf{q} = \mathbf{R}\beta - \mathbf{q} + \mathbf{R}(\mathbf{X'X})^{-1}\mathbf{X}'\varepsilon$. But, $\mathbf{R}\beta - \mathbf{q} = \mathbf{0}$, so under the hypothesis, $\mathbf{Rb} - \mathbf{q} = \mathbf{R}(\mathbf{X'X})^{-1}\mathbf{X}'\varepsilon$. Insert this in the result above to obtain
$E[\mathbf{e}_*'\mathbf{e}_*] = (n-K)\sigma^2 + E[\varepsilon'\mathbf{X}(\mathbf{X'X})^{-1}\mathbf{R}'[\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{R}']^{-1}\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{X}'\varepsilon]$. The quantity in square brackets is a scalar, so it is equal to its trace. Permute $\varepsilon'\mathbf{X}(\mathbf{X'X})^{-1}\mathbf{R}'$ in the trace to obtain

$$E[\mathbf{e}_*'\mathbf{e}_*] = (n - K)\sigma^2 + E[tr\{[\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{R}']^{-1}\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{X}'\varepsilon\varepsilon'\mathbf{X}(\mathbf{X'X})^{-1}\mathbf{R}']\}$$

We may now carry the expectation inside the trace and use $E[\varepsilon\varepsilon'] = \sigma^2\mathbf{I}$ to obtain

$$E[\mathbf{e}_*'\mathbf{e}_*] = (n - K)\sigma^2 + \text{tr}\{[\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{R}']^{-1}\mathbf{R}(\mathbf{X'X})^{-1}\mathbf{X}'\sigma^2\mathbf{I}\mathbf{X}(\mathbf{X'X})^{-1}\mathbf{R}']\}$$

Carry the σ^2 outside the trace operator, and after cancellation of the products of matrices times their inverses, we obtain $E[\mathbf{e}_*'\mathbf{e}_*] = (n - K)\sigma^2 + \sigma^2\text{tr}[\mathbf{I}_J] = (n - K + J)\sigma^2$.

10. Show that in the multiple regression of $\mathbf{y}$ on a constant, $\mathbf{x}_1$, and $\mathbf{x}_2$, while imposing the restriction $\beta_1 + \beta_2 = 1$ leads to the regression of $\mathbf{y} - \mathbf{x}_1$ on a constant and $\mathbf{x}_2 - \mathbf{x}_1$.

For convenience, we put the constant term last instead of first in the parameter vector. The constraint is $\mathbf{Rb} - \mathbf{q} = \mathbf{0}$ where $\mathbf{R} = [1\ \ 1\ \ 0]$ so $\mathbf{R}_1 = [1]$ and $\mathbf{R}_2 = [1,0]$. Then, $\beta_1 = [1]^{-1}[1 - \beta_2] = 1 - \beta_2$. Thus, $\mathbf{y} = (1 - \beta_2)\mathbf{x}_1 + \beta_2\mathbf{x}_2 + \alpha\mathbf{i} + \boldsymbol{\varepsilon}$ or $\mathbf{y} - \mathbf{x}_1 = \beta_2(\mathbf{x}_2 - \mathbf{x}_1) + \alpha\mathbf{i} + \boldsymbol{\varepsilon}$.

Applications

```
?=======================================================================
? Application 5.1 Wage Equation
?=======================================================================
Read;File="F:\Text-Revision\edition6\Solutions-and-Applications\time_var.dat";
nvar=5;nobs=17919$
? This creates the group count variable.
Regress ; Lhs = one ; Rhs = one ; Str = ID ; Panel $
? This READ merges the smaller file into the larger one.
Read;File="F:\Text-Revision\edition6\Solutions-and-Applications\time_invar.dat";
names=ability,med,fed,bh,sibs? ; group=_groupti ;nvar=5;nobs=2178$
Names=id,educ,lwage,pexp,t;
namelist ; x1=one,educ,pexp,ability$
namelist ; x2=med,fed,bh,sibs$
?=======================================================================
? a.  Long regression
?=======================================================================
regress ; lhs= lwage ; rhs = x1,x2 $
+----------------------------------------------------+
| Ordinary    least squares regression               |
| LHS=LWAGE    Mean                 =   2.296821     |
|              Standard deviation   =   .5282364     |
| WTS=none     Number of observs.   =      17919     |
| Model size   Parameters           =          8     |
|              Degrees of freedom   =      17911     |
| Residuals    Sum of squares       =   4119.734     |
|              Standard error of e  =   .4795950     |
| Fit          R-squared            =   .1760081     |
|              Adjusted R-squared   =   .1756861     |
| Model test   F[  7, 17911] (prob) = 546.55 (.0000) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|     .98965433       .03389449    29.198   .0000
 EDUC    |     .07118866       .00225722    31.538   .0000   12.6760422
 PEXP    |     .03951038       .00089858    43.970   .0000   8.36268765
 ABILITY |     .07736880       .00493359    15.682   .0000    .05237402
 MED     |    .709887D-04      .00169543      .042   .9666   11.4719013
 FED     |     .00531681       .00133795     3.974   .0001   11.7092472
 BH      |    -.05286954       .00999042    -5.292   .0000    .15385903
 SIBS    |     .00487138       .00179116     2.720   .0065   3.15620291
?=======================================================================
? b.  F test
?=======================================================================
Calc ; list ; fstat = Rsqrd/(kreg-1)/((1-rsqrd)/(n-kreg)) $
+-----------------------------------+
 FSTAT   =     14.025040
Calc ; r1 = rsqrd ; df1=n-kreg$
Matrix ; b1 = b ; v1 = varb $
Matrix ; b1 =b1(5:8) ; v1=varb(5:8,5:8)$
Regress ; lhs = lwage ; rhs = x1 $
+----------------------------------------------------+
```

```
| Ordinary    least squares regression                    |
| LHS=LWAGE    Mean                  =   2.296821     |
|              Standard deviation   =   .5282364     |
| WTS=none     Number of observs.   =      17919     |
| Model size   Parameters           =          4     |
|              Degrees of freedom   =      17915     |
| Residuals    Sum of squares       =   4132.637     |
|              Standard error of e  =   .4802919     |
| Fit          R-squared            =   .1734272     |
|              Adjusted R-squared   =   .1732888     |
| Model test   F[  3, 17915] (prob) =1252.94 (.0000) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    1.02722913       .03004146    34.194   .0000
 EDUC    |     .07376210       .00221425    33.312   .0000   12.6760422
 PEXP    |     .03948955       .00089835    43.958   .0000   8.36268765
 ABILITY |     .08289072       .00459996    18.020   .0000    .05237402
?=====================================================================
? c.  F test for hypothesis that coefficients on X2 are zero
?=====================================================================
Calc ; list ; fstat = (r1-rsqrd)/(col(x2))/((1-r1)/(df1)) $
+------------------------------------+
 FSTAT   =     14.025040
?=====================================================================
? c.  Wald test for hypothesis that coefficients on X2 are zero
?=====================================================================
Matrix ; List ; Wald = b1'<v1>b1 $
Matrix WALD     has  1 rows and  1 columns.
               1
        +--------------
       1|   56.10016
Note Wald = 4*F, as expected.

?=====================================================================
? Application 5.2 Translog Cost Function
?=====================================================================
? First prepare the data
?
Create ; lpk=log(pk);lpl=log(pl);lpf=log(pf)$
create ; lpk2=.5*lpk^2 ; lpl2=.5*lpl^2 ; lpf2=.5*lpf^2$
Create ; lpkf=lpk*lpf ; lplf=lpl*lpf ; lpkl=lpk*lpl $
Create ; lq = log(q) ; lq2 = .5*lq^2 $
Create ; lqk=lq*lpk ; lql=lq*lpl ; lqf=lq*lpf $
Create ; lc = log(cost) $
Create ; lcpf = log(cost/pf) $
Create ; lpkpf=log(pk/pf) ; lplpf=log(pl/pf) $
Create ; lpkpf2=.5*lpkpf^2 ; lplpf2=.5*lplpf^2 ; lplfpkf=lplpf*lpkpf $
Create ; lqlpkf=lq*lpkpf ; lqlplf=lq*lplf $
?=====================================================================
? a.  Beta is a,b,dk,dl,df,pkk,pll,pff,pkl,pkf,plf,c,tqk,tql,tqf
?=====================================================================
Restrictions are
            0,0,1,1,1,0,0,0,0,0,0,0,0,0,0      1
            0,0,0,0,0,1,0,0,1,1,0,0,0,0,0      0
    R =     0,0,0,0,0,0,1,0,1,0,1,0,0,0,0  q = 0
            0,0,0,0,0,0,0,1,0,1,1,0,0,0,0      0
            0,0,0,0,0,0,0,0,0,0,0,0,1,1,1      0
?=====================================================================
? b. Testing the theory
?=====================================================================
Namelist ; X1=one,lq,lpk,lpl,lpf,lpk2,lpl2,lpf2,lpkl,lpkf,lplf,lq2,lqk,lq...
Namelist ; X0=one,lq,lpkf,lplf,lpkpf2,lplpf2,lplfpkf,lq2,lqlpkf,lqlplf$
Regress ; lhs = lc ; rhs=x0 $

+----------------------------------------------------+
```

```
| Ordinary    least squares regression               |
| LHS=LC       Mean                 =   3.071619     |
|              Standard deviation   =   1.542734     |
| WTS=none     Number of observs.   =        158     |
| Model size   Parameters           =         10     |
|              Degrees of freedom   =        148     |
| Residuals    Sum of squares       =   2.634416     |
|              Standard error of e  =   .1334170     |
| Fit          R-squared            =   .9929498     |
|              Adjusted R-squared   =   .9925211     |
| Model test   F[  9,   148] (prob) =2316.03 (.0000) |
+----------------------------------------------------+
+--------+-------------+---------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+-------------+---------------+--------+--------+----------+
 Constant|   -1.13340208      1.04296294    -1.087   .2789
 LQ      |     .02244828       .12717485      .177   .8601   8.26548908
 LPKF    |    -.02309567       .14153592     -.163   .8706   14.4192992
 LPLF    |    -.01690697       .09185395     -.184   .8542   30.4387314
 LPKPF2  |    -.04730093       .21017152     -.225   .8222    .42211776
 LPLPF2  |    -.03419034       .06850142     -.499   .6184   15.6173009
 LPLFPKF |    -.00741233       .11649585     -.064   .9494   4.84868706
 LQ2     |     .05544306       .00446607    12.414   .0000   35.7912728
 LQLPKF  |     .03562155       .02862683     1.244   .2153   7.15696461
 LQLPLF  |     .01279036       .00375187     3.409   .0008   251.570118
Calc ; ee0 = sumsqdev $
Regress ; lhs = lcpf ; rhs = x1 $
+----------------------------------------------------+
| Ordinary    least squares regression               |
| LHS=LCPF     Mean                 =  -.3195570     |
|              Standard deviation   =   1.542364     |
| WTS=none     Number of observs.   =        158     |
| Model size   Parameters           =         15     |
|              Degrees of freedom   =        143     |
| Residuals    Sum of squares       =   2.464348     |
|              Standard error of e  =   .1312753     |
| Fit          R-squared            =   .9934018     |
|              Adjusted R-squared   =   .9927558     |
| Model test   F[ 14,   143] (prob) =1537.82 (.0000) |
+----------------------------------------------------+
+--------+-------------+---------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+-------------+---------------+--------+--------+----------+
 Constant|   -76.2592615      38.2800363    -1.992   .0483
 LQ      |   -1.08042535       .37554512    -2.877   .0046   8.26548908
 LPK     |    6.38079702      4.52920686     1.409   .1611   4.25096457
 LPL     |    14.7182926      7.08482345     2.077   .0395   8.97279814
 LPF     |   -1.89473291      2.84231282     -.667   .5061   3.39117564
 LPK2    |    -.32741427       .44070869     -.743   .4587   9.05539681
 LPL2    |   -1.53852735       .69240298    -2.222   .0279   40.2700121
 LPF2    |    -.07350556       .18203881     -.404   .6870   5.78602018
 LPKL    |    -.57205049       .37189026    -1.538   .1262   38.1346773
 LPKF    |    -.02402470       .24632928     -.098   .9224   14.4192992
 LPLF    |     .16228289       .27007181      .601   .5489   30.4387314
 LQ2     |     .05297849       .00471336    11.240   .0000   35.7912728
 LQK     |     .04014440       .02979137     1.348   .1799   35.1677247
 LQL     |     .13104059       .03828401     3.423   .0008   74.2063474
 LQF     |     .05865220       .02554928     2.296   .0232   28.0107601
Calc ; ee1 = sumsqdev $
Calc ; list ; Fstat = ((ee0 - ee1)/5)/(ee1/(158-15))$
+------------------------------------+
FSTAT   =      1.973714
--> Calc ; list ; ftb(.95,5,143)$
+------------------------------------+
Result  =      2.277490
The F statistic is small; the theory is not rejected.
```

```
?=======================================================================
? c. Testing homotheticity
?=======================================================================
+----------------------------------------------------+
| Ordinary    least squares regression               |
| LHS=LCPF     Mean                 =  -.3195570     |
|              Standard deviation   =   1.542364     |
| WTS=none     Number of observs.   =        158     |
| Model size   Parameters           =         10     |
|              Degrees of freedom   =        148     |
| Residuals    Sum of squares       =   2.634223     |
|              Standard error of e  =   .1334121     |
| Fit          R-squared            =   .9929469     |
|              Adjusted R-squared   =   .9925180     |
| Model test   F[  9,   148] (prob) =2315.08 (.0000) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|   -2.78239562      1.04292476    -2.668   .0085
 LQ      |     .01362521       .12717020      .107   .9148   8.26548908
 LPKF    |    -.06044098       .14153074     -.427   .6700   14.4192992
 LPLF    |    -.07639000       .09185059     -.832   .4069   30.4387314
 LPKPF2  |    -.10507269       .21016383     -.500   .6178    .42211776
 LPLPF2  |    -.00146323       .06849891     -.021   .9830   15.6173009
 LPLFPKF |     .01806822       .11649158      .155   .8770   4.84868706
 LQ2     |     .05565578       .00446590    12.462   .0000   35.7912728
 LQLPKF  |     .03824257       .02862578     1.336   .1836   7.15696461
 LQLPLF  |     .01296202       .00375173     3.455   .0007   251.570118
Regress ; lhs = lcpf ; Rhs = x0 ; cls:b(9)=0,b(10)=0$
+----------------------------------------------------+
| Linearly restricted regression                     |
| Ordinary    least squares regression               |
| LHS=LCPF     Mean                 =  -.3195570     |
|              Standard deviation   =   1.542364     |
| WTS=none     Number of observs.   =        158     |
| Model size   Parameters           =          8     |
|              Degrees of freedom   =        150     |
| Residuals    Sum of squares       =   2.896172     |
|              Standard error of e  =   .1389526     |
| Fit          R-squared            =   .9922456     |
|              Adjusted R-squared   =   .9918837     |
| Model test   F[  7,   150] (prob) =2741.96 (.0000) |
| Restrictns.  F[  2,   148] (prob) =   7.36 (.0009) |
| Not using OLS or no constant. Rsqd & F may be < 0. |
| Note, with restrictions imposed,  Rsqd may be < 0. |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|   -6.20547247       .37175165   -16.693   .0000
 LQ      |     .40111764       .03208201    12.503   .0000   8.26548908
 LPKF    |    -.05918207       .14502101     -.408   .6838   14.4192992
 LPLF    |     .03234530       .08668866      .373   .7096   30.4387314
 LPKPF2  |    -.20340518       .21249945     -.957   .3400    .42211776
 LPLPF2  |    -.00516132       .06888408     -.075   .9404   15.6173009
 LPLFPKF |     .08684971       .10534811      .824   .4110   4.84868706
 LQ2     |     .06103878       .00440807    13.847   .0000   35.7912728
 LQLPKF  |   -.138778D-16     .517639D-09      .000  1.0000   7.15696461
 LQLPLF  |       .000000     .915064D-10      .000  1.0000   251.570118
Calc ; list ; ftb(.95,2,148)$
+------------------------------------+
Result  =      3.057197
```

The F statistic of 7.36 is larger than the critical value of 3.057. The hypothesis is rejected.

```
?======================================================================
? d.  Testing generalized Cobb-Douglas against full translog.
?======================================================================
Regress ; lhs = lcpf ; rhs = x0 ;cls:b(5)=0,b(6)=0,b(7)=0,b(9)=0,b(10)=0$
+----------------------------------------------------+
| Linearly restricted regression                     |
| Ordinary    least squares regression               |
| LHS=LCPF     Mean                 =  -.3195570     |
|              Standard deviation   =   1.542364     |
| WTS=none     Number of observs.   =        158     |
| Model size   Parameters           =          5     |
|              Degrees of freedom   =        153     |
| Residuals    Sum of squares       =   3.191949     |
|              Standard error of e  =   .1444383     |
| Fit          R-squared            =   .9914536     |
|              Adjusted R-squared   =   .9912302     |
| Model test   F[  4,   153] (prob) =4437.33 (.0000) |
| Restrictns.  F[  5,   148] (prob) =   6.27 (.0000) |
| Not using OLS or no constant. Rsqd & F may be < 0. |
| Note, with restrictions imposed,  Rsqd may be < 0. |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|   -5.07718678       .18072495   -28.093   .0000
 LQ      |     .41724916       .03285950    12.698   .0000   8.26548908
 LPKF    |     .00903097       .01466874      .616   .5391   14.4192992
 LPLF    |    -.03131901       .00770196    -4.066   .0001   30.4387314
 LPKPF2  |   -.582867D-15    .127559D-07      .000  1.0000    .42211776
 LPLPF2  |   -.328730D-15    .986857D-08      .000  1.0000   15.6173009
 LPLFPKF |    .461436D-15    .201473D-07      .000  1.0000   4.84868706
 LQ2     |     .05956626       .00452575    13.162   .0000   35.7912728
 LQLPKF  |   -.555112D-16    .538074D-09      .000  1.0000   7.15696461
 LQLPLF  |   -.693889D-17    .223074D-09      .000  1.0000   251.570118

Calc ; list ; ftb(.95,5,148)$
+------------------------------------+
| Listed Calculator Results          |
+------------------------------------+
 Result  =      2.275319
```

The F statistic of 6.27 is larger than the critical value of 2.275. The hypothesis is rejected.

```
?======================================================================
? e.  Testing Cobb-Douglas against full translog.
?======================================================================
Matrix ; b2=b(5:10) ; v2=varb(5:10,5:10) $
Matrix ; list ; Fcd = 1/6 * b2'<v2>b2 $
Matrix FCD      has  1 rows and  1 columns.
               1
        +--------------
       1|   28.87144
Calc ; list ; ftb(.95,6,148)$
+------------------------------------+
| Listed Calculator Results          |
+------------------------------------+
 Result  =      2.160352
```

The F statistic of 28.871 is larger than the critical value of 2.16. The hypothesis is rejected.

```
?======================================================================
? f. Testing generalized Cobb-Douglas against homothetic translog.
?======================================================================
Regress ; Lhs = lcpf ; rhs = one,lq,lpkf,lplf,lpkpf2,lplpf2,lplfpkf,lq2
        ; cls:b(5)=0,b(6)=0,b(7)=0$
+----------------------------------------------------+
| Linearly restricted regression                     |
```

```
| Ordinary    least squares regression                |
| LHS=LCPF     Mean                 =  -.3195570      |
|              Standard deviation   =   1.542364      |
| WTS=none     Number of observs.   =        158      |
| Model size   Parameters           =          5      |
|              Degrees of freedom   =        153      |
| Residuals    Sum of squares       =   3.191949      |
|              Standard error of e  =   .1444383      |
| Fit          R-squared            =   .9914536      |
|              Adjusted R-squared   =   .9912302      |
| Model test   F[  4,   153] (prob) =4437.33 (.0000) |
| Restrictns.  F[  3,   150] (prob) =   5.11 (.0022) |
| Not using OLS or no constant. Rsqd & F may be < 0. |
| Note, with restrictions imposed,  Rsqd may be < 0. |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|   -5.07718678       .18072495    -28.093   .0000
 LQ      |     .41724916       .03285950     12.698   .0000   8.26548908
 LPKF    |     .00903097       .01466874       .616   .5391   14.4192992
 LPLF    |    -.03131901       .00770196     -4.066   .0001   30.4387314
 LPKPF2  |   -.199840D-14    .243505D-07       .000  1.0000    .42211776
 LPLPF2  |   -.746798D-15    .608762D-08       .000  1.0000   15.6173009
 LPLFPKF |    .140166D-14    .121752D-07       .000  1.0000   4.84868706
 LQ2     |     .05956626       .00452575     13.162   .0000   35.7912728

Calc ; list ; ftb(.95,3,150) $
+------------------------------------+
| Listed Calculator Results          |
+------------------------------------+
 Result  =      2.664907
?
?=========================================================================
? g.  We have not rejected the theory, but we have rejected all the
? functional forms
? except the nonhomothetic translog.  Just like Christensen and Greene.
?=========================================================================


?=========================================================================
? Application 5.3 Nonlinear restrictions
?=========================================================================
sample;1-52$
name;x=one,logpg,logi,logpnc,logpuc,logppt,t,logpd,logpn,logps$
?=========================================================================
? a.  Simple hypothesis test
?=========================================================================
Regr;lhs=logg;rhs=x$
+----------------------------------------------------+
| Ordinary    least squares regression                |
| LHS=LOGG     Mean                 =   1.570475      |
|              Standard deviation   =   .2388115      |
| WTS=none     Number of observs.   =         52      |
| Model size   Parameters           =         10      |
|              Degrees of freedom   =         42      |
| Residuals    Sum of squares       =   .3812817E-01 |
|              Standard error of e  =   .3012994E-01 |
| Fit          R-squared            =   .9868911      |
|              Adjusted R-squared   =   .9840821      |
| Model test   F[  9,    42] (prob) = 351.33 (.0000) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|   -7.28719016      2.52056245     -2.891   .0061
 LOGPG   |     .06051812       .05401018      1.120   .2689   3.72930296
```

LOGI	|	.99299135	.25037574	3.966	.0003	9.67214751
LOGPNC	|	-.15471632	.26696298	-.580	.5653	4.38036654
LOGPUC	|	-.48909058	.08519952	-5.741	.0000	4.10544881
LOGPPT	|	.01926966	.13644891	.141	.8884	4.14194132
T	|	.03797198	.00751371	5.054	.0000	26.5000000
LOGPD	|	1.73205775	.25988611	6.665	.0000	4.23906603
LOGPN	|	-.72953933	.26506853	-2.752	.0087	4.23689080
LOGPS	|	-.86798166	.35291106	-2.459	.0181	4.17535768

```
Calc;r1=rsqrd$
Regr;lhs=logg;rhs=one,logpg,logi,logpnc,logpuc,logppt,t$
+-----------------------------------------------+
| Ordinary    least squares regression           |
| LHS=LOGG     Mean                 =   1.570475   |
|              Standard deviation   =   .2388115   |
| WTS=none     Number of observs.   =         52   |
| Model size   Parameters           =          7   |
|              Degrees of freedom   =         45   |
| Residuals    Sum of squares       =   .1014368   |
|              Standard error of e  =   .4747790E-01 |
| Fit          R-squared            =   .9651249   |
|              Adjusted R-squared   =   .9604749   |
| Model test   F[  6,    45] (prob) = 207.55 (.0000) |
+-----------------------------------------------+
```

Variable	Coefficient	Standard Error	t-ratio	P[|T|>t]	Mean of X
Constant	-13.1396625	2.09171186	-6.282	.0000	
LOGPG	-.05373342	.04251099	-1.264	.2127	3.72930296
LOGI	1.64909204	.20265477	8.137	.0000	9.67214751
LOGPNC	-.03199098	.20574296	-.155	.8771	4.38036654
LOGPUC	-.07393002	.10548982	-.701	.4870	4.10544881
LOGPPT	-.06153395	.12343734	-.499	.6206	4.14194132
T	-.01287615	.00525340	-2.451	.0182	26.5000000

```
Calc;r0=rsqrd$
Calc;list;f=((r1-r0)/2)/((1-r1)/(n-10))$
+------------------------------------+
| Listed Calculator Results          |
+------------------------------------+
 F       =     34.868735
```

The critical value from the *F* table is 2.827, so we would reject the hypothesis.

```
?======================================================================
? b.  Nonlinear restriction
?======================================================================
```

Since the restricted model is quite nonlinear, it would be quite cumbersome to estimate and examine the loss in fit. We can test the restriction using the unrestricted model. For this problem,

$$\mathbf{f} = [\gamma_{nc} - \gamma_{uc}, \gamma_{nc}\delta_s - \gamma_{pt}\delta_d]'$$

The matrix of derivatives, using the order given above and " to represent the entire parameter vector, is

$$\mathbf{G} = \begin{bmatrix} \partial f_1 / \partial \boldsymbol{\alpha} \\ \partial f_2 / \partial \boldsymbol{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_s & 0 & -\delta_d & 0 & -\gamma_{pt} & 0 & \gamma_{nc} \end{bmatrix}.$$

The parameter estimates are

Thus, $\mathbf{f} = [-.17399, .10091]'$. The covariance matrix to use for the tests is $\mathbf{G}s^2(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}'$

The statistic for the joint test is $\chi^2 = \mathbf{f}'[\mathbf{G}s^2(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}\mathbf{f} = .4772$. This is less than the critical value for a chi-squared with two degrees of freedom, so we would not reject the joint hypothesis. For the individual hypotheses,

we need only compute the equivalent of a *t* ratio for each element of **f**. Thus,

$$z_1 = -.6053$$

and $$z_2 = .2898$$

Neither is large, so neither hypothesis would be rejected. (Given the earlier result, this was to be expected.)

```
?=========================================================================
? c.  Computations for nonlinear restriction
?=========================================================================
sample;1-52$
name;x=one,logpg,logi,logpnc,logpuc,logppt,t,logpd,logpn,logps$
Regr;lhs=logg;rhs=x$
+-----------------------------------------------------+
| Ordinary    least squares regression                |
| LHS=LOGG     Mean                 =   1.570475      |
|              Standard deviation   =   .2388115      |
| WTS=none     Number of observs.   =         52      |
| Model size   Parameters           =          7      |
|              Degrees of freedom   =         45      |
| Residuals    Sum of squares       =   .1014368      |
|              Standard error of e  =   .4747790E-01 |
| Fit          R-squared            =   .9651249      |
|              Adjusted R-squared   =   .9604749      |
| Model test   F[  6,    45] (prob) = 207.55 (.0000) |
+-----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|   -13.1396625      2.09171186    -6.282   .0000
 LOGPG   |    -.05373342       .04251099    -1.264   .2127   3.72930296
 LOGI    |    1.64909204       .20265477     8.137   .0000   9.67214751
 LOGPNC  |    -.03199098       .20574296     -.155   .8771   4.38036654
 LOGPUC  |    -.07393002       .10548982     -.701   .4870   4.10544881
 LOGPPT  |    -.06153395       .12343734     -.499   .6206   4.14194132
 T       |    -.01287615       .00525340    -2.451   .0182   26.5000000
Calc;r1=rsqrd$
Regr;lhs=logg;rhs=one,logpg,logi,logpnc,logpuc,logppt,t$
+-----------------------------------------------------+
| Ordinary    least squares regression                |
| LHS=LOGG     Mean                 =   1.570475      |
|              Standard deviation   =   .2388115      |
| WTS=none     Number of observs.   =         52      |
| Model size   Parameters           =          7      |
|              Degrees of freedom   =         45      |
| Residuals    Sum of squares       =   .1014368      |
|              Standard error of e  =   .4747790E-01 |
| Fit          R-squared            =   .9651249      |
|              Adjusted R-squared   =   .9604749      |
| Model test   F[  6,    45] (prob) = 207.55 (.0000) |
+-----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|   -13.1396625      2.09171186    -6.282   .0000
 LOGPG   |    -.05373342       .04251099    -1.264   .2127   3.72930296
 LOGI    |    1.64909204       .20265477     8.137   .0000   9.67214751
 LOGPNC  |    -.03199098       .20574296     -.155   .8771   4.38036654
 LOGPUC  |    -.07393002       .10548982     -.701   .4870   4.10544881
 LOGPPT  |    -.06153395       .12343734     -.499   .6206   4.14194132
 T       |    -.01287615       .00525340    -2.451   .0182   26.5000000
Calc;r0=rsqrd$
Calc;list;fstat=((r1-r0)/2)/((1-r1)/(n-10))$
+----------------------------------+
 FSTAT   =     34.868735
Calc;list;ftb(.95,3,42)$
+----------------------------------+
 Result  =      2.827049

REGR;Lhs=logg;rhs=x$
Calc ; ds=b(10);dd=-b(8);gpt=-b(6);gnc=b(4)$
Matr;gm=[0,0,0,1,-1,0,0,0,0,0 / 0,0,0,ds,0,dd,0,gpt,0,gnc]$
Calc;f1=b(4)-b(6) ; f2=b(4)*b(10)-b(6)*b(8)$
Matrix;list;f=[f1/f2]$
```

```
Matrix F         has  2 rows and  1 columns.
               1
        +--------------
       1|    -.17399
       2|     .10091
Matrix;list;vf=gm*varb*gm'$
Matrix VF        has  2 rows and  2 columns.
               1             2
        +----------------------------
       1|     .08263      -.08059
       2|    -.08059       .12129
Matrix;list;Wald=f'<vf>f$
Matrix WALD      has  1 rows and  1 columns.
               1
        +--------------
       1|     .47716
Calc;list;z1=f(1)/sqr(vf(1,1))$
+------------------------------------+
 Z1      =      -.605278
Calc;list;z2=f(2)/sqr(vf(2,2))$
+------------------------------------+
 Z2      =       .289760
```