Chapter 6

Functional Form and Structural Change

Exercises

1. T he F statistic could be computed as

F = {[1425 - (104 + 88 + ... + 211)] / (70 - 16)}/[(104 + 88 + ... + 211) / (570 - 70)] = 1.343

The 95% critical value for the F distribution with 54 and 500 degrees of freedom is 1.363.

2. a. Using the hint, we seek the c_* which is the slope on **d** in the regression of **q** = **y** - c**d** - **e** on **y** and **d**. The regression coefficients are $\begin{bmatrix} \mathbf{y}'\mathbf{y} & \mathbf{y}'\mathbf{d} \\ \mathbf{d}'\mathbf{y} & \mathbf{d}'\mathbf{d} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y}'(\mathbf{y} - c\mathbf{d} - \mathbf{e}) \\ \mathbf{d}'(\mathbf{y} - c\mathbf{d} - \mathbf{e}) \end{bmatrix} = \begin{bmatrix} \mathbf{y}'\mathbf{y} & \mathbf{y}'\mathbf{d} \\ \mathbf{d}'\mathbf{y} & \mathbf{d}'\mathbf{d} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y}'\mathbf{y} - c\mathbf{y}'\mathbf{d} - \mathbf{y}'\mathbf{e} \\ \mathbf{d}'\mathbf{y} - c\mathbf{d}'\mathbf{d} - \mathbf{d}'\mathbf{e} \end{bmatrix}$. In the preceding, note that $(\mathbf{y}'\mathbf{y},\mathbf{d}'\mathbf{y})'$ is the first column of the matrix being inverted while $c(\mathbf{y}'\mathbf{d},\mathbf{d}'\mathbf{d})'$ is c times the second. An inverse matrix times the first column of the original matrix is the first column of an identity matrix, and likewise for the second. Also, since **d** was one of the original regressors in (1), $\mathbf{d}'\mathbf{e} = 0$, and, of course, $\mathbf{y}'\mathbf{e} = \mathbf{e}'\mathbf{e}$. If we combine all of these, the coefficient vector is

$$-\begin{pmatrix}1\\0\end{pmatrix} - c\begin{pmatrix}0\\1\end{pmatrix} - \begin{bmatrix} \mathbf{y}'\mathbf{y} & \mathbf{y}'\mathbf{d} \\ \mathbf{d}'\mathbf{y} & \mathbf{d}'\mathbf{d} \end{bmatrix}^{-1} \begin{pmatrix}\mathbf{e}'\mathbf{e}\\0\end{pmatrix} = -\begin{pmatrix}1\\0\end{pmatrix} - c\begin{pmatrix}0\\1\end{pmatrix} - \begin{bmatrix} \mathbf{y}'\mathbf{y} & \mathbf{y}'\mathbf{d} \\ \mathbf{d}'\mathbf{y} & \mathbf{d}'\mathbf{d} \end{bmatrix}^{-1} \begin{pmatrix}1\\0\end{pmatrix}\mathbf{e}'\mathbf{e}.$$

We are interested in the second (lower) of the two coefficients. The matrix product at the end is $\mathbf{e}'\mathbf{e}$ times the first column of the inverse matrix, and we wish to find its second (bottom) element. Therefore, collecting what we have thus far, the desired coefficient is $c_* = -c - \mathbf{e}'\mathbf{e}$ times the off diagonal element in the inverse matrix. The off diagonal element is

$$-\mathbf{d}'\mathbf{y} / [(\mathbf{y}'\mathbf{y})(\mathbf{d}'\mathbf{d}) - (\mathbf{y}'\mathbf{d})^2] = -\mathbf{d}'\mathbf{y} / \{[(\mathbf{y}'\mathbf{y})(\mathbf{d}'\mathbf{d})][1 - (\mathbf{y}'\mathbf{d})^2/[(\mathbf{y}'\mathbf{y})(\mathbf{d}'\mathbf{d})]]\}$$
$$= -\mathbf{d}'\mathbf{y} / [(\mathbf{y}'\mathbf{y})(\mathbf{d}'\mathbf{d})(1 - r_{yd}^2)].$$

Therefore, $$c_* = [(\mathbf{e}'\mathbf{e})(\mathbf{d}'\mathbf{y})] / [(\mathbf{y}'\mathbf{y})(\mathbf{d}'\mathbf{d})(1 - r_{yd}^2)] - c$$

(The two negative signs cancel.) This can be further reduced. Since all variables are in deviation form, $\mathbf{e}'\mathbf{e}/\mathbf{y}'\mathbf{y}$ is $(1 - R^2)$ in the full regression. By multiplying it out, you can show that $\bar{d} = P$ so that

$$\mathbf{d}'\mathbf{d} = \Sigma_i (d_i - P)^2 = nP(1-P)$$

and $$\mathbf{d}'\mathbf{y} = \Sigma_i (d_i - P)(y_i - \bar{y}) = \Sigma_i(d_i - P)y_i = n_1(\bar{y}_1 - \bar{y})$$

where n_1 is the number of observations which have $d_i = 1$. Combining terms once again, we have

$$c_* = \{[n_1(\bar{y}_1 - \bar{y})(1 - R^2)\} / \{nP(1-P)(1 - r_{yd}^2)\} - c$$

Finally, since $P = n_1/n$, this further simplifies to the result claimed in the problem,

$$c_* = \{(\bar{y}_1 - \bar{y})(1 - R^2)\} / \{(1-P)(1 - r_{yd}^2)\} - c$$

The problem this creates for the theory is that in the present setting, if, indeed, c is negative, $(\bar{y}_1 - \bar{y})$ will almost surely be also. Therefore, the sign of c_* is ambiguous.

3. We first find the joint distribution of the observed variables. $\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{bmatrix} \beta & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{pmatrix} x* \\ \varepsilon \\ u \end{pmatrix}$ so $[y,x]$ have a joint normal distribution with mean vector $E\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{bmatrix} \beta & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{pmatrix} \mu* \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha + \beta\mu* \\ \mu* \end{pmatrix}$ and covariance matrix $Var\begin{pmatrix} y \\ x \end{pmatrix} = \begin{bmatrix} \beta & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_*^2 & 0 & 0 \\ 0 & \sigma_\varepsilon^2 & 0 \\ 0 & 0 & \sigma_u^2 \end{bmatrix} \begin{bmatrix} \beta & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \beta^2\sigma_*^2 + \sigma_\varepsilon^2 & \beta\sigma_*^2 \\ \beta\sigma_*^2 & \sigma_*^2 + \sigma_u^2 \end{bmatrix}$, The probability limit of the slope in the linear regression of y on x is, as usual,

$$\text{plim } b = \text{Cov}[y,x]/\text{Var}[x] = \beta/(1 + \sigma_u^2/\sigma_*^2) < \beta.$$

The probability limit of the intercept is plim

$$a = E[y] - (\text{plim } b)E[x] = \alpha + \beta\mu^* - \beta\mu^*/(1 + \sigma_u^2/\sigma_*^2)$$
$$= \alpha + \beta[\mu^*\sigma_u / (\sigma_*^2 + \sigma_u^2)] > \alpha \quad (\text{assuming } \beta > 0).$$

If x is regressed on y instead, the slope will estimate $\text{plim}[b'] = \text{Cov}[y,x]/\text{Var}[y] = \beta\sigma_*^2/(\beta^2\sigma_*^2 + \sigma_\varepsilon^2)$. Then, $\text{plim}[1/b'] = \beta + \sigma_\varepsilon^2/\beta^2\sigma_*^2 > \beta$. Therefore, b and b' will bracket the true parameter (at least in their probability limits). Unfortunately, without more information about σ_u^2, we have no idea how wide this bracket is. Of course, if the sample is large and the estimated bracket is narrow, the results will be strongly suggestive.

4. In the regression of **y** on **x** and **d**, if **d** and **x** are independent, we can invoke the familiar result for least squares regression. The results are the same as those obtained by two simple regressions. It is instructive to verify this. $plim\begin{bmatrix} \mathbf{x}'\mathbf{x}/n & \mathbf{x}'\mathbf{d}/n \\ \mathbf{d}'\mathbf{x}/n & \mathbf{d}'\mathbf{d}/n \end{bmatrix}^{-1} \begin{pmatrix} \mathbf{x}'\mathbf{y}/n \\ \mathbf{d}'\mathbf{y}/n \end{pmatrix} = \begin{bmatrix} \sigma_*^2 + \sigma_u^2 & 0 \\ 0 & \pi \end{bmatrix}^{-1} \begin{pmatrix} \beta\sigma_*^2 \\ \gamma\pi \end{pmatrix} = \begin{pmatrix} \beta/\left(1 + \sigma_u^2/\sigma_*^2\right) \\ \gamma \end{pmatrix}$. Therefore, although the coefficient on **x** is distorted, the effect of interest, namely, γ, is correctly measured. Now consider what happens if x^* and d are not independent. With the second assumption, we must replace the off diagonal zero above with plim($\mathbf{x}'\mathbf{d}/n$). Since u and d are still uncorrelated, this equals $\text{Cov}[x^*,d]$. This is

$$\text{Cov}[x^*,d] = E[x^*d] = \pi E[x^*d|d=1] + (1-\pi)E[x^*d|d=0] = \pi\mu^1.$$

Also, $\text{plim}[\mathbf{y}'\mathbf{d}/n]$ is now $\beta\text{Cov}[x^*,d] + \gamma\text{plim}(\mathbf{d}'\mathbf{d}/n) = \beta\pi\mu^1 + \gamma\pi$ and $\text{plim}[\mathbf{y}'\mathbf{x}^*/n]$ equals $\beta\text{plim}[\mathbf{x}^{*\prime}\mathbf{x}^*/n] + \gamma\text{plim}[\mathbf{x}^{*\prime}\mathbf{d}/n] = \beta\sigma_*^2 + \gamma\pi\mu^1$. Then, the probability limits of the least squares coefficient estimators is

$$plim\begin{bmatrix} \mathbf{x}'\mathbf{x}/n & \mathbf{x}'\mathbf{d}/n \\ \mathbf{d}'\mathbf{x}/n & \mathbf{d}'\mathbf{d}/n \end{bmatrix}^{-1} \begin{pmatrix} \mathbf{x}'\mathbf{y}/n \\ \mathbf{d}'\mathbf{y}/n \end{pmatrix} = \begin{bmatrix} \sigma_*^2 + \sigma_u^2 & \pi\mu^1 \\ \pi\mu^1 & \pi \end{bmatrix}^{-1} \begin{pmatrix} \beta\sigma_*^2 + \gamma\pi\mu^1 \\ \beta\pi\mu^1 + \gamma\pi \end{pmatrix} = \begin{pmatrix} \beta/\left(1 + \sigma_u^2/\sigma_*^2\right) \\ \gamma \end{pmatrix}$$

$$= \frac{1}{\pi(\sigma_*^2 + \sigma_u^2) + \pi^2(\mu^1)^2} \begin{bmatrix} \pi & -\pi\mu^1 \\ -\pi\mu^1 & \sigma_*^2 + \sigma_u^2 \end{bmatrix} \begin{pmatrix} \beta\sigma_*^2 + \gamma\pi\mu^1 \\ \beta\pi\mu^1 + \gamma\pi \end{pmatrix}$$

$$= \frac{1}{\pi(\sigma_*^2 + \sigma_u^2) + \pi^2(\mu^1)^2} \begin{pmatrix} \beta(\pi\sigma_*^2 + \pi^2(\mu^1)^2) \\ \gamma(\pi(\sigma_*^2 + \sigma_u^2) + \pi^2(\mu^1)^2) + \beta\pi\sigma_u^2 \end{pmatrix}.$$

The second expression does reduce to $\text{plim } c = \gamma + \beta\pi\mu^1\sigma_u^2/[\pi(\sigma_*^2 + \sigma_u^2) - \pi^2(\mu^1)^2]$, but the upshot is that in the presence of measurement error, the two estimators become an unredeemable hash of the underlying parameters. Note that both expressions reduce to the true parameters if σ_u^2 equals zero.

Finally, the two means are estimators of

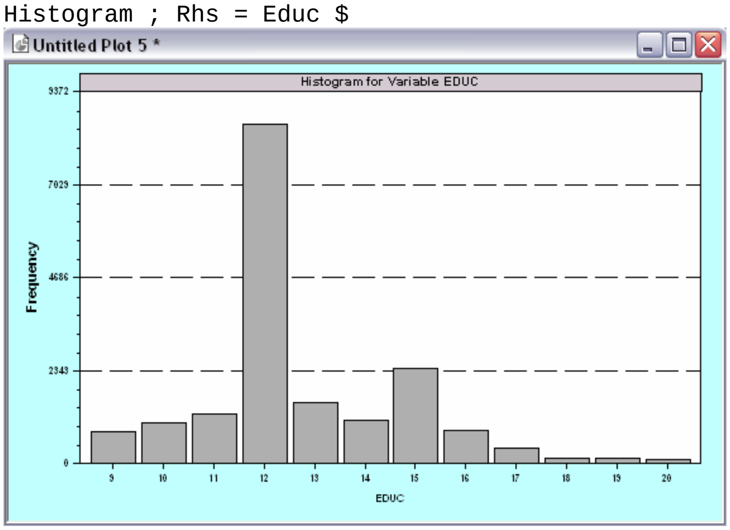
$$E[y|d=1] = \beta E[x^*|d=1] + \gamma = \beta\mu^1 + \gamma$$

and $$E[y|d=0] = \beta E[x^*|d=0] = \beta\mu^0,$$

so the difference is $\beta(\mu^1 - \mu^0) + \gamma$, which is a mixture of two effects. Which one will be larger is entirely indeterminate, so it is reasonable to conclude that this is *not* a good way to analyze the problem. If γ equals zero, this difference will merely reflect the differences in the values of x^*, which may be entirely unrelated to the issue under examination here. (This is, unfortunately, what is usually reported in the popular press.) □

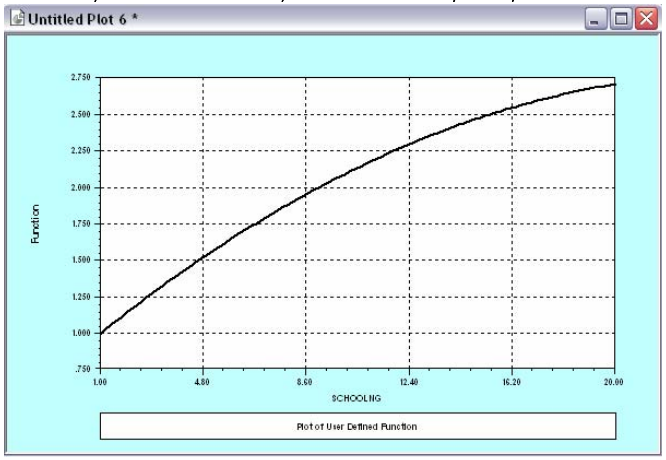
Applications

```
?=======================================================================
? Application 6.1
?=======================================================================
a. Wage equation
Namelist ; X = one,educ,ability,pexp,med,fed,bh,sibs$
Regress ; Lhs = lwage ; Rhs = x $
Calc ; xb = b(1)+b(2)*12+b(3)*xbr(ability)+b(4)*xbr(med)
           +b(5)*xbr(fed)+b(6)*0+b(7)*xbr(sibs) $
Calc ; list ; mv = exp(xb) * b(2) $
+-----------------------------------------------------+
| Ordinary    least squares regression                |
| LHS=LWAGE    Mean                 =   2.296821      |
|              Standard deviation   =   .5282364      |
| WTS=none     Number of observs.   =      17919      |
| Model size   Parameters           =          7      |
|              Degrees of freedom   =      17912      |
| Residuals    Sum of squares       =   4126.175      |
|              Standard error of e  =   .4799564      |
| Fit          R-squared            =   .1747197      |
|              Adjusted R-squared   =   .1744433      |
| Model test   F[  6, 17912] (prob) = 632.02 (.0000) |
+-----------------------------------------------------+
+--------+-------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|
+--------+-------------+----------------+--------+--------+----------+
 Constant|     .96950956       .03370543    28.764   .0000
 EDUC    |     .07220350       .00225076    32.080   .0000   12.6760422
 ABILITY |     .07746781       .00493727    15.690   .0000    .05237402
 PEXP    |     .03950928       .00089926    43.936   .0000   8.36268765
 MED     |    -.00011702       .00169634     -.069   .9450   11.4719013
 FED     |     .00545695       .00133870     4.076   .0000   11.7092472
 SIBS    |     .00476557       .00179240     2.659   .0078   3.15620291

+------------------------------------+
| Listed Calculator Results          |
+------------------------------------+
 MV      =       .725176b. Step function

?=======================================================================
? b.
?=======================================================================
Histogram ; Rhs = Educ $
```



```
Create ; HS = Educ <= 12 $
Create ; Col = (Educ>12) * (educ <=16) $
Create ; Grad = Educ > 16 $
Regress ; Lhs=lwage ; Rhs = one,Col,Grad,ability,pexp,med,fed,bh,sibs $
+----------------------------------------------------+
| Ordinary    least squares regression               |
| LHS=LWAGE    Mean                 =   2.296821     |
|              Standard deviation   =   .5282364     |
| WTS=none     Number of observs.   =      17919     |
| Model size   Parameters           =          9     |
|              Degrees of freedom   =      17910     |
| Residuals    Sum of squares       =   4215.033     |
|              Standard error of e  =   .4851239     |
| Fit          R-squared            =   .1569472     |
|              Adjusted R-squared   =   .1565706     |
| Model test   F[  8, 17910] (prob) = 416.78 (.0000) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    1.81124933       .02069456    87.523   .0000
 COL     |     .17467913       .00872506    20.020   .0000    .32183716
 GRAD    |     .36244740       .02086328    17.373   .0000    .03493499
 ABILITY |     .10097636       .00486713    20.747   .0000    .05237402
 PEXP    |     .03814088       .00090643    42.078   .0000   8.36268765
 MED     |     .00081934       .00171488      .478   .6328   11.4719013
 FED     |     .00700641       .00135096     5.186   .0000   11.7092472
 BH      |    -.06962521       .01007870    -6.908   .0000    .15385903
 SIBS    |     .00371191       .00181156     2.049   .0405   3.15620291


c. Education squared
Create ; educsq = educ*educ $
Regress ; Lhs = lwage;rhs=one,educ,educsq,ability,pexp,med,fed,bh,sibs$
+----------------------------------------------------+
| Ordinary    least squares regression               |
| LHS=LWAGE    Mean                 =   2.296821     |
|              Standard deviation   =   .5282364     |
| WTS=none     Number of observs.   =      17919     |
| Model size   Parameters           =          9     |
|              Degrees of freedom   =      17910     |
| Residuals    Sum of squares       =   4114.269     |
|              Standard error of e  =   .4792902     |
| Fit          R-squared            =   .1771010     |
|              Adjusted R-squared   =   .1767334     |
| Model test   F[  8, 17910] (prob) = 481.81 (.0000) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|     .42778242       .12008093     3.562   .0004
 EDUC    |     .15590624       .01751608     8.901   .0000   12.6760422
 EDUCSQ  |    -.00313261       .00064230    -4.877   .0000   164.377588
 ABILITY |     .07433494       .00496954    14.958   .0000    .05237402
 PEXP    |     .03962214       .00089830    44.108   .0000   8.36268765
 MED     |     .00030520       .00169504      .180   .8571   11.4719013
 FED     |     .00519423       .00133734     3.884   .0001   11.7092472
 BH      |    -.04957434       .01000691    -4.954   .0000    .15385903
 SIBS    |     .00499325       .00179020     2.789   .0053   3.15620291
Namelist ; x1 = one,educ,educsq,ability,pexp,med,fed,bh,sibs $
Matrix ; means = mean(x1)$
Matrix ; means(2)=0 $
Matrix ; means(3)=0$
Calc   ; a=means'b $
Calc   ; b2=b(2) ; b3=b(3) $
Sample ; 1 $
```

```
Fplot ; fcn = a + b2*schoolng + b3*schoolgn^2 ; pts=100
      ; start = 12 ; limits = 1,20 ; labels=schoolng ; plot(schoolng) $
```

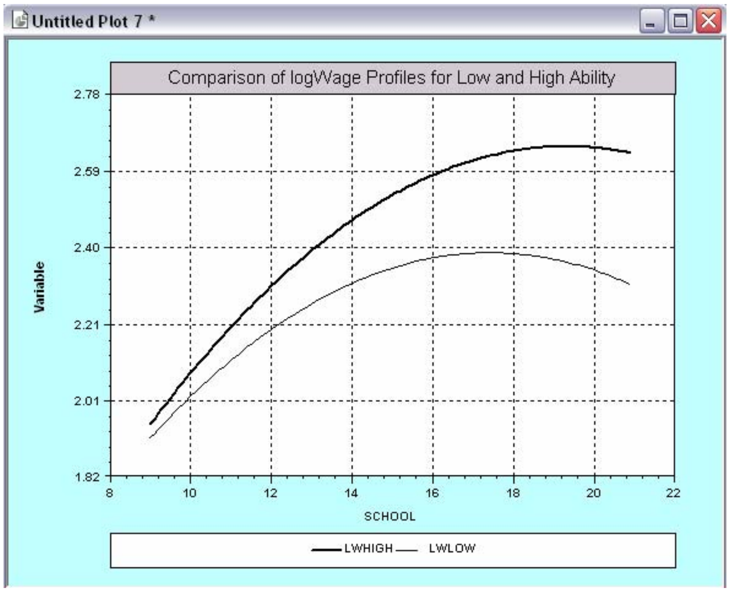


d. Interaction.

```
Sample ; All $
Create ; EA = Educ*ability $
Regress ; Lhs = lwage;rhs=one,educ,ability,ea,pexp,med,fed,bh,sibs$
Calc ; abar =xbr(ability) $
Calc ; list ; me = b(2)+b(4)*abar $
Calc ; sdme = sqr(varb(2,2)+abar^2*varb(4,4) + 2*abar*varb(2,4))$
Calc ; list ; lower = me - 1.96*sdme ; upper = me + 1.96*sdme $
+----------------------------------------------------+
| Ordinary    least squares regression               |
| LHS=LWAGE    Mean                 =   2.296821     |
|              Standard deviation   =   .5282364     |
| WTS=none     Number of observs.   =      17919     |
| Model size   Parameters           =          9     |
|              Degrees of freedom   =      17910     |
| Residuals    Sum of squares       =   4119.377     |
|              Standard error of e  =   .4795877     |
| Fit          R-squared            =   .1760794     |
|              Adjusted R-squared   =   .1757113     |
| Model test   F[  8, 17910] (prob) = 478.44 (.0000) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    1.00190276       .03529335    28.388   .0000
 EDUC    |     .07006221       .00243183    28.811   .0000   12.6760422
 ABILITY |     .04693108       .02494471     1.881   .0599    .05237402
 EA      |     .00253975       .00204029     1.245   .2132   1.60372621
 PEXP    |     .03947437       .00089903    43.908   .0000   8.36268765
 MED     |    .542277D-04      .00169546      .032   .9745   11.4719013
 FED     |     .00534599       .00133813     3.995   .0001   11.7092472
 BH      |    -.05314420       .00999271    -5.318   .0000    .15385903
 SIBS    |     .00479076       .00179231     2.673   .0075   3.15620291

+------------------------------------+
| Listed Calculator Results          |
+------------------------------------+
 ME      =       .070195
 LOWER   =       .065503
 UPPER   =       .074888
```

e.

```
Regress ; Lhs = lwage;rhs=one,educ,educsq,ability,ea,pexp,med,fed,bh,sibs$
+----------------------------------------------------+
| Ordinary    least squares regression               |
| LHS=LWAGE    Mean                 =   2.296821     |
|              Standard deviation   =   .5282364     |
| WTS=none     Number of observs.   =      17919     |
| Model size   Parameters           =         10     |
|              Degrees of freedom   =      17909     |
| Residuals    Sum of squares       =   4106.031     |
|              Standard error of e  =   .4788235     |
| Fit          R-squared            =   .1787487     |
|              Adjusted R-squared   =   .1783360     |
| Model test   F[  9, 17909] (prob) = 433.11 (.0000) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    -.10514525       .14931731     -.704   .4813
 EDUC    |     .24088793       .02252126    10.696   .0000   12.6760422
 EDUCSQ  |    -.00654261       .00085754    -7.630   .0000   164.377588
 ABILITY |    -.12453442       .03354596    -3.712   .0002    .05237402
 EA      |     .01631824       .00272231     5.994   .0000   1.60372621
 PEXP    |     .03951247       .00089761    44.020   .0000   8.36268765
 MED     |     .00045246       .00169356      .267   .7893   11.4719013
 FED     |     .00524829       .00133606     3.928   .0001   11.7092472
 BH      |    -.04775208       .01000179    -4.774   .0000    .15385903
 SIBS    |     .00460796       .00178961     2.575   .0100   3.15620291
+-----------------------------------+
| Listed Calculator Results         |
+-----------------------------------+
 AVGLOW  =      -.798563
 AVGHIGH =       .717891
Create ; lowa = ability < xbr(ability) ; higha = 1 - lowa $
Calc   ; list ; avglow= lowa'ability / lowa'lowa ;
avghigh=higha'ability/higha'higha $
Calc   ; a = b(1) + b(6)*xbr(pexp)+b(7)*xbr(med)+
             b(8)*xbr(fed)+b(9)*xbr(bh)+b(10)*xbr(sibs)$
Calc   ; al=a+b(4)*avglow ; ah = a+b(4)*avghigh$
Samp;1-120$
Create ; school = trn(9,.1)$
Create ; lwlow  = al + b(2)*school+b(3)*school^2 + b(5)*avglow*school $
Create ; lwhigh = ah + b(2)*school+b(3)*school^2 + b(5)*avghigh*school $
Plot ; lhs = school ; rhs =lwhigh,lwlow ;fill ;grid
;Title=Comparison of logWage Profiles for Low and High Ability$
```



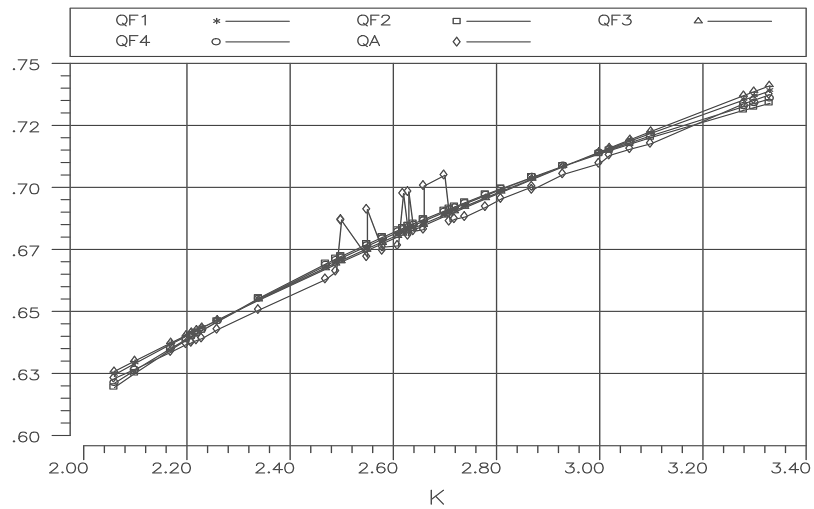

```
?========================================================================
? Application 6.2
?========================================================================
Sample ; All $
Namelist ; X = one,educ,ability,pexp,med,fed,sibs$
Regress ; For [bh=0] ; Lhs = lwage ; Rhs = x $
Calc ; ee0=sumsqdev $
Matrix ; b0=b ; v0=varb $
Regress ; For [bh=1] ; Lhs = lwage ; Rhs = x $
Calc ; ee1=sumsqdev $
Matrix ; b1=b ; v1=varb $
Regress ; Lhs = lwage ; Rhs = x $
Calc ; ee=sumsqdev $
Calc ; list ; chow = ((ee-ee0-ee1)/col(x))/ ((ee0+ee1)/(n-2*col(x))) $
+-----------------------------------+
| Listed Calculator Results          |
+-----------------------------------+
 CHOW    =      7.348379
Matrix ; db=b0-b1 ; vdb=v0+v1 $
Matrix ; list ; Wald = db'<vdb>db $
Matrix WALD     has  1 rows and  1 columns.
               1
        +--------------
       1|   50.57114
```

```
?=====================================================================
? Application 6.3
?=====================================================================
```

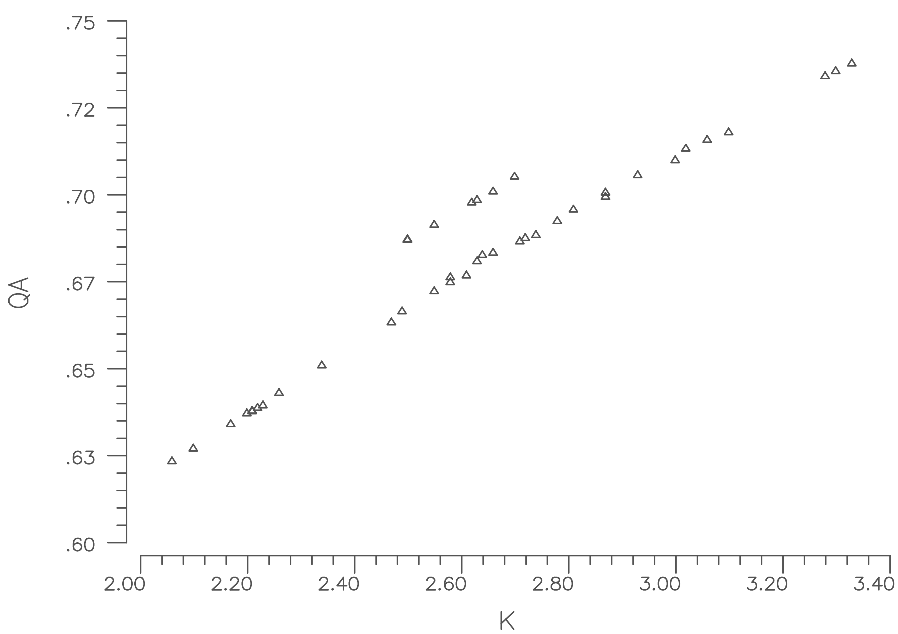
a. The least squares estimates of the four models are

$$q/A = .45237 + .23815\ln k$$
$$q/A = .91967 - .61863/k$$
$$\ln(q/A) = -.72274 + .35160\ln k$$
$$\ln(q/A) = -.032194 - .91496/k$$

At these parameter values, the four functions are nearly identical. A plot of the four sets of predictions from the regressions and the actual values appears below.



b. The scatter diagram is shown below. The last seven years of the data set show clearly the effect observed by Solow.



.

c. The regression results for the various models are listed below. (d is the dummy variable equal to 1 for the last seven years of the data set. Standard errors for parameter estimates are given in parentheses.)

α	β	γ	δ	R^2	**e′e**
Model 1: $q/A = \alpha + \beta\ln k + \gamma d + \delta(d\ln k) + \varepsilon$					
.4524	.2381			.94355	.00213
(.00903)	(.00932)				
.4477	.2396	.01900		.99914	.000032
(.00113)	(.00117)	(.000384)			
.4476	.2397	.02746	-.08883	.99915	.000032
(.00115)	(.00118)	(.0119)	(.0126)		
Model 2: $q/A = \alpha - \beta(1/k) + \gamma d + \delta(d/k) + \varepsilon$					
.9168	.6186			.94915	.001915
(.00891)	(.0229)				
.9167	.6185	.01961		.99321	.000256
(.00331)	(.00849)	(.00108)			
.9168	.6187	.008651	.02140	.99322	.000255
(.00336)	(.00863)	(.0354)	(.0917)		
Model 3: $\ln(q/A) = \alpha + \beta\ln k + \gamma d + \delta(d\ln k) + \varepsilon$					
-.7227	.3516			.94069	.004882
(.0137)	(.0141)				
-.7298	.3538	.002881		.99918	.000068
(.00164)	(.00169)	(.000554)			
-.7300	.3540	.04961	-.02182	.99921	.000065
(.00164)	(.00148)	(.0171)	(.0179)		
Model 4: $\ln(q/A) = \alpha - \beta(1/k) + \gamma d + \delta(d/k) + \varepsilon$					
-.03219	.9150			.94964	.004146
(.0131)	(.0337)				
-.03665	.9148	.02572		.99629	.000305
(.00361)	(.00928)	(.00118)			
-.03646	.9153	.004290	.05556	.99632	.000303
(.00366)	(.00941)	(.0386)	(.0999)		

d. For the four models, the F test of the third specification against the first is equivalent to the Chow-test. The statistics are:

Model 1: F = (.002126 - .000032)/2 / (.000032/37) = 1210.6
Model 2: F = = 120.43
Model 3: F = = 1371.0
Model 4: F = = 234.64

The critical value from the F table for 2 and 37 degrees of freedom is 3.26, so all of these are statistically significant. The hypothesis that the same model applies in both subperiods must be rejected. □

```
?=================================================================
? Application 6.4
?=================================================================
```
According to the full model, the expected number of incidents for a ship of the base type A built in the base period 1960 to 1964, is 3.4. The other 19 predicted values follow from the previous results and are left as an exercise. The relevant test statistics for differences across ship type and year are as follows:

$$\text{type}: F[4,12] = \frac{(3925.2 - 660.9)/4}{660.9/12} = 14.82,$$

$$\text{year}: F[3,12] = \frac{(1090.3 - 660.9)/3}{660.9/12} = 2.60.$$

The 5 percent critical values from the *F* table with these degrees of freedom are 3.26 and 3.49, respectively, so we would conclude that the average number of incidents varies significantly across ship types but not across years.

Regression Coefficients				
	Full Model	***Time Effects***	***Type Effects***	***No Effects***
Constant	3.4	6.0	8.25	10.85
B	27.75	0	27.75	0
C	–7.0	0	–7.0	0
D	–4.5	0	–4.5	0
E	–3.25	0	–3.25	0
65–69	7.0	7.0	0	0
70–74	11.4	11.4	0	0
75–79	1.0	1.0	0	0
R^2	0.84823	0.0986	0.74963	0
$\mathbf{e}'\mathbf{e}$	660.9	3925.2	1090.2	4354.5

Chapter 7

Specification Analysis and Model Selection

Exercises

1. The result cited is E[$\mathbf{b}_1$] = β_1 + $\mathbf{P}_{1.2}\beta_2$ where $\mathbf{P}_{1.2} = (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2$, so the coefficient estimator is biased. If the conditional mean function $E[\mathbf{X}_2|\mathbf{X}_1]$ is a linear function of $\mathbf{X}_1$, then the sample estimator $P_{1.2}$ actually is an unbiased estimator of the slopes of that function. (That result is Theorem B.3, equation (B-68), in another form). Now, write the model in the form

$$\mathbf{y} = \mathbf{X}_1\beta_1 + E[\mathbf{X}_2|\mathbf{X}_1]\beta_2 + \varepsilon + (\mathbf{X}_2 - E[\mathbf{X}_2|\mathbf{X}_1])\beta_2$$

So, when we regress $\mathbf{y}$ on $\mathbf{X}_1$ alone and compute the predictions, we are computing an estimator of $\mathbf{X}_1(\beta_1 + \mathbf{P}_{1.2}\beta_2) = \mathbf{X}_1\beta_1 + E[\mathbf{X}_2|\mathbf{X}_1]\beta_2$. Both parts of the compound disturbance in this regression ε and $(X_2 - E[X_2|X_1])\beta_2$ have mean zero and are uncorrelated with $\mathbf{X}_1$ and E[$\mathbf{X}_2|\mathbf{X}_1$], so the prediction error has mean zero. The implication is that the forecast is unbiased. Note that this is not true if E[$\mathbf{X}_2|\mathbf{X}_1$] is nonlinear, since $\mathbf{P}_{1.2}$ does not estimate the slopes of the conditional mean in that instance. The generality is that leaving out variables wil bias the coefficients, but need not bias the forecasts. It depends on the relationship between the conditional mean function E[$\mathbf{X}_2|\mathbf{X}_1$] and $\mathbf{X}_1\mathbf{P}_{1.2}$.

2. The "long" estimator, $\mathbf{b}_{1.2}$ is unbiased, so its mean squared error equals its variance, $\sigma^2(\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1)^{-1}$
The short estimator, $\mathbf{b}_1$ is biased; E[$\mathbf{b}_1$] = β_1 + $\mathbf{P}_{1.2}\beta_2$. It's variance is $\sigma^2(\mathbf{X}_1'\mathbf{X}_1)^{-1}$. It's easy to show that this latter variance is smaller. You can do that by comparing the inverses of the two matrices. The inverse of the first matrix equals the inverse of the second one minus a positive definite matrix, which makes the inverse smaller hence the original matrix is larger - Var[$\mathbf{b}_{1.2}$] $\geq$ Var[$\mathbf{b}_1$]. But, since $\mathbf{b}_1$ is biased, the variance is not its mean squared error. The mean squared error of $\mathbf{b}_1$ is Var[$\mathbf{b}_1$] + **bias**$\times$**bias′**. The second term is $\mathbf{P}_{1.2}\beta_2\beta_2'\mathbf{P}_{1.2}'$. When this is added to the variance, the sum may be larger or smaller than Var[$\mathbf{b}_{1.2}$]; it depends on the data and on the parameters, β_2. The important point is that the mean squared error of the biased estimator *may* be smaller than that of the unbiased estimator.

3. The log likelihood function at the maximum is

$$\begin{aligned}\ln L &= -n/2[1 + \ln 2\pi + \ln(\mathbf{e}'\mathbf{e}/n)]\\ &= -n/2\{1 + \ln 2\pi + \ln[nS_{yy}(1 - R^2)]\}\\ &= -n/2\{1 + \ln 2\pi + \ln(nS_{yy}) + \ln(1-R^2)\} \text{ where } S_{yy} = \Sigma_{i=1}^{n}(y_i - \overline{y})^2\end{aligned}$$

since $R^2 = 1 - e'e/S_{yy}$. The derivative of this expression is $\partial\ln L/\partial R^2 = (-n/2)\{1/(1-R^2)\}(-1)$ which is always positive. Therefore, the log likelihood increases when R^2 increases.

4. An inconvenient way to obtain the result is by repeated substitution of C_{t-1}, then C_{t-2} and so on. It is much easier and faster to introduce the lag operator used in Chapter 20. Thus, the alternative model is

$$C_t = \gamma_1 + \gamma_2 Y_t + \gamma_3 LC_t + \varepsilon_{1t} \text{ where } LC_t = C_{t-1}.$$

Then, $(1 - \gamma_3 L)C_t = \gamma_1 + \gamma_2 Y_t + \varepsilon_{1t}$.

Now, multiply both sides of the equation by $1/(1-\gamma_3 L) = 1 + \gamma_3 L + \gamma_3^2 L^2 + \ldots$ to obtain

$$\boldsymbol{C_t = \gamma_1/(1 - \gamma_3) + \gamma_2 Y_t + \gamma_2\gamma_3 Y_{t-1} + \Sigma_{s=2}^{\infty}\gamma_2\gamma_3^s Y_{t-s} + \Sigma_{s=0}^{\infty}\gamma_3^s\varepsilon_{t-s}}.$$

Application

The J test in Example is carried out using over 50 years of data. It is optimistic to hope that the underlying structure of the economy did not change in 50 years. Does the result of the test carried out in Example 8.2 persist if it is based on data only from 1980 to 2000? Repeat the computation with this subset of the data.

```
?====================================
? Example 7.2 and Application 7.1
?====================================
Dates  ; 1950.1 $
Period ; 1950.1 - 2000.4 $
Create ; Ct = Realcons ; Yt = RealDPI $
Create ; Ct1 = Ct[-1] ; Yt1 = Yt[-1] $
? Example 7.2
Period ; 1950.2 - 2000.4 $
Regress; Lhs = Ct ; Rhs = one,Yt,Yt1 ; Keep = CY $
Regress; Lhs = Ct ; Rhs = one,Yt,Ct1 ; Keep = CC $
Regress; Lhs = Ct ; Rhs = one,Yt,Yt1,CC $
+-----------------------------------------------------+
| Ordinary    least squares regression                |
| Model was estimated May 12, 2007 at 08:56:19AM      |
| LHS=CT       Mean                 =   3008.995      |
|              Standard deviation   =   1456.900      |
| WTS=none     Number of observs.   =        203      |
| Model size   Parameters           =          4      |
|              Degrees of freedom   =        199      |
| Residuals    Sum of squares       =   73550.21      |
|              Standard error of e  =   19.22496      |
| Fit          R-squared            =   .9998285      |
|              Adjusted R-squared   =   .9998259      |
| Model test   F[  3,   199] (prob) =******* (.0000) |
| Diagnostic   Log likelihood       =  -886.1351      |
|              Restricted(b=0)      =  -1766.209      |
|              Chi-sq [  3]  (prob) =1760.15 (.0000) |
| Info criter. LogAmemiya Prd. Crt. =   5.931932      |
|              Akaike Info. Criter. =   5.931926      |
| Autocorrel   Durbin-Watson Stat.  =  2.0256102      |
|              Rho = cor[e,e(-1)]   =  -.0128051      |
+-----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    -.60444607      3.43245774     -.176   .8604
 YT      |     .31456542       .04619552     6.809   .0000   3352.09360
 YT1     |    -.33004915       .04591940    -7.188   .0000   3325.25222
 CC      |    1.01450597       .01613899    62.861   .0000   3008.99507
Regress; Lhs = Ct ; Rhs = one,Yt,Ct1,CY $
+-----------------------------------------------------+
| Ordinary    least squares regression                |
| Model was estimated May 12, 2007 at 08:56:19AM      |
| LHS=CT       Mean                 =   3008.995      |
|              Standard deviation   =   1456.900      |
| WTS=none     Number of observs.   =        203      |
| Model size   Parameters           =          4      |
|              Degrees of freedom   =        199      |
| Residuals    Sum of squares       =   73550.21      |
|              Standard error of e  =   19.22496      |
| Fit          R-squared            =   .9998285      |
|              Adjusted R-squared   =   .9998259      |
| Model test   F[  3,   199] (prob) =******* (.0000) |
| Diagnostic   Log likelihood       =  -886.1351      |
|              Restricted(b=0)      =  -1766.209      |
|              Chi-sq [  3]  (prob) =1760.15 (.0000) |
| Info criter. LogAmemiya Prd. Crt. =   5.931932      |
|              Akaike Info. Criter. =   5.931926      |
```

```
| Autocorrel   Durbin-Watson Stat.  =  2.0256102     |
|              Rho = cor[e,e(-1)]   =  -.0128051     |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|   -865.712368      120.569071    -7.180   .0000
 YT      |    9.82505250      1.36759557     7.184   .0000   3352.09360
 CT1     |    1.02780685       .01635059    62.861   .0000   2982.97438
 CY      |   -10.6765577      1.48541853    -7.188   .0000   3008.99507
? Application 7.1.  We use only the 1980 data, so we
? start in quarter 2 of 1980 even though data are
? available for the last quarter of 1979.
Period ; 1980.2 - 2000.4 $
Regress; Lhs = Ct ; Rhs = one,Yt,Yt1 ; Keep = CY $
Regress; Lhs = Ct ; Rhs = one,Yt,Ct1 ; Keep = CC $
Regress; Lhs = Ct ; Rhs = one,Yt,Yt1,CC $
+----------------------------------------------------+
| Ordinary    least squares regression               |
| Model was estimated May 12, 2007 at 08:58:19AM     |
| LHS=CT       Mean                 =   4503.230     |
|              Standard deviation   =   879.3593     |
| WTS=none     Number of observs.   =         83     |
| Model size   Parameters           =          4     |
|              Degrees of freedom   =         79     |
| Residuals    Sum of squares       =   43603.43     |
|              Standard error of e  =   23.49345     |
| Fit          R-squared            =   .9993123     |
|              Adjusted R-squared   =   .9992862     |
| Model test   F[  3,    79] (prob) =******* (.0000) |
| Diagnostic   Log likelihood       =  -377.7300     |
|              Restricted(b=0)      =  -679.9419     |
|              Chi-sq [  3]  (prob) = 604.42 (.0000) |
| Info criter. LogAmemiya Prd. Crt. =   6.360511     |
|              Akaike Info. Criter. =   6.360436     |
| Autocorrel   Durbin-Watson Stat.  =  1.8153241     |
|              Rho = cor[e,e(-1)]   =   .0923379     |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    39.6958824      37.1402619     1.069   .2884
 YT      |     .20222923       .07364203     2.746   .0075   4987.32410
 YT1     |    -.25661196       .07221392    -3.553   .0006   4951.70482
 CC      |    1.04938412       .04670690    22.467   .0000   4503.23012
Regress; Lhs = Ct ; Rhs = one,Yt,Ct1,CY $
+----------------------------------------------------+
| Ordinary    least squares regression               |
| Model was estimated May 12, 2007 at 08:58:19AM     |
| LHS=CT       Mean                 =   4503.230     |
|              Standard deviation   =   879.3593     |
| WTS=none     Number of observs.   =         83     |
| Model size   Parameters           =          4     |
|              Degrees of freedom   =         79     |
| Residuals    Sum of squares       =   43603.43     |
|              Standard error of e  =   23.49345     |
| Fit          R-squared            =   .9993123     |
|              Adjusted R-squared   =   .9992862     |
| Model test   F[  3,    79] (prob) =******* (.0000) |
| Diagnostic   Log likelihood       =  -377.7300     |
|              Restricted(b=0)      =  -679.9419     |
|              Chi-sq [  3]  (prob) = 604.42 (.0000) |
| Info criter. LogAmemiya Prd. Crt. =   6.360511     |
|              Akaike Info. Criter. =   6.360436     |
| Autocorrel   Durbin-Watson Stat.  =  1.8153241     |
|              Rho = cor[e,e(-1)]   =   .0923379     |
+----------------------------------------------------+
```

```
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|   -856.107861      221.141722    -3.871   .0002
 YT      |    1.21490273       .32340906     3.757   .0003   4987.32410
 CT1     |     .98759074       .04395654    22.467   .0000   4465.65542
 CY      |   -1.13474451       .31933175    -3.553   .0006   4503.23012
?
? The results are essentially the same.  This suggests
? that neither model is right.
```

The regressions are based on real consumption and real disposable income. Results for 1950 to 2000 are given in the text. Repeating the exercise for 1980 to 2000 produces: for the first regression, the estimate of α is 1.03 with a t ratio of 23.27 and for the second, the estimate is -1.24 with a t ratio of -3.062. Thus, as before, both models are rejected. This is qualitatively the same results obtained with the full 51 year data set.

Chapter 8

The Generalized Regression Model and Heteroscedasticity

Exercises

1. Write the two estimators as $\hat{\boldsymbol\beta} = \boldsymbol\beta + (\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol\Omega^{-1}\boldsymbol\varepsilon$ and $\mathbf{b} = \boldsymbol\beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol\varepsilon$. Then, $(\hat{\boldsymbol\beta} - \mathbf{b}) = [(\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol\Omega^{-1} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\boldsymbol\varepsilon$ has $E[\hat{\boldsymbol\beta} - \mathbf{b}] = \mathbf{0}$ since both estimators are unbiased. Therefore,

$$\text{Cov}[\hat{\boldsymbol\beta}, \hat{\boldsymbol\beta} - \mathbf{b}] = E[(\hat{\boldsymbol\beta} - \boldsymbol\beta)(\hat{\boldsymbol\beta} - \mathbf{b})'].$$

Then,

$$\begin{aligned} &E\{(\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol\Omega^{-1}\boldsymbol\varepsilon\boldsymbol\varepsilon'[(\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol\Omega^{-1} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']'\} \\ &= (\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol\Omega^{-1}(\sigma^2\boldsymbol\Omega)[\boldsymbol\Omega^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{X})^{-1} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}] \\ &= \sigma^2(\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol\Omega^{-1}\boldsymbol\Omega\boldsymbol\Omega^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{X})^{-1} - (\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol\Omega^{-1}\boldsymbol\Omega\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\ &= (\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{X})^{-1}(\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{X})(\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{X})^{-1} - (\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1} = \mathbf{0}\end{aligned}$$

once the inverse matrices are multiplied.

2 First, $(\mathbf{R}\hat{\boldsymbol\beta} - \mathbf{q}) = \mathbf{R}[\boldsymbol\beta + (\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol\Omega^{-1}\boldsymbol\varepsilon)] - \mathbf{q} = \mathbf{R}(\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol\Omega^{-1}\boldsymbol\varepsilon$ if $\mathbf{R}\boldsymbol\beta - \mathbf{q} = \mathbf{0}$.
Now, use the inverse square root matrix of $\boldsymbol\Omega$, $\mathbf{P} = \boldsymbol\Omega^{-1/2}$ to obtain the transformed data,

$$\mathbf{X}^* = \mathbf{P}\mathbf{X} = \boldsymbol\Omega^{-1/2}\mathbf{X},\quad \mathbf{y}^* = \mathbf{P}\mathbf{y} = \boldsymbol\Omega^{-1/2}\mathbf{y},\ \text{and}\ \boldsymbol\varepsilon^* = \mathbf{P}\boldsymbol\varepsilon = \boldsymbol\Omega^{-1/2}\boldsymbol\varepsilon.$$

Then, $E[\boldsymbol\varepsilon^*\boldsymbol\varepsilon^{*\prime}] = E[\boldsymbol\Omega^{-1/2}\boldsymbol\varepsilon\boldsymbol\varepsilon'\boldsymbol\Omega^{-2}] = \boldsymbol\Omega^{-1/2}(\sigma^2\boldsymbol\Omega)\boldsymbol\Omega^{-1/2} = \sigma^2\mathbf{I}$,

and,

$$\begin{aligned}\hat{\boldsymbol\beta} &= (\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol\Omega^{-1}\mathbf{y} = (\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{X}^{*\prime}\mathbf{y}^* \\ &= \text{the OLS estimator in the regression of } \mathbf{y}^* \text{ on } \mathbf{X}^*.\end{aligned}$$

Then, $\mathbf{R}\hat{\boldsymbol\beta} - \mathbf{q} = \mathbf{R}(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{X}^{*\prime}\boldsymbol\varepsilon^*$

and the numerator is $\boldsymbol\varepsilon^{*\prime}\mathbf{X}^*(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{R}'[\mathbf{R}(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{R}']^{-1}\mathbf{R}(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{X}^{*\prime}\boldsymbol\varepsilon^* / J$. By multiplying it out, we find that the matrix of the quadratic form above is idempotent. Therefore, this is an idempotent quadratic form in a normally distributed random vector. Thus, its distribution is that of σ^2 times a chi-squared variable with degrees of freedom equal to the rank of the matrix. To find the rank of the matrix of the quadratic form, we can find its trace. That is

$$\begin{aligned} &\text{tr}\{\mathbf{X}^*(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{R}'[\mathbf{R}(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{R}']^{-1}\mathbf{R}(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{X}^*\} \\ &= \text{tr}\{(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{R}'[\mathbf{R}(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{R}']^{-1}\mathbf{R}(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{X}^{*\prime}\mathbf{X}^*\} \\ &= \text{tr}\{(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{R}'[\mathbf{R}(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{R}']^{-1}\mathbf{R}\} \\ &= \text{tr}\{[\mathbf{R}(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{R}'][\mathbf{R}(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{R}']^{-1}\} = \text{tr}\{\mathbf{I}_J\} = J,\end{aligned}$$

which might have been expected. Before proceeding, we should note, we could have deduced this outcome from the form of the matrix. The matrix of the quadratic form is of the form $\mathbf{Q} = \mathbf{X}^*\mathbf{A}\mathbf{B}\mathbf{A}'\mathbf{X}^{*\prime}$ where $\mathbf{B}$ is the nonsingular matrix in the square brackets and $\mathbf{A} = (\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{R}'$, which is a $K\times J$ matrix which cannot have rank higher than J. Therefore, the entire product cannot have rank higher than J. Continuing, we now find that the numerator (apart from the scale factor, σ^2) is the ratio of a chi-squared[J] variable to its degrees of freedom.

We now turn to the denominator. By multiplying it out, we find that the denominator is $(\mathbf{y}^* - \mathbf{X}^*\hat{\boldsymbol\beta})'(\mathbf{y}^* - \mathbf{X}^*\hat{\boldsymbol\beta})/(n - K)$. This is exactly the sum of squared residuals in the least squares regression of $\mathbf{y}^*$ on $\mathbf{X}^*$. Since $\mathbf{y}^* = \mathbf{X}^*\boldsymbol\beta + \boldsymbol\varepsilon^*$ and $\hat{\boldsymbol\beta} = (\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{X}^{*\prime}\mathbf{y}^*$ the denominator is $\boldsymbol\varepsilon^{*\prime}\mathbf{M}^*\boldsymbol\varepsilon^*/(n - K)$, the familiar form of the sum of squares. Once again, this is an idempotent quadratic form in a normal vector (and, again, apart from the scale factor, σ^2, which now cancels). The rank of the **M** matrix is $n - K$, as always, so the denominator is also a chi-squared variable divided by its degrees of freedom.

It remains only to show that the two chi-squared variables are independent. We know they are if the two matrices are orthogonal. They are since $\mathbf{M}^*\mathbf{X}^* = \mathbf{0}$. This completes the proof, since all of the requirements for the F distribution have been shown.

3. First, we know that the denominator of the F statistic converges to σ^2. Therefore, the limiting distribution of the F statistic is the same as the limiting distribution of the statistic which results when the denominator is replaced by σ^2. It is useful to write this modified statistic as

$$W^* = (1/\sigma^2)(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{q})'[\mathbf{R}(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{R}']^{-1}(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{q})/J.$$

Now, incorporate the results from the previous problem to write this as

$$W^* = \boldsymbol{\varepsilon}^{*\prime}\mathbf{X}^*(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{R}'[\mathbf{R}\sigma^2(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{R}']^{-1}\mathbf{R}(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{X}^{*\prime}\boldsymbol{\varepsilon}/J$$

Let $\boldsymbol{\varepsilon}^0 = \mathbf{R}(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{X}^{*\prime}\boldsymbol{\varepsilon}^*$.

Note that this is a $J\times 1$ vector. By multiplying it out, we find that $E[\boldsymbol{\varepsilon}^0\boldsymbol{\varepsilon}^{0\prime}] = \text{Var}[\boldsymbol{\varepsilon}^0] = \mathbf{R}\{\sigma^2(\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\}\mathbf{R}'$. Therefore, the modified statistic can be written as $W^* = \boldsymbol{\varepsilon}^{0\prime}\text{Var}[\boldsymbol{\varepsilon}^0]^{-1}\boldsymbol{\varepsilon}^0/J$. This is the 'full rank quadratic form' discussed in Appendix B. For convenience, let $\mathbf{C} = \text{Var}[\boldsymbol{\varepsilon}^0]$, $\mathbf{T} = \mathbf{C}^{-1/2}$, and $\mathbf{v} = \mathbf{T}\boldsymbol{\varepsilon}^0$. Then, $W^* = \mathbf{v}'\mathbf{v}$. By construction, $\mathbf{v} = \text{Var}[\boldsymbol{\varepsilon}^0]^{-1/2}\boldsymbol{\varepsilon}^0$, so $E[\mathbf{v}] = \mathbf{0}$ and $\text{Var}[\mathbf{v}] = \mathbf{I}$. The limiting distribution of $\mathbf{v}'\mathbf{v}$ is chi-squared J if the limiting distribution of $\mathbf{v}$ is standard normal. All of the conditions for the central limit theorem apply to $\mathbf{v}$, so we do have the result we need. This implies that as long as the data are well behaved, the numerator of the F statistic will converge to the ratio of a chi-squared variable to its degrees of freedom. □

4. The development is unchanged. As long as the limiting behavior of $(1/n)\hat{\mathbf{X}}'\hat{\mathbf{X}} = (1/n)\mathbf{X}'\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}$ is the same as that of $(1/n)\mathbf{X}^{*\prime}\mathbf{X}^*$, the limiting distribution of the test statistic will be the same as if the true $\boldsymbol{\Omega}$ were used instead of the estimate $\hat{\boldsymbol{\Omega}}$.

5. First, in order to simplify the algebra somewhat without losing any generality, we will scale the columns of **X** so that for each $\mathbf{x}_k$, $\mathbf{x}_k'\mathbf{x}_k = 1$. We do this by beginning with our original data matrix, say, $\mathbf{X}^0$ and obtaining **X** as $\mathbf{X} = \mathbf{X}^0\mathbf{D}^{-1/2}$, where **D** is a diagonal matrix with diagonal elements $\mathbf{D}_{kk} = \mathbf{x}_k^{0\prime}\mathbf{x}_k^0$. By multiplying it out, we find that the GLS slopes based on **X** instead of $\mathbf{X}^0$ are

$$\hat{\boldsymbol{\beta}} = [(\mathbf{X}^0\mathbf{D}^{-1/2})'\boldsymbol{\Omega}^{-1}(\mathbf{X}^0\mathbf{D}^{-1/2})]^{-1}[(\mathbf{X}^0\mathbf{D}^{-1/2})'\boldsymbol{\Omega}^{-1}\mathbf{y}] = \mathbf{D}^{1/2}[\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X}](\mathbf{D}')^{1/2}(\mathbf{D}')^{-1/2}\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{y} = \mathbf{D}^{1/2}\,\hat{\boldsymbol{\beta}}^0$$

with variance $\text{Var}[\hat{\boldsymbol{\beta}}] = \mathbf{D}^{1/2}\sigma^2[\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X}]^{-1}(\mathbf{D}')^{1/2} = \mathbf{D}^{1/2}\text{Var}[\hat{\boldsymbol{\beta}}^0](\mathbf{D}')^{1/2}$. Likewise, the OLS estimator based on **X** instead of $\mathbf{X}^0$ is $\mathbf{b} = \mathbf{D}^{1/2}\mathbf{b}^0$ and has variance $\text{Var}[\mathbf{b}] = \mathbf{D}^{1/2}\text{Var}[\mathbf{b}^0](\mathbf{D}')^{1/2}$. Since the scaling affects both estimators identically, we may ignore it and simply assume that $\mathbf{X}'\mathbf{X} = \mathbf{I}$.

If each column of **X** is a characteristic vector of $\boldsymbol{\Omega}$, then, for the kth column, $\mathbf{x}_k$, $\boldsymbol{\Omega}\mathbf{x}_k = \lambda_k\mathbf{x}_k$. Further, $\mathbf{x}_k'\boldsymbol{\Omega}\mathbf{x}_k = \lambda_k$ and $\mathbf{x}_k'\boldsymbol{\Omega}\mathbf{x}_j = 0$ for any two different columns of **X**. (We neglect the scaling of **X**, so that $\mathbf{X}'\mathbf{X} = \mathbf{I}$, which we would usually assume for a set of characteristic vectors. The implicit scaling of **X** is absorbed in the characteristic roots.) Recall that the characteristic vectors of $\boldsymbol{\Omega}^{-1}$ are the same as those of $\boldsymbol{\Omega}$ while the characteristic roots are the reciprocals. Therefore, $\mathbf{X}'\boldsymbol{\Omega}\mathbf{X} = \boldsymbol{\Lambda}_K$, the diagonal matrix of the K characteristic roots which correspond to the columns of **X**. In addition, $\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X} = \boldsymbol{\Lambda}_K^{-1}$, so $(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1} = \boldsymbol{\Lambda}_K$, and $\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{y} = \boldsymbol{\Lambda}_K^{-1}\mathbf{X}'\mathbf{y}$. Therefore, the GLS estimator is simply $\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{y}$ with variance $\text{Var}[\hat{\boldsymbol{\beta}}] = \sigma^2\boldsymbol{\Lambda}_K$. The OLS estimator is $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \mathbf{X}'\mathbf{y}$. Its variance is $\text{Var}[\mathbf{b}] = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Omega}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} = \sigma^2\boldsymbol{\Lambda}_K$, which means that OLS and GLS are identical in this case.

6. Write $\mathbf{b} = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}$ and $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + (\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{\varepsilon}$. The covariance matrix is

$$E[(\mathbf{b} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'] = E[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'\boldsymbol{\Omega}^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\sigma^2\boldsymbol{\Omega})\boldsymbol{\Omega}^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1} = \sigma^2(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}.$$

For part (b), $\mathbf{e} = \mathbf{M}\boldsymbol{\varepsilon}$ as always, so $E[\mathbf{e}\mathbf{e}'] = \sigma^2\mathbf{M}\boldsymbol{\Omega}\mathbf{M}$. No further simplification is possible for the general case.

For part (c),

$$\begin{aligned}\hat{\boldsymbol{\varepsilon}} = \mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}} &= \mathbf{y} - \mathbf{X}[\boldsymbol{\beta} + (\mathbf{X}'\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\Omega^{-1}\boldsymbol{\varepsilon}] \\ &= \mathbf{X}\boldsymbol{\beta} + \varepsilon - \mathbf{X}[\boldsymbol{\beta} + (\mathbf{X}'\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\Omega^{-1}\boldsymbol{\varepsilon}] \\ &= [\mathbf{I} - \mathbf{X}(\mathbf{X}'\Omega^{-1}\mathbf{X})^{-1}\mathbf{X}'\Omega^{-1}]\boldsymbol{\varepsilon}.\end{aligned}$$

Thus, $E[\hat{\boldsymbol{\varepsilon}}\,\hat{\boldsymbol{\varepsilon}}'] = [\mathbf{I} - \mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Omega}^{-1}]E[\boldsymbol{\varepsilon\varepsilon}'][\mathbf{I} - \mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Omega}^{-1}]'$

$$= [\mathbf{I} - \mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Omega}^{-1}](\sigma^2\boldsymbol{\Omega})[\mathbf{I} - \mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Omega}^{-1}]'$$
$$= [\sigma^2\boldsymbol{\Omega} - \sigma^2\mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}'][\mathbf{I} - \mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Omega}^{-1}]'$$
$$= [\sigma^2\boldsymbol{\Omega} - \sigma^2\mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}'][\mathbf{I} - \boldsymbol{\Omega}^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}']$$
$$= \sigma^2\boldsymbol{\Omega} - \sigma^2\mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}' - \sigma^2\mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}' + \sigma^2\mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1})\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}'$$
$$= \sigma^2[\boldsymbol{\Omega} - \mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}']$$

The GLS residual vector appears in the preceding part. As always, the OLS residual vector is $\mathbf{e} = \mathbf{M}\boldsymbol{\varepsilon} = [\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\boldsymbol{\varepsilon}$. The covariance matrix is

$$E[\mathbf{e}\,\hat{\boldsymbol{\varepsilon}}'] = E[(\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\boldsymbol{\varepsilon\varepsilon}'(\mathbf{I} - \mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Omega}^{-1})']$$
$$= (\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')(\sigma^2\boldsymbol{\Omega})(\mathbf{I} - \boldsymbol{\Omega}^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}')$$
$$= \sigma^2\boldsymbol{\Omega} - \sigma^2\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Omega} - \sigma^2\boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}' + \sigma^2\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}'$$
$$= \sigma^2\boldsymbol{\Omega} - \sigma^2\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$$
$$= \sigma^2\mathbf{M}\boldsymbol{\Omega}. \quad \square$$

7. The GLS estimator is $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}'^{-1}\mathbf{y} = [\Sigma_i\mathbf{x}_i\mathbf{x}_i'/(\boldsymbol{\beta}'\mathbf{x}_i)^2]^{-1}[\Sigma_i\mathbf{x}_i y_i/(\boldsymbol{\beta}'\mathbf{x}_i)^2]$. The log-likelihood for this model is $\ln L = -\Sigma_i\ln(\boldsymbol{\beta}'\mathbf{x}_i) - \Sigma_i y_i/(\boldsymbol{\beta}'\mathbf{x}_i)$.

The likelihood equations are

$$\partial\ln L/\partial\boldsymbol{\beta} = -\Sigma_i(1/\boldsymbol{\beta}'\mathbf{x}_i)\mathbf{x}_i + \Sigma_i[y_i/(\boldsymbol{\beta}'\mathbf{x}_i)^2]\mathbf{x}_i = \mathbf{0}$$

or $\quad \Sigma_i(\mathbf{x}_i y_i/(\boldsymbol{\beta}'\mathbf{x}_i)^2) = \Sigma_i\mathbf{x}_i/(\boldsymbol{\beta}'\mathbf{x}_i)$.

Now, write $\quad \Sigma_i\mathbf{x}_i/(\boldsymbol{\beta}'\mathbf{x}_i) = \Sigma_i\mathbf{x}_i\mathbf{x}_i'\boldsymbol{\beta}/(\boldsymbol{\beta}'\mathbf{x}_i)^2$,

so the likelihood equations are equivalent to $\Sigma_i(\mathbf{x}_i y_i/(\boldsymbol{\beta}'\mathbf{x}_i).^2) = \Sigma_i\mathbf{x}_i\mathbf{x}_i'\boldsymbol{\beta}/(\boldsymbol{\beta}'\mathbf{x}_i).^2$, or $\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{y} = (\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})\boldsymbol{\beta}$. These are the normal equations for the GLS estimator, so the two estimators are the same. We should note, the solution is only implicit, since $\boldsymbol{\Omega}$ is a function of $\boldsymbol{\beta}$. For another more common application, see the discussion of the FIML estimator for simultaneous equations models in Chapter 13.

8. The covariance matrix is

$$\sigma^2\boldsymbol{\Omega} = \sigma^2\begin{bmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \rho & \rho & 1 & \cdots & \rho \\ & & & \vdots & \\ \rho & \rho & \rho & \cdots & 1 \end{bmatrix}.$$

The matrix **X** is a column of 1s, so the least squares estimator of μ is $\bar{y}$. Inserting this $\boldsymbol{\Omega}$ into (10-5), we obtain $\text{Var}[\bar{y}] = \dfrac{\sigma^2}{n}(1-\rho+n\rho)$. The limit of this expression is $\rho\sigma^2$, not zero. Although ordinary least squares is unbiased, it is not consistent. For this model, $\mathbf{X}'\boldsymbol{\Omega}\mathbf{X}/n = 1 + \rho(n-1)$, which does not converge. Using Theorem 8.2 instead, **X** is a column of 1s, so $\mathbf{X}'\mathbf{X} = n$, a scalar, which satisfies condition 1. To find the characteristic roots, multiply out the equation $\boldsymbol{\Omega}\mathbf{x} = \lambda\mathbf{x} = (1-\rho)\mathbf{I}\mathbf{x} + \rho\mathbf{i}\mathbf{i}'\mathbf{x} = \lambda\mathbf{x}$. Since $\mathbf{i}'\mathbf{x} = \Sigma_i x_i$, consider any vector **x** whose elements sum to zero. If so, then it's obvious that $\lambda = \rho$. There are n-1 such roots. Finally, suppose that $\mathbf{x} = \mathbf{i}$. Plugging this into the equation produces $\lambda = 1 - \rho + n\rho$. The characteristic roots of $\boldsymbol{\Omega}$ are $(1-\rho)$ with multiplicity $n-1$ and $(1-\rho+n\rho)$, which violates condition 2.

9. This is a heteroscedastic regression model in which the matrix **X** is a column of ones. The efficient estimator is the GLS estimator, $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{y} = [\Sigma_i 1y_i/x_i^2] / [\Sigma_i 1^2/\mathbf{x}_i^2] = [\Sigma_i(y_i/x_i^2)] / [\Sigma_i(1/x_i^2)]$. As always, the variance of the estimator is $\text{Var}[\hat{\boldsymbol{\beta}}] = \sigma^2(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1} = \sigma^2/[\Sigma_i(1/x_i^2)]$. The ordinary least squares estimator is $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \bar{y}$. The variance of $\bar{y}$ is $\sigma^2(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\boldsymbol{\Omega}\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1} = (\sigma^2/n^2)\Sigma_i x_i^2$. To show that the variance of the OLS estimator is greater than or equal to that of the GLS estimator, we must show that $(\sigma^2/n^2)\Sigma_i x_i^2 \geq \sigma^2/\Sigma_i(1/x_i^2)$ or $(1/n^2)(\Sigma_i x_i^2)(\Sigma_i(1/x_i^2)) \geq 1$ or $\Sigma_i\Sigma_j(x_i^2/x_j^2) \geq n^2$. The double sum contains n terms equal to one. There remain $n(n$-1)/2 pairs of the form $(x_i^2/x_j^2 + x_j^2/x_i^2)$. If it can be shown that each of these

sums is greater than or equal to 2, the result is proved. Just let $z_i = x_i^2$. Then, we require $z_i/z_j + z_j/z_i - 2 \geq 0$. But, this is equivalent to $(z_i^2 + z_j^2 - 2z_iz_j) / z_iz_j \geq 0$ or $(z_i - z_j)^2/z_iz_j \geq 0$, which is certainly true if z_i and z_j are positive. They are since z_i equals x_i^2. This completes the proof.

10. Consider, first, $\bar{y}$. We saw earlier that Var[$\bar{y}$] = $(\sigma^2/n^2)\Sigma_i x_i^2 = (\sigma^2/n)(1/n)\Sigma_i x_i^2$. The expected value is $E[\bar{y}] = E[(1/n)\Sigma_i y_i] = \alpha$. If the mean square of x converges to something finite, then $\bar{y}$ is consistent for α. That is, if $\text{plim}(1/n)\Sigma_i x_i^2 = \bar{q}$ where $\bar{q}$ is some finite number, then, plim $\bar{y} = \alpha$. As such, it follows that s^2 and $s_*^2 = (1/(n-1))\Sigma_i(y_i - \alpha)^2$ have the same probability limit. We consider, therefore, plim $s_*^2 = \text{plim}(1/(n-1))\Sigma_i\varepsilon_i^2$. The expected value of s_*^2 is $E[(1/(n-1))\ \Sigma_i\varepsilon_i^2] = \sigma^2(1/\Sigma_i x_i^2)$. Once again, nothing more can be said without some assumption about x_i. Thus, we assume again that the average square of x_i converges to a finite, positive constant, $\bar{q}$. Of course, the result is unchanged by division by $(n-1)$ instead of n, so $\lim_{n\to\infty} E[s_*^2] = \sigma^2\bar{q}$. The variance of s_*^2 is $\text{Var}[s_*^2] = \Sigma_i\text{Var}[\varepsilon_i^2]/(n - 1)^2$. To characterize this, we will require the variances of the squared disturbances, which involves their fourth moments. But, if we assume that every fourth moment is finite, then the preceding is $(n/(n-1)^2)$ times the average of these fourth moments. If every fourth moment is finite, then the term is dominated by the leading $(n/(n-1)^2)$ which converges to zero. It follows that plim $s_*^2 = \sigma^2\bar{q}$. Therefore, the conventional estimator estimates Asy.Var[$\bar{y}$]= $\sigma^2\bar{q}/n$.

The appropriate variance of the least squares estimator is Var[$\bar{y}$]= $(\sigma^2/n^2)\Sigma_i x_i^2$, which is, of course, precisely what we have been analyzing above. It follows that the conventional estimator of the variance of the OLS estimator in this model is an appropriate estimator of the true variance of the least squares estimator. This follows from the fact that the regressor in the model, **i**, is unrelated to the source of heteroscedasticity, as discussed in the text.

11. The sample moments are obtained using, for example, $S_{xx} = \mathbf{x}'\mathbf{x} - n\bar{x}^2$ and so on. For the two samples, we obtain

	$\bar{y}$	$\bar{x}$	S_{xx}	S_{yy}	S_{xy}
Sample 1	6	6	300	300	200
Sample 2	6	6	300	1000	400

The parameter estimates are computed directly using the results of Chapter 6.

	Intercept	Slope	R^2	s^2
Sample 1	2	2/3	4/9	(1500/9)/48 = 3.472
Sample 2	-2	4/3	16/30	(4200/9)/48 = 9.722

The pooled moments based on 100 observations are $\mathbf{X}'\mathbf{X} = \begin{bmatrix} 100 & 600 \\ 600 & 4200 \end{bmatrix}$, $\mathbf{X}'\mathbf{y} = \begin{bmatrix} 600 \\ 4200 \end{bmatrix}$, $\mathbf{y}'\mathbf{y} = 4900$. The coefficient vector based on these data is $[a,b] = [0,1]$. This might have been predicted since the two **X′X** matrices are identical. OLS which ignores the heteroscedasticity would simply average the estimates. The sum of squared residuals would be $\mathbf{e}'\mathbf{e} = \mathbf{y}'\mathbf{y} - \mathbf{b}'\mathbf{X}'\mathbf{y} = 4900 - 4200 = 700$, so the estimate of σ^2 is $s^2 = 700/98 = 7.142$. Note that the earlier values obtained were 3.472 and 9.722, so the pooled estimate is between the two, once again, as might be expected. The asymptotic covariance matrix of these estimates is $s^2(\mathbf{X}'\mathbf{X})^{-1}$ $= 7.142\begin{bmatrix} .07 & -.01 \\ -.01 & .167 \end{bmatrix}$.

To test the equality of the variances, we can use the Goldfeld and Quandt test. Under the null hypothesis of equal variances, the ratio $F = [\mathbf{e}_1'\mathbf{e}_1/(n_1 - 2)]/[\mathbf{e}_2'\mathbf{e}_2/(n_2 - 2)]$ (or vice versa for the subscripts) is the ratio of two independent chi-squared variables each divided by their respective degrees of freedom. Although it might seem so from the discussion in the text (and the literature) there is nothing in the test which requires that the coefficient vectors be assumed equal across groups. Since for our data, the second sample has the larger residual variance, we refer $F[48,48] = s_2^2/s_1^2 = 9.722 / 3.472 = 2.8$ to the F table. The critical value for 95% significance is 1.61, so the hypothesis of equal variances is rejected.

The method of Example 8.5 can be applied to this groupwise heteroscedastic model. The two step estimator is $\hat{\boldsymbol{\beta}} = [(1/s_1^2)\mathbf{X}_1'\mathbf{X}_1 + (1/s_2^2)\mathbf{X}_2'\mathbf{X}_2]^{-1}[(1/s_1^2)\mathbf{X}_1'\mathbf{y}_1 + (1/s_2^2)\mathbf{X}_2'\mathbf{y}_2]$. The **X′X** matrices are the same in this problem, so this simplifies to $\hat{\beta} = [(1/s_1^2 + 1/s_2^2)\mathbf{X}'\mathbf{X}]^{-1}[(1/s_1^2)\mathbf{X}_1'\mathbf{y}_1 + (1/s_2^2)\mathbf{X}_2'\mathbf{y}_2]$. The estimator is, therefore $\left[\left(\frac{1}{3.472}+\frac{1}{9.722}\right)\begin{pmatrix}50 & 300\\300 & 2100\end{pmatrix}\right]^{-1}\left[\frac{1}{3.472}\begin{pmatrix}300\\2000\end{pmatrix}+\frac{1}{9.722}\begin{pmatrix}300\\2200\end{pmatrix}\right]=\begin{pmatrix}.9469\\.8422\end{pmatrix}$.

```
?=======================================================
? Application 8.1
?=======================================================
```

a. The ordinary least squares regression of Y on a constant, X_1, and X_2 produces the following results:

Sum of squared residuals	1911.9275		
R^2	.03790		
Standard error of regression	6.3780		
Variable	**Coefficient**	**Standard Error**	**t-ratio**
One	.190394	.9144	.208
X_1	1.13113	.9826	1.151
X_2	.376825	.4399	.857

b.

Covariance Matrix			**White's Corrected Matrix**		
.836212			.524589		
-.115451	.96551		.076578	.282366	
-.047133	.051081	.193532	.399218	-.091608	1.14447

c. To apply White's test, we first obtain the residuals from the regression of Y on a constant, X_1, and X_2. Then, we regress the squares of these residuals on a constant, X_1, X_2, X_1^2, X_2^2, and X_1X_2. The R^2 in this regression is .78296, so the chi-squared statistic is $50\times0.78296 = 39.148$. The critical value from the table of chi-squared with 5 degrees of freedom is 11.08, so we would conclude that there is evidence of heteroscedasticity.

d. Lagrange multiplier test.

```
Regress;Lhs=y;rhs=one,x1,x2 ; Res=e ; het $
create ; lmi=e*e/(sumsqdev/n) - 1 $
Name ; x=one,x1,x2 $
Calc ; list ; .5*xss(x,lmi)$
The result was reported with the regression,
| Br./Pagan LM Chi-sq [  2]  (prob) =  72.78 (.0000) |
```

e. Two step estimator

read;nobs=50;nvar=1;names=y;byva $

```
   -1.42   2.75   2.10  -5.08   1.49   1.00    .16  -1.11   1.66
    -.26  -4.87   5.94   2.21  -6.87    .90   1.61   2.11  -3.82
    -.62   7.01  26.14   7.39    .79   1.93   1.97 -23.17  -2.52
   -1.26   -.15   3.41  -5.45   1.31   1.52   2.04   3.00   6.31
    5.51 -15.22  -1.47  -1.48   6.66   1.78   2.62  -5.16  -4.71
    -.35   -.48   1.24    .69   1.91
```

read;nobs=50;nvar=1;names=x1;byva $

```
   -1.65   1.48    .77    .67    .68    .23   -.40  -1.13    .15
    -.63    .34    .35    .79    .77  -1.04    .28    .58   -.41
   -1.78   1.25    .22   1.25   -.12    .66   1.06   -.66  -1.18
    -.80  -1.32    .16   1.06   -.60    .79    .86   2.04   -.51
     .02    .33  -1.99    .70   -.17    .33    .48   1.90   -.18
    -.18  -1.62    .39    .17   1.02
```

read;nobs=50;nvar=1;names=x2;byva $

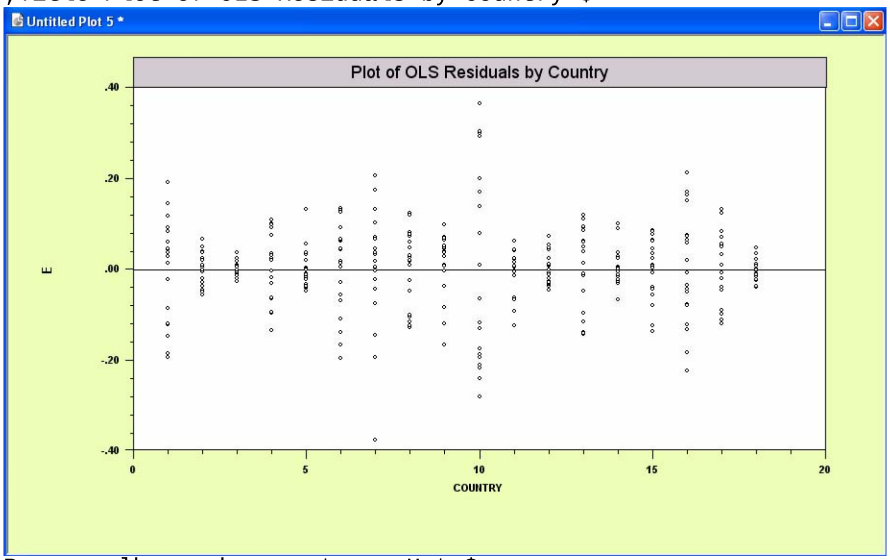
```
    -.67    .70    .32   2.88   -.19  -1.28  -2.72   -.70  -1.55
    -.74  -1.87   1.56    .37  -2.07   1.20    .26  -1.34  -2.10
     .61   2.32   4.38   2.16   1.51    .30   -.17   7.82  -1.15
    1.77   2.92  -1.94   2.09   1.50   -.46    .19   -.39   1.54
    1.87  -3.45   -.88  -1.53   1.42  -2.70   1.77  -1.89  -1.85
    2.01   1.26  -2.02   1.91  -2.23
Regress;Lhs=y;rhs=one,x1,x2 ; Res=e $
+-----------------------------------------------+
| Ordinary    least squares regression          |
| Model was estimated May 12, 2007 at 08:33:20PM |
| LHS=Y        Mean                 =   .3938000 |
|              Standard deviation   =   6.368374 |
| WTS=none     Number of observs.   =         50 |
| Model size   Parameters           =          3 |
```

```
|               Degrees of freedom   =        47      |
| Residuals     Sum of squares       =   1911.928     |
|               Standard error of e  =   6.378033     |
| Fit           R-squared            =   .3790450E-01 |
|               Adjusted R-squared   =  -.3035736E-02 |
| Model test    F[  2,    47] (prob) =    .93 (.4033) |
| Diagnostic    Log likelihood       =  -162.0430     |
|               Restricted(b=0)      =  -163.0091     |
|               Chi-sq [  2]  (prob) =   1.93 (.3806) |
| Info criter. LogAmemiya Prd. Crt. =   3.763988     |
|               Akaike Info. Criter. =   3.763844     |
| Autocorrel    Durbin-Watson Stat.  =  1.8560359     |
|               Rho = cor[e,e(-1)]   =   .0719820     |
+-----------------------------------------------------+
+--------+-------------+---------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+-------------+---------------+--------+--------+----------+
 Constant|     .19039401       .91444640      .208   .8360
 X1      |    1.13113339       .98260352     1.151   .2555    .10820000
 X2      |     .37682493       .43992218      .857   .3960    .21500000
Create ; e2 = e*e $
Create ; loge2 = log(e2) $
Regress ; lhs = loge2 ; Rhs = one,x1,x2 ; keep=vi $
Create ; vi = 1/exp(vi) $
Regress ; Lhs = y ; rhs = one,x1,x2 ; wts = vi $
+-----------------------------------------------------+
| Ordinary    least squares regression                |
| Model was estimated May 12, 2007 at 08:33:20PM      |
| LHS=Y         Mean                 =  -.5316339     |
|               Standard deviation   =   4.535703     |
| WTS=VI        Number of observs.   =        50      |
| Model size    Parameters           =         3      |
|               Degrees of freedom   =        47      |
| Residuals     Sum of squares       =   890.9017     |
|               Standard error of e  =   4.353775     |
| Fit           R-squared            =   .1162193     |
|               Adjusted R-squared   =   .7861157E-01 |
| Model test    F[  2,    47] (prob) =   3.09 (.0548) |
| Diagnostic    Log likelihood       =  -150.0732     |
|               Restricted(b=0)      =  -153.1619     |
|               Chi-sq [  2]  (prob) =   6.18 (.0456) |
| Info criter. LogAmemiya Prd. Crt. =   3.000355     |
|               Akaike Info. Criter. =   3.285051     |
| Autocorrel    Durbin-Watson Stat.  =  1.9978648     |
|               Rho = cor[e,e(-1)]   =   .0010676     |
+-----------------------------------------------------+
+--------+-------------+---------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+-------------+---------------+--------+--------+----------+
 Constant|     .16662621       .71981411      .231   .8179
 X1      |     .77648745       .63883379     1.215   .2303   -.51884171
 X2      |     .84717700       .36328984     2.332   .0240   -.34867101
```

Applications

```
?=======================================================
? Application 8.2  Gasoline Consumption
?=======================================================
? Rename variable for convenience
Create ; y=lgaspcar $
? RHS of new regression
Namelist ; x = one,lincomep,lrpmg,lcarpcap $
? Base regression.  Is cars per capita significant?
Regress ; Lhs = y ; Rhs = x $
+-----------------------------------------------------+
| Ordinary    least squares regression                |
| LHS=Y        Mean                 =   4.296242      |
|              Standard deviation   =   .5489071      |
| WTS=none     Number of observs.   =        342      |
| Model size   Parameters           =          4      |
|              Degrees of freedom   =        338      |
| Residuals    Sum of squares       =   14.90436      |
|              Standard error of e  =   .2099898      |
| Fit          R-squared            =   .8549355      |
|              Adjusted R-squared   =   .8536479      |
| Model test   F[  3,   338] (prob) = 664.00 (.0000) |
+-----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    2.39132562       .11693429    20.450   .0000
 LINCOMEP|     .88996166       .03580581    24.855   .0000  -6.13942544
 LRPMG   |    -.89179791       .03031474   -29.418   .0000   -.52310321
 LCARPCAP|    -.76337275       .01860830   -41.023   .0000  -9.04180473Calc ; r0
= rsqrd $
Namelist ; Cntry=c2,c3,c4,c5,c6,c7,c8,c9,c10,c11,c12,c13,c14,c15,c16,c17,c18$
Regress;lhs=y;rhs=x,cntry ; Res = e $
+-----------------------------------------------------+
| Ordinary    least squares regression                |
| LHS=Y        Mean                 =   4.296242      |
|              Standard deviation   =   .5489071      |
| WTS=none     Number of observs.   =        342      |
| Model size   Parameters           =         21      |
|              Degrees of freedom   =        321      |
| Residuals    Sum of squares       =   2.736491      |
|              Standard error of e  =   .9233035E-01 |
| Fit          R-squared            =   .9733657      |
|              Adjusted R-squared   =   .9717062      |
| Model test   F[ 20,   321] (prob) = 586.56 (.0000) |
+-----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    2.28585577       .22832349    10.011   .0000
 LINCOMEP|     .66224966       .07338604     9.024   .0000  -6.13942544
 LRPMG   |    -.32170246       .04409925    -7.295   .0000   -.52310321
 LCARPCAP|    -.64048288       .02967885   -21.580   .0000  -9.04180473
 C2      |    -.12030455       .03414942    -3.523   .0005    .05555556
 C3      |     .75598453       .04074554    18.554   .0000    .05555556
 C4      |     .10360026       .03660467     2.830   .0049    .05555556
 C5      |    -.08108439       .03356343    -2.416   .0163    .05555556
 C6      |    -.13598740       .03187957    -4.266   .0000    .05555556
 C7      |     .05125389       .04152961     1.234   .2180    .05555556
 C8      |     .30646950       .03529373     8.683   .0000    .05555556
 C9      |    -.05330785       .03711258    -1.436   .1519    .05555556
 C10     |     .09007170       .03860659     2.333   .0203    .05555556
 C11     |    -.05106438       .03357607    -1.521   .1293    .05555556
 C12     |    -.06915517       .04040779    -1.711   .0880    .05555556
```

```
 C13      |     -.60407878       .09122015     -6.622   .0000      .05555556
 C14      |      .74048679       .18008419      4.112   .0000      .05555556
 C15      |      .11664698       .03471246      3.360   .0009      .05555556
 C16      |      .22413229       .04764432      4.704   .0000      .05555556
 C17      |      .05959184       .03018816      1.974   .0492      .05555556
 C18      |      .76939510       .04457642     17.260   .0000      .05555556
Calc ; r1 = rsqrd $
Calc ; list ; Fstat = ((r1 - r0)/17) / ((1-r1)/(n-4-17)) $
Calc ; list ; Fc    =ftb(.95,17,(n-4-17)) $
+--------------------------------------+
| Listed Calculator Results            |
+--------------------------------------+
 FSTAT    =     83.960798
 FC       =      1.654675
Plot ; lhs = country ; rhs = e ; Bars = 0
;Title=Plot of OLS Residuals by Country $
```




```
Regress;lhs=y;rhs=x,cntry ; Het $
+----------------------------------------------------+
| Ordinary    least squares regression               |
| LHS=Y        Mean                 =   4.296242     |
|              Standard deviation   =   .5489071     |
| WTS=none     Number of observs.   =        342     |
| Model size   Parameters           =         21     |
|              Degrees of freedom   =        321     |
| Residuals    Sum of squares       =   2.736491     |
|              Standard error of e  =   .9233035E-01 |
| Fit          R-squared            =   .9733657     |
|              Adjusted R-squared   =   .9717062     |
| Model test   F[ 20,    321] (prob) = 586.56 (.0000) |
| White heteroscedasticity robust covariance matrix  |
| Br./Pagan LM Chi-sq [ 20]  (prob) = 338.94 (.0000) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|     2.28585577       .22608070     10.111   .0000
 LINCOMEP|      .66224966       .07277408      9.100   .0000   -6.13942544
 LRPMG   |     -.32170246       .05381258     -5.978   .0000    -.52310321
 LCARPCAP|     -.64048288       .03876145    -16.524   .0000   -9.04180473
 C2      |     -.12030455       .03160815     -3.806   .0002     .05555556
 C3      |      .75598453       .03692877     20.471   .0000     .05555556
 C4      |      .10360026       .03642008      2.845   .0047     .05555556
 C5      |     -.08108439       .03252022     -2.493   .0132     .05555556
 C6      |     -.13598740       .03504274     -3.881   .0001     .05555556
 C7      |      .05125389       .05768530       .889   .3749     .05555556
 C8      |      .30646950       .03516370      8.716   .0000     .05555556
```

```
 C9      |    -.05330785       .04078467    -1.307   .1921    .05555556
 C10     |     .09007170       .05606508     1.607   .1091    .05555556
 C11     |    -.05106438       .03228064    -1.582   .1147    .05555556
 C12     |    -.06915517       .03857838    -1.793   .0740    .05555556
 C13     |    -.60407878       .09798870    -6.165   .0000    .05555556
 C14     |     .74048679       .18836593     3.931   .0001    .05555556
 C15     |     .11664698       .03500336     3.332   .0010    .05555556
 C16     |     .22413229       .08147015     2.751   .0063    .05555556
 C17     |     .05959184       .03166823     1.882   .0608    .05555556
 C18     |     .76939510       .04121364    18.668   .0000    .05555556

Create ; e2 = e*e $
Regress ; Lhs = e2 ; Rhs = one,cntry $
Calc ; List ; White = n*rsqrd ;  ctb(.95,17) $
+-----------------------------------+
| Listed Calculator Results         |
+-----------------------------------+
 WHITE   =    131.209847
 Result  =     27.587112
Calc ; s2 = e'e/n $
Matrix ; s2g = {1/19} * cntry'e2
       ; s2g = 1/s2 * s2g
       ; g = s2g - 1
       ; List ; lmstat = {19/2}*g'g  $
Matrix LMSTAT   has  1 rows and  1 columns.
        +--------------
       1|  277.00947
Name   ; All = c1,cntry $
Matrix ; vg = 1/19*all'e2 $
Create ; wt = 1/vg(country) $
Regress ; Lhs = y ; rhs = x,cntry;wts=wt $
+-----------------------------------------------------+
| Ordinary    least squares regression                |
| LHS=Y        Mean                 =   4.460122      |
|              Standard deviation   =   .4535009      |
| WTS=WT       Number of observs.   =        342      |
| Model size   Parameters           =         21      |
|              Degrees of freedom   =        321      |
| Residuals    Sum of squares       =   .5901434      |
|              Standard error of e  =   .4287719E-01  |
| Fit          R-squared            =   .9915851      |
|              Adjusted R-squared   =   .9910608      |
| Model test   F[ 20,   321] (prob) =1891.29 (.0000)  |
+-----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    2.43706653       .11308370    21.551   .0000
 LINCOMEP|     .57506962       .02926687    19.649   .0000  -5.84790214
 LRPMG   |    -.27967108       .03518536    -7.949   .0000   -.87736963
 LCARPCAP|    -.56540465       .01613491   -35.042   .0000  -8.34742189
 C2      |    -.12007208       .02789011    -4.305   .0000    .08866789
 C3      |     .76945446       .03011060    25.554   .0000    .34252221
 C4      |     .11000512       .03169158     3.471   .0006    .01995470
 C5      |    -.09845013       .02921659    -3.370   .0008    .05724878
 C6      |    -.13641007       .03387520    -4.027   .0001    .01079455
 C7      |     .13502296       .04413211     3.060   .0024    .00604952
 C8      |     .28669153       .03200056     8.959   .0000    .01577251
 C9      |    -.08901681       .03324265    -2.678   .0078    .01701683
 C10     |     .15281210       .05659004     2.700   .0073    .00228044
 C11     |    -.04087890       .02882321    -1.418   .1571    .03809105
 C12     |    -.05220341       .02952832    -1.768   .0780    .09438377
 C13     |    -.53400193       .06166458    -8.660   .0000    .01328985
 C14     |     .64117855       .10737812     5.971   .0000    .06594614
 C15     |     .12783552       .03189740     4.008   .0001    .02454617
 C16     |     .38638811       .05013313     7.707   .0000    .00712693
 C17     |     .04507072       .03121765     1.444   .1498    .01629698
```

```
 C18      |      .77812476       .03277077    23.744   .0000    .17152029


?========================================================
? Application 8.3  Iterative estimator
?========================================================
create ; logc = log(c) ; logq=log(q) ; logq2=logq^2 ; logp=log(pf) $
Name ; x = one,logq,logq2,logp $
Regress ; lhs = logc ; rhs = x ; Res = e $
Matrix ; b0=b $
Procedure$
Create ; e2 = e*e
; le = e2/(sumsqdev/n)-1 $ (MLE)
?le = log(e2) $            (Iterative two step)
Regress ; quiet ; lhs=le ; rhs=one,lf ; keep = s2i $
Create ; wi = 1/exp(s2i) $
Regress ; lhs = logc ; rhs = x ; wts=wi ; res=e $
Matrix ; db = b-b0 ; b0 = b $
Calc ; list ; db2 = db'db $
Endproc $
Exec ; n = 10 $
These are the two step estimators from Example 8.4
+----------------------------------------------------+
| Ordinary    least squares regression               |
| LHS=LOGC     Mean                 =   12.92005     |
|              Standard deviation   =   1.192244     |
| WTS=WI       Number of observs.   =         90     |
| Model size   Parameters           =          4     |
|              Degrees of freedom   =         86     |
| Residuals    Sum of squares       =   1.212889     |
|              Standard error of e  =   .1187576     |
| Fit          R-squared            =   .9904126     |
|              Adjusted R-squared   =   .9900782     |
| Model test   F[  3,    86] (prob) =2961.37 (.0000) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    9.27731457       .20978736    44.222   .0000
 LOGQ    |     .91610564       .03299348    27.766   .0000  -1.56779393
 LOGQ2   |     .02164855       .01101812     1.965   .0527   3.87530677
 LOGP    |     .40174171       .01633292    24.597   .0000   12.4336185
These are the maximum likelihood estimates
+----------------------------------------------------+
| Ordinary    least squares regression               |
| Residuals    Sum of squares       =   1.347926     |
|              Standard error of e  =   .1251941     |
| Fit          R-squared            =   .9892110     |
|              Adjusted R-squared   =   .9888346     |
| Model test   F[  3,    86] (prob) =2628.35 (.0000) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    9.24395222       .21962091    42.090   .0000
 LOGQ    |     .92163069       .03302261    27.909   .0000  -1.43646434
 LOGQ2   |     .02461767       .01143734     2.152   .0342   3.46800689
 LOGP    |     .40366011       .01701993    23.717   .0000   12.5455161
```

Chapter 9

Models for Panel Data

1. The pooled least squares estimator is

$$\hat{y} = \underset{(.95595)}{-.747476} + \underset{(.058656)}{1.058959}x, \quad \mathbf{e}'\mathbf{e} = 120.6687$$

The fixed effects regression can be computed just by including the three dummy variables since the sample sizes are quite small. The results are

$$\hat{y} = -1.4684i_1 - 2.8362i_2 + .12166i_3 + \underset{(.050719)}{1.102192}x \qquad \mathbf{e}'\mathbf{e} = 79.183.$$

The F statistic for testing the hypothesis that the constant terms are all the same is

$F[26,2] = [(120.6687 - 79.183)/2]/[79.183/26] = 6.811.$

The critical value from the F table is 19.458, so the hypothesis is not rejected.

In order to estimate the random effects model, we need some additional parameter estimates. The group means are

	$\bar{y}$	$\bar{x}$
Group 1	15.502	14.962
Group 2	15.415	16.559
Group 3	14.373	12.930

In the group means regression using these three observations, we obtain

$$\bar{y}_{i.} = 10.665 + .29909\,\bar{x}_{i.} \text{ with } \mathbf{e}_{**}'\mathbf{e}_{**} = .19747.$$

There is only one degree of freedom, so this is the candidate for estimation of $\sigma_\varepsilon^2/T + \sigma_u^2$. In the least squares dummy variable (fixed effects) regression, we have an estimate of σ_ε^2 of 79.183/26 = 3.045. Therefore, our estimate of σ_u^2 is $\hat{\sigma}_u^2 = .19747/1 - 3.045/10 = -.6703$. Obviously, this won't do. Before abandoning the random effects model, we consider an alternative consistent estimator of the constant and slope, the pooled ordinary least squares estimator. Using the group means above, we find

$$\Sigma_{i=1}^{3} [\,\bar{y}_{i.} - (-.747476) - 1.058959\,\bar{x}_{i.}]^2 = 3.9273.$$

One ought to proceed with some caution at this point, but it is difficult to place much faith in the group means regression with but a single degree of freedom, so this is probably a preferable estimator in any event. (The true model underlying these data -- using a random number generator -- has a slope, β of 1.000 and a true constant of zero. Of course, this would not be known to the analyst in a real world situation.) Continuing, we now use $\hat{\sigma}_u^2 = 3.9273 - 3.045/10 = 3.6227$ as the estimator. (The true value of $\rho = \sigma_u^2/(\sigma_u^2+\sigma_\varepsilon^2)$ is .5.) This leads to $\theta = 1 - [3.0455^{1/2}/(10(3.6227) + 3.045)^{1/2}] = .721524$. Finally, the FGLS estimator computed according to (16-48) is $\hat{y} = -1.3415(.786) + 1.0987\,(.028998)x$.

For the LM test, we return to the pooled ordinary least squares regression. The necessary quantities are $\mathbf{e}'\mathbf{e} = 120.6687$, $\Sigma_t e_{1t} = -.55314$, $\Sigma_t e_{2t} = -13.72824$, $\Sigma_t e_{3t} = 14.28138$. Therefore,

$LM = \{[3(10)]/[2(9)]\}\{[(-.55314)^2 + (13.72824)^2 + (14.28138)^2]/120.687 - 1\}^2 = 8.4683$

The statistic has one degree of freedom. The critical value from the chi-squared distribution is 3.84, so the hypothesis of no random effect is rejected. Finally, for the Hausman test, we compare the FGLS and least squares dummy variable estimators. The statistic is $\chi^2 = [(1.0987 - 1.058959)^2]/[(.058656)^2 - (.05060)^2] = 1.794373$. This is relatively small and argues (once again) in favor of the random effects model. □

2. There is no effect on the coefficients of the other variables. For the dummy variable coefficients, with the full set of n dummy variables, each coefficient is

$\bar{y}_i$ * = mean residual for the ith group in the regression of y on the xs omitting the dummy variables.

(We use the partitioned regression results of Chapter 6.) If an overall constant term and n-1 dummy variables (say the last n-1) are used, instead, the coefficient on the ith dummy variable is simply $\bar{y}_i^* - \bar{y}_1^*$ while the constant term is still $\bar{y}_1^*$ For a full proof of these results, see the solution to Exercise 5 of Chapter 8 earlier in this book.

3. (a) The pooled OLS estimator will be $\mathbf{b} = \left[\Sigma_{i=1}^n \mathbf{X}_i'\mathbf{X}_i\right]^{-1}\left[\Sigma_{i=1}^n \mathbf{X}_i'\mathbf{y}_i\right]$ where X_i and y_i have T_i observations. It remains true that $\mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta} + \boldsymbol{\varepsilon}_i + u_i\mathbf{i}$, where $\text{Var}[\boldsymbol{\varepsilon}_i + u_i\mathbf{i}|X_i] = \text{Var}[\mathbf{w}_i|\mathbf{X}_i] = \sigma_\varepsilon^2\mathbf{I} + \sigma_u^2\mathbf{ii}'$ and, maintaining the assumptions, both ε_i and u_i are uncorrelated with X_i. Substituting the expression for y_i into that of b and collecting terms, we have

$$\mathbf{b} = \boldsymbol{\beta} + \left[\Sigma_{i=1}^n \mathbf{X}_i'\mathbf{X}_i\right]^{-1}\left[\Sigma_{i=1}^n \mathbf{X}_i'\mathbf{w}_i\right].$$

Unbiasedness follows immediately as long as $E[w_i|X_i]$ equals zero, which it does by assumption. Consistency, as mentioned in Section 9.3.2, is covered in the discussion of Chapter 4. We would need for the matrix $\mathbf{Q}$ $= \left[\frac{1}{n}\Sigma_{i=1}^n \frac{1}{T_i}\mathbf{X}_i'\mathbf{X}_i\right]$ to converge to a matrix of constants, or not to degenerate to a matrix of zeros. The requirements for the large sample behavior of the vector in the second set of brackets is quite the same as in our earlier discussions of consistency. The vector $(1/n)\Sigma_{i=1}^n \mathbf{X}_i'\mathbf{w}_i = (1/n)\Sigma_{i=1}^n \mathbf{v}_i$ has mean zero. We would require the conditions of the Lindeberg-Feller version of the central theorem to apply, which could be expected.

(b) We seek to establish consistency, not unbiasedness. As such, we will ignore the degrees of freedom correction, -K, in (9-37). Use n(T-1) as the denominator. Thus, the question is whether

$$\text{plim}\frac{\Sigma_{i=1}^n\Sigma_{t=1}^T(e_{it} - \bar{e}_{i.})^2}{n(T-1)} = \sigma_\varepsilon^2$$

If so, then the estimator in (9-37) will be consistent. Using (9-33) and e_{it} - $\bar{e}_i = \bar{y}_i - \bar{\mathbf{x}}_i'\mathbf{b} - a_i$, it follows that $e_{it} - \bar{e}_i = \varepsilon_{it} - \bar{\varepsilon}_i - (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\mathbf{b} - \boldsymbol{\beta})$. Summing the squares in (9-37), we find that the estimator in (9-37)

$$\frac{\Sigma_{i=1}^n\Sigma_{t=1}^T(e_{it} - \bar{e}_{i.})^2}{n(T-1)} = \frac{1}{n}\sum_{i=1}^n\hat{\sigma}^2(i) + (\mathbf{b}-\boldsymbol{\beta})'\left[\frac{1}{n}\sum_{i=1}^n\frac{1}{T}\sum_{t=1}^T(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)'\right](\mathbf{b}-\boldsymbol{\beta})$$

$$- 2(\mathbf{b}-\boldsymbol{\beta})'\left[\frac{1}{n}\sum_{i=1}^n\frac{1}{T}\sum_{t=1}^T(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)(\varepsilon_{it} - \bar{\varepsilon}_{i.})'\right]$$

The second term will converge to zero as the center matrix converges to a constant Q and the vectors converge to zero as b converges to β. (We use the Slutsky theorem.) The third term will converge to zero as both the leading vector converges to zero and the covariance vector between the regressors and the disturbances converges to zero. That leaves the first term, which is the average of the estimators in (9-34). The terms in the average are independent. Each has expected value exactly equal to σ_ε^2. So, if each estimator has finite variance, then the average will converge to its expectation. Appendix D discusses various different conditions underwhich a sample average will converge to its expectation. For example, finite fouth moment of ε_{it} would be sufficient here (though weaker conditions would also suffice). Note that this derivation follows through for any consistent estimator of β, not just for **b**.

4. To find plim(1/n)LM = plim $[T/(2(T\text{-}1))]\{[\Sigma_i(\Sigma_t e_{it})^2]/[\Sigma_i\Sigma_t e_{it}^2] - 1\}^2$ we can concentrate on the sums inside the curled brackets. First, $\Sigma_i(\Sigma_t e_{it})^2 = nT^2\{(1/n)\Sigma_i[(1/T)\Sigma_t e_{it}]^2\}$ and $\Sigma_i\Sigma_t e_{it}^2 = nT(1/(nT))\Sigma_i\Sigma_t e_{it}^2$. The ratio equals $[\Sigma_i(\Sigma_t e_{it})^2]/[\Sigma_i\Sigma_t e_{it}^2] = T\{(1/n)\Sigma_i[(1/T)\Sigma_t e_{it}]^2\}/\{(1/(nT))\Sigma_i\Sigma_t e_{it}^2\}$. Using the argument used in Exercise 8 to establish consistency of the variance estimator, the limiting behavior of this statistic is the same as that which is computed using the true disturbances since the OLS coefficient estimator is consistent. Using the true disturbances, the numerator may be written $(1/n)\Sigma_i[(1/T)\Sigma_t\varepsilon_{it}]^2 = (1/n)\Sigma_i\bar{\varepsilon}_{i.}^2$. Since $E[\bar{\varepsilon}_{i.}] = 0$,

$\text{plim}(1/n)\Sigma_i \bar{\varepsilon}_{i.}^2$ = Var[$\bar{\varepsilon}_{i.}$] = $\sigma_\varepsilon^2 T + \sigma_u^2$The denominator is simply the usual variance estimator, so $\text{plim}(1/(nT))\Sigma_i\Sigma_t\varepsilon_{it}^2$ = Var[ε_{it}] = $\sigma_\varepsilon^2 + \sigma_u^2$Therefore, inserting these results in the expression for LM, we find that plim (1/n)LM = $[T/(2(T-1))]\{[T(\sigma_\varepsilon^2 T + \sigma_u^2)]/[\sigma_\varepsilon^2 + \sigma_u^2] - 1\}^2$. Under the null hypothesis that $\sigma_u^2 = 0$, this equals 0. By expanding the inner term then collecting terms, we find that under the alternative hypothesis that σ_u^2 is not equal to 0, plim (1/n)LM = $[T(T-1)/2][\sigma_u^2/(\sigma_\varepsilon^2+\sigma_u^2)]^2$. Within group i, $\text{Corr}^2[\varepsilon_{it},\varepsilon_{is}] = \rho^2 = \sigma_u^2/(\sigma_u^2+\sigma_\varepsilon^2)$ so plim (1/n)LM = $[T(T-1)/2](\rho^2)^2$. It is worth noting what is obtained if we do not divide the LM statistic by n at the outset. Under the null hypothesis, the limiting distribution of LM is chi-squared with one degree of freedom. This is a random variable with mean 1 and variance 2, so the statistic, itself, does not converge to a constant; it converges to a random variable. Under the alternative, the LM statistic has mean and variance of order n (as we see above) and hence, explodes. It is this latter attribute which makes the test a consistent one. As the sample size increases, the power of the LM test must go to 1. □

5. The ordinary least squares regression results are

R^2 = .92803, **e′e** = 146.761, 40 observations

Variable	Coefficient	Standard Error
X_1	.446845	.07887
X_2	1.83915	.1534
Constant	3.60568	2.555
Period 1	-3.57906	1.723
Period 2	-1.49784	1.716
Period 3	2.00677	1.760
Period 4	-3.03206	1.731
Period 5	-5.58937	1.768
Period 6	-1.49474	1.714
Period 7	1.52021	1.714
Period 8	-2.25414	1.737
Period 9	-3.29360	1.722
Group 1	-.339998	1.135
Group 2	4.39271	1.183
Group 3	5.00207	1.125

Estimated covariance matrix for the slopes:

	β_1	β_2
β_1	.0062209	
β_2	.00030947	.023523

For testing the hypotheses that the sets of dummy variable coefficients are zero, we will require the sums of squared residuals from the restrictions. These are

Regression	Sum of squares
All variables included	146.761
Period variables omitted	318.503
Group variables omitted	369.356
Period and group variables omitted	585.622

The F statistics are therefore,

(1) $F[9,25]$ = [(318.503 - 146.761)/9]/[146.761/25] = 3.251

(2) $F[3,25]$ = [(369.356 - 146.761)/3]/[146.761/25] = 12.639

(3) $F[12,25]$ = [(585.622 - 146.761)/12]/[146.761/25] = 6.23

The critical values for the three distributions are 2.283, 2.992, and 2.165, respectively. All sample statistics are larger than the table value, so all of the hypotheses are rejected. □

6. The covariance matrix would be

	$i=1,t=1$	$i=1,t=2$	$i=2,t=1$	$i=2,t=2$
$i=1,t=1$	$\sigma_\varepsilon^2+\sigma_u^2+\sigma_v^2$	σ_u^2	σ_v^2	0
$i=1,t=2$	σ_u^2	$\sigma_\varepsilon^2+\sigma_u^2+\sigma_v^2$	0	σ_v^2
$i=2,t=1$	σ_v^2	0	$\sigma_\varepsilon^2+\sigma_u^2+\sigma_v^2$	σ_u^2
$i=2,t=2$	0	σ_v^2	σ_u^2	$\sigma_\varepsilon^2+\sigma_u^2+\sigma_v^2$

7. The two separate regressions are as follows:

	Sample 1	**Sample 2**
$b = \mathbf{x}'\mathbf{y}/\mathbf{x}'\mathbf{x}$	4/5 = .8	6/10 = .6
$\mathbf{e}'\mathbf{e} = \mathbf{y}'\mathbf{y} - b\mathbf{x}'\mathbf{y}$	20 - 4(4/5) = 84/5	10 - 6(6/10) = 64/10
$R^2 = 1 - \mathbf{e}'\mathbf{e}/\mathbf{y}'\mathbf{y}$	1 - (84/5)/20 = .16	1 - (64/10)/10 = .36
$s^2 = \mathbf{e}'\mathbf{e}/(n\text{-}1)$	(84/5)/19 = .88421	(64/10)/19 = .33684
Est.Var$[b] = s^2/\mathbf{x}'\mathbf{x}$	.88421/5 = .17684	.33684/10 = .033684

To carry out a Lagrange multiplier test of the hypothesis of equal variances, we require the separate and common variance estimators based on the restricted slope estimator. This, in turn, is the pooled least squares estimator. For the combined sample, we obtain

$$b = [\mathbf{x}_1'\mathbf{y}_1 + \mathbf{x}_2'\mathbf{y}_2]/[\mathbf{x}_1'\mathbf{x}_1 + \mathbf{x}_2'\mathbf{x}_2] = (4 + 6) / (5 + 10) = 2/3.$$

Then, the variance estimators are based on this estimate. For the hypothesized common variance,

$$\mathbf{e}'\mathbf{e} = (\mathbf{y}_1'\mathbf{y}_1 + \mathbf{y}_2'\mathbf{y}_2) - b(\mathbf{x}_1'\mathbf{y}_1 + \mathbf{x}_2'\mathbf{y}_2) = (20 + 10) - (2/3)(4 + 6) = 70/3,$$

so the estimate of the common variance is $\mathbf{e}'\mathbf{e}/40 = (70/3)/40 = .58333$. Note that the divisor is 40, not 39, because we are comptuting maximum likelihood estimators. The individual estimators are

$$\mathbf{e}_1'\mathbf{e}_1/20 = (\mathbf{y}_1'\mathbf{y}_1 - 2b(\mathbf{x}_1'\mathbf{y}_1) + b^2(\mathbf{x}_1'\mathbf{x}_1))/20 = (20 - 2(2/3)4 + (2/3)^2 5)/20 = .84444$$

and $\mathbf{e}_2'\mathbf{e}_2/20 = (\mathbf{y}_2'\mathbf{y}_2 - 2b(\mathbf{x}_2'\mathbf{y}_2) + b^2(\mathbf{x}_2'\mathbf{x}_2))/20 = (10 - 2(2/3)6 + (2/3)^2 10)/20 = .32222.$

The LM statistic is given in Example 16.3,

$$LM = (T/2)[(s_1^2/s^2 - 1)^2 + (s_2^2/s^2 - 1)^2] = 10[(.84444/.58333 - 1)^2 + (.32222/.58333 - 1)^2] = 4.007.$$

This has one degree of freedom for the single restriction. The critical value from the chi-squared table is 3.84, so we would reject the hypothesis.

In order to compute a two step GLS estimate, we can use either the original variance estimates based on the separate least squares estimates or those obtained above in doing the LM test. Since both pairs are consistent, both FGLS estimators will have all of the desirable asymptotic properties. For our estimator, we used $\hat{\sigma}_1^2 = \mathbf{e}_j'\mathbf{e}_j/T$ from the original regressions. Thus, $\hat{\sigma}_1^2 = .84$ and $\hat{\sigma}_2^2 = .32$. The GLS estimator is

$$\hat{\boldsymbol{\beta}} = [(1/\hat{\sigma}_1^2)\mathbf{x}_1'\mathbf{y}_1 + (1/\hat{\sigma}_2^2)\mathbf{x}_2'\mathbf{y}_2]/[(1/\hat{\sigma}_1^2)\mathbf{x}_1'\mathbf{x}_1 + (1/\hat{\sigma}_2^2)\mathbf{x}_2'\mathbf{x}_2] = [4/.84 + 6/.32]/[5/.84 + 10/.32] = .632.$$

The estimated sampling variance is $1/[(1/\hat{\sigma}_1^2)\mathbf{x}_1'\mathbf{x}_1 + (1/\hat{\sigma}_2^2)\mathbf{x}_2'\mathbf{x}_2] = .02688$. This implies an asymptotic standard error of $(.02688)^2 = .16395$. To test the hypothesis that $\beta = 1$, we would refer $z = (.632 - 1) / .16395 = -2.245$ to a standard normal table. This is reasonably large, and at the usual significance levels, would lead to rejection of the hypothesis.

The Wald test is based on the unrestricted variance estimates. Using $b = .632$, the variance estimators are $\hat{\sigma}_1^2 = [\mathbf{y}_1'\mathbf{y}_1 - 2b(\mathbf{x}_1'\mathbf{y}_1) + b^2(\mathbf{x}_1'\mathbf{x}_1)]/20 = .847056$

and $\hat{\sigma}_2^2 = [\mathbf{y}_2'\mathbf{y}_2 - 2b(\mathbf{x}_2'\mathbf{y}_2) + b^2(\mathbf{x}_2'\mathbf{x}_2)]/20 = .320512$

while the pooled estimator would be $\hat{\sigma}^2 = [\mathbf{y}'\mathbf{y} - 2b(\mathbf{x}'\mathbf{y}) + b^2(\mathbf{x}'\mathbf{x})]/40 = .583784$. The statistic is given at the end of Example 16.3,

$$W = (T/2)[(\hat{\sigma}/\hat{\sigma}_1^2 - 1)^2 + (\hat{\sigma}/\hat{\sigma}_2^2 - 1)^2]$$
$$= 10[(.583784/.847056 - 1)^2 + (.583784/.320512 - 1)^2] = 7.713.$$

We reach the same conclusion as before.

To compute the maximum likelihood estimators, we begin our iterations from the two separate ordinary least squares estimates of b which produce estimates $\hat{\sigma}_1^2 = .84$ and $\hat{\sigma}_2^2 = .32$. The iterations are

Iteration	$\hat{\sigma}_1^2$	$\hat{\sigma}_2^2$	$\hat{\beta}$
0	.840000	.320000	.632000

1	.847056	.320512	.631819
2	.847071	.320506	.631818
3	.847071	.320506	converged

Now, to compute the likelihood ratio statistic for a likelihood ratio test of the hypothesis of equal variances, we refer χ^2 = 40ln.58333 - 20ln.847071 - 20ln.320506 to the chi-squared table. (Under the null hypothesis, the pooled least squares estimator is maximum likelihood.) Thus, χ^2 = 4.5164, which is roughly equal to the LM statistic and leads once again to rejection of the null hypothesis.

Finally, we allow for cross sectional correlation of the disturbances. Our initial estimate of b is the pooled least squares estimator, 2/3. The estimates of the two variances are .84444 and .32222 as before while the cross sectional covariance estimate is

$$\mathbf{e}_1'\mathbf{e}_2/20 = [\mathbf{y}_1'\mathbf{y}_2 - b(\mathbf{x}_1'\mathbf{y}_2 + \mathbf{x}_2'\mathbf{y}_1) + b^2(\mathbf{x}_1'\mathbf{x}_2)]/20 = .14444.$$

Before proceeding, we note, the estimated squared correlation of the two disturbances is

$$r = .14444 / [(.84444)(.32222)]^{1/2} = .277,$$

which is not particularly large. The LM test statistic given in (16-14) is 1.533, which is well under the critical value of 3.84. Thus, we would not reject the hypothesis of zero cross section correlation. Nonetheless, we proceed. The estimator is shown in (16-6). The two step FGLS and iterated maximum likelihood estimates appear below.

Iteration	$\hat{\sigma}_1^2$	$\hat{\sigma}_2^2$	$\hat{\sigma}_{12}$	$\hat{\beta}$
0	.84444	.32222	.14444	.5791338
1	.8521955	.3202177	.1597994	.5731058
2	.8528702	.3203616	.1609133	.5727069
3	.8529155	.3203725	.1609873	.5726805
4	.8529185	.3203732	.1609921	.5726788
5	.8529187	.3203732	.1609925	converged

Because the correlation is relatively low, the effect on the previous estimate is relatively minor. □

8. If all of the regressor matrices are the same, the estimator in (8-35) reduces to

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\Sigma_{i=1}^{n}\ \{(1/\sigma_i^2)/[\Sigma_{j=1}^{n}\ (1/\sigma_j^2)]\}\mathbf{X}'\mathbf{y}_i = \Sigma_{i=1}^{n}\ w_i\mathbf{b}_i$$

a weighted average of the ordinary least squares estimators, $\mathbf{b}_i = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}_i$ with weights $w_i = (1/\sigma_i^2)/[\Sigma_{j=1}^{n}(1/\sigma_j^2)]$. If it were necessary to estimate the weights, a simple two step estimator could be based on individual variance estimators. Either of $s_i^2 = \mathbf{e}_i'\mathbf{e}_i/T$ based on separate least squares regressions (with different estimators of β) or based on residuals computed from a common pooled ordinary least squares slope estimator could be used. □

9. The various least squares estimators of the parameters are

	Sample 1	**Sample 2**	**Sample 3**	**Pooled**
a	11.6644	5.42213	1.41116	8.06392
	(9.658)	(10.46)	(7.328)	
b	.926881	1.06410	1.46885	1.05413
	(.4328)	(.4756)	(.3590)	
e′e	452.206	673.409	125.281	
	(464.288)	(732.560)	(171.240)	(1368.088)

(Values of **e′e** in parentheses above are based on the pooled slope estimator.) The FGLS estimator and its estimated asymptotic covariance matrix are

$$\mathbf{b} = \begin{pmatrix} 7.17889 \\ 1.13792 \end{pmatrix},\quad \text{Est.Asy.Var}[\mathbf{b}] = \begin{bmatrix} 22.8049 & -1.0629 \\ -1.0629 & 0.05197 \end{bmatrix}$$

Note that the FGLS estimator of the slope is closer to the 1.46885 of sample 3 (the highest of the three OLS estimates). This is to be expected since the third group has the smallest residual variance. The LM test statistic is based on the pooled regression,

$$LM = (10/2)\{[(464.288/10)/(1368.088/30) - 1]^2 + ...\} = 3.7901$$

To compute the Wald statistic, we require the unrestricted regression. The parameter estimates are given above. The sums of squares are 465.708, 785.399, and 145.055 for i = 1, 2, and 3, respectively. For the common estimate of σ^2, we use the total sum of squared GLS residuals, 1396.162. Then,

$$W = (10/2)\{[(1396.162/30)/(465.708/10) - 1]^2 + ...\} = 25.21.$$

The Wald statistic is far larger than the *LM* statistic. Since there are two restrictions, at significance levels of 95% or 99% with critical values of 5.99 or 9.21, the two tests lead to different conclusions. The likelihood ratio statistic based on the FGLS estimates is $\chi^2 = 30\ln(1396.162/30) - 10\ln(465.708/10) ... = 6.42$ which is between the previous two and between the 95% and 99% critical values.

Applications

As usual, the applications below require econometric software. The computations can be done with any modern software package, so no specific program is recommended.

```
--> read $
Last observation read from data file was     200
End of data listing in edit window was reached
--> REGRESS ; Lhs = I ; Rhs = F,C,one $
+---------------------------------------------------+
| Ordinary    least squares regression              |
| LHS=I        Mean                 =   145.9582     |
|              Standard deviation   =   216.8753     |
| WTS=none     Number of observs.   =        200     |
| Model size   Parameters           =          3     |
|              Degrees of freedom   =        197     |
| Residuals    Sum of squares       =   1755850.     |
|              Standard error of e  =   94.40840     |
| Fit          R-squared            =   .8124080     |
|              Adjusted R-squared   =   .8105035     |
| Model test   F[  2,   197] (prob) = 426.58 (.0000) |
+---------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 F       |     .11556216       .00583571    19.803   .0000   1081.68110
 C       |     .23067849       .02547580     9.055   .0000   276.017150
 Constant|   -42.7143694      9.51167603    -4.491   .0000

--> CALC    ; R0=Rsqrd $
--> REGRESS ; Lhs = I ; Rhs = F,C,one ; Cluster = 20 $


+---------------------------------------------------+
| Ordinary    least squares regression              |
| LHS=I        Mean                 =   145.9582     |
|              Standard deviation   =   216.8753     |
| WTS=none     Number of observs.   =        200     |
| Model size   Parameters           =          3     |
|              Degrees of freedom   =        197     |
| Residuals    Sum of squares       =   1755850.     |
|              Standard error of e  =   94.40840     |
| Fit          R-squared            =   .8124080     |
|              Adjusted R-squared   =   .8105035     |
| Model test   F[  2,   197] (prob) = 426.58 (.0000) |
+---------------------------------------------------+
+---------------------------------------------------------------------+
| Covariance matrix for the model is adjusted for data clustering.    |
| Sample of    200 observations contained     10 clusters defined by  |
|     20 observations (fixed number) in each cluster.                 |
| Sample of    200 observations contained      1 strata defined by    |
|    200 observations (fixed number) in each stratum.                 |
+---------------------------------------------------------------------+
```

```
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 F       |     .11556216       .01589434     7.271   .0000   1081.68110
 C       |     .23067849       .08496711     2.715   .0072   276.017150
 Constant|   -42.7143694      20.4252029    -2.091   .0378
```

The standard errors increase substantially. This is at least suggestive that there is correlation across observations within the groups. A formal test would be based on one of the panel models below. When the random effects model is fit by maximum likelihood, for example, the log likelihood function is -1095.257. The log likelihood function for the pooled model is -1191.802. Thus, the correlation is highly significant. The Lagrange multiplier statistic reported below is 798.16, which is far larger than the critical value of 3.84. Once again, these results do suggest within groups correlation.

```
--> REGRESS ; Lhs = I ; Rhs = F,C,one  ; Panel ; Pds=20 ; Fixed $
+----------------------------------------------------+
| Least Squares with Group Dummy Variables           |
| Ordinary    least squares regression               |
| LHS=I        Mean                 =   145.9583     |
|              Standard deviation   =   216.8753     |
| WTS=none     Number of observs.   =        200     |
| Model size   Parameters           =         12     |
|              Degrees of freedom   =        188     |
| Residuals    Sum of squares       =   523478.1     |
|              Standard error of e  =   52.76797     |
| Fit          R-squared            =   .9440725     |
|              Adjusted R-squared   =   .9408002     |
| Model test   F[ 11,   188] (prob) = 288.50 (.0000) |
+----------------------------------------------------+
+----------------------------------------------------+
| Panel:Groups   Empty       0,   Valid data      10 |
|                Smallest   20,   Largest         20 |
|                Average group size           20.00 |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 F       |     .11012380       .01185669     9.288   .0000   1081.68110
 C       |     .31006534       .01735450    17.867   .0000   276.017150
+--------------------------------------------------------------------+
|             Test Statistics for the Classical Model                |
+--------------------------------------------------------------------+
|       Model            Log-Likelihood  Sum of Squares    R-squared |
|(1)  Constant term only    -1359.15096  .9359943929D+07    .0000000 |
|(2)  Group effects only    -1216.34872  .2244352274D+07    .7602173 |
|(3)  X - variables only    -1191.80236  .1755850484D+07    .8124080 |
|(4)  X and group effects   -1070.78103  .5234781474D+06    .9440725 |
+--------------------------------------------------------------------+
|                        Hypothesis Tests                            |
|         Likelihood Ratio Test           F Tests                    |
|         Chi-squared   d.f.  Prob.       F    num. denom.   P value |
|(2) vs (1)   285.604      9  .00000   66.932     9     190   .00000 |
|(3) vs (1)   334.697      2  .00000  426.576     2     197   .00000 |
|(4) vs (1)   576.740     11  .00000  288.500    11     188   .00000 |
|(4) vs (2)   291.135      2  .00000  309.014     2     188   .00000 |
|(4) vs (3)   242.043      9  .00000   49.177     9     188   .00000 |
+--------------------------------------------------------------------+
--> CALC    ; R1 = Rsqrd $
--> MATRIX  ; bf = b(1:2) ; vf = varb(1:2,1:2) $
--> CALC    ; List ; Fstat=((R1-R0)/9)/((1-R1)/(n-2-10))
    ; FC=Ftb(.95,9,(n-2-10)) $
+-----------------------------------+
| Listed Calculator Results         |
+-----------------------------------+
 FSTAT   =     49.176625
```

```
 FC      =      1.929957
```

The F statistic of 49.18 is far larger than the critical value, so the hypothesis of equal constant terms is rejected.

```
--> REGRESS ; Lhs = I ; Rhs = F,C,one
    ; Panel ; Pds=20 ; Random $
+--------------------------------------------------+
| Random Effects Model: v(i,t) = e(i,t) + u(i)     |
| Estimates:  Var[e]              =   .278446D+04  |
|             Var[u]              =   .612849D+04  |
|             Corr[v(i,t),v(i,s)] =   .687594      |
| Lagrange Multiplier Test vs. Model (3) =  798.16 |
| ( 1 df, prob value =  .000000)                   |
| (High values of LM favor FEM/REM over CR model.) |
|             Sum of Squares          .184029D+07  |
|             R-squared               .803387D+00  |
+--------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 F       |     .10974919       .01031952    10.635   .0000   1081.68110
 C       |     .30780890       .01715154    17.946   .0000   276.017150
 Constant|   -57.7159079      27.1118671    -2.129   .0333
```

The LM statistic, as noted earlier, is very large, so the hypothesis of no effects is rejected.

```
--> MATRIX  ; br = b(1:2) ; vr = varb(1:2,1:2) $
--> MATRIX  ; db = bf-br ; vdb = vf-vr ; List ; Hausman=db'<vdb>db $
               1
        +--------------
       1|    2.45500
--> CALC    ; List ; Ctb(.95,2) $
+------------------------------------+
| Listed Calculator Results          |
+------------------------------------+
 Result  =      5.991465
```

The Hausman statistic is quite small, which suggests that the random effects approach is consistent with the data.

2.

```
create ; logc=log(cost/pfuel)
       ; logp1=log(pmtl/pfuel)
       ; logp2=log(peqpt/pfuel)
       ; logp3=log(plabor/pfuel)
       ; logp4=log(pprop/pfuel)
       ; logp5=log(kprice/pfuel)
       ; logq=log(output)
       ; logq2=.5*logq^2 $
Namelist ; cd = logp1,logp2,logp3,logp4,logp5 $
create
       ; p11=.5* logp1^2
       ; p22=.5* logp2^2
       ; p33=.5* logp3^2
       ; p44=.5* logp4^2
       ; p55=.5* logp5^2
       ; p12=logp1*logp2
       ; p13=logp1*logp3
       ; p14=logp1*logp4
       ; p15=logp1*logp5
       ; p23=logp2*logp3
       ; p24=logp2*logp4
       ; p25=logp2*logp5
       ; p34=logp3*logp4
       ; p35=logp3*logp5
       ; p45=logp4*logp5 $
Namelist ; tl = p11,p12,p13,p14,p15,p22,p23,p24,p25,p33,p34,p35,p44,p45,p55$
Namelist ; z = loadfctr,stage,points $
regress;lhs=logc;rhs=one,logq,logq2,cd,z $
+----------------------------------------------------+
| Ordinary    least squares regression               |
| LHS=LOGC     Mean                 =   .7723984     |
|              Standard deviation   =   1.074424     |
| WTS=none     Number of observs.   =        256     |
| Model size   Parameters           =         11     |
|              Degrees of freedom   =        245     |
| Residuals    Sum of squares       =   2.965806     |
|              Standard error of e  =   .1100242     |
| Fit          R-squared            =   .9899249     |
|              Adjusted R-squared   =   .9895136     |
| Model test   F[ 10,   245] (prob) =2407.23 (.0000) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 Constant|    20.3856176      22.8643711      .892   .3735
 LOGQ    |     .95227889       .01832119    51.977   .0000  -1.11237037
 LOGQ2   |     .06568531       .01060839     6.192   .0000   1.45687077
 LOGP1   |    -.32662031      1.17956412     -.277   .7821    .37999226
 LOGP2   |    -.28619766       .56614750     -.506   .6136   -.25308254
 LOGP3   |     .16012937       .08634095     1.855   .0649    .66688211
 LOGP4   |    -.00519153       .07328859     -.071   .9436  -2.14504306
 LOGP5   |    1.43718160      1.78896723      .803   .4225  -12.6860637
 LOADFCTR|    -.94688632       .18441822    -5.134   .0000    .54786115
 STAGE   |    -.00021794     .402227D-04    -5.418   .0000   507.879666
 POINTS  |     .00199712       .00031682     6.304   .0000   72.9843750
?
? Turns out the translog model cannot be computed with the firm
? dummy variables.  I'll use the Cobb Douglas form.
?
regress;lhs=logc;rhs=  one,logq,logq2,cd ; panel ; pds=ti $
+----------------------------------------------------+
| OLS Without Group Dummy Variables                  |
| Ordinary    least squares regression               |
| LHS=LOGC     Mean                 =   .7723984     |
|              Standard deviation   =   1.074424     |
| WTS=none     Number of observs.   =        256     |
```

```
| Model size   Parameters           =         8      |
|              Degrees of freedom   =       248      |
| Residuals    Sum of squares       =   4.190133     |
|              Standard error of e  =   .1299834     |
| Fit          R-squared            =   .9857657     |
|              Adjusted R-squared   =   .9853639     |
| Model test   F[  7,   248] (prob) =2453.53 (.0000) |
+----------------------------------------------------+
+----------------------------------------------------+
| Panel Data Analysis of LOGC        [ONE way]        |
|           Unconditional ANOVA (No regressors)      |
| Source      Variation   Deg. Free.     Mean Square |
| Between      272.013          24.     11.3339      |
| Residual     22.3551         231.     .967752E-01  |
| Total        294.368         255.     1.15439      |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 LOGQ    |      .93708702       .01772733    52.861   .0000  -1.11237037
 LOGQ2   |      .07754607       .01211431     6.401   .0000   1.45687077
 LOGP1   |    -.94586281      1.38855410     -.681   .4964    .37999226
 LOGP2   |    -.79081045       .66530892    -1.189   .2357   -.25308254
 LOGP3   |      .01998606       .09963618      .201   .8412    .66688211
 LOGP4   |      .08893118       .08543313     1.041   .2989  -2.14504306
 LOGP5   |     2.63118115      2.10504302     1.250   .2125  -12.6860637
 Constant|     35.4178566      26.9017806     1.317   .1892
+----------------------------------------------------+
| Least Squares with Group Dummy Variables           |
| Ordinary    least squares regression               |
| LHS=LOGC     Mean                 =   .7723984     |
|              Standard deviation   =   1.074424     |
| WTS=none     Number of observs.   =       256      |
| Model size   Parameters           =        32      |
|              Degrees of freedom   =       224      |
| Residuals    Sum of squares       =   .9373686     |
|              Standard error of e  =   .6468911E-01 |
| Fit          R-squared            =   .9968157     |
|              Adjusted R-squared   =   .9963750     |
| Model test   F[ 31,   224] (prob) =2261.94 (.0000) |
+----------------------------------------------------+
+----------------------------------------------------+
| Panel:Groups   Empty       0,   Valid data      25 |
|                Smallest    2,   Largest         15 |
|                Average group size            10.24 |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 LOGQ    |      .66448665       .03580894    18.556   .0000  -1.11237037
 LOGQ2   |    -.00955723       .01280811     -.746   .4563   1.45687077
 LOGP1   |     1.84750938       .76113884     2.427   .0159    .37999226
 LOGP2   |      .73986763       .37612716     1.967   .0503   -.25308254
 LOGP3   |    -.05323942       .06396335     -.832   .4060    .66688211
 LOGP4   |      .22763995       .04625120     4.922   .0000  -2.14504306
 LOGP5   |    -1.83738098      1.16995945    -1.570   .1176  -12.6860637
+------------------------------------------------------------------+
|              Test Statistics for the Classical Model               |
+------------------------------------------------------------------+
|       Model            Log-Likelihood  Sum of Squares    R-squared |
|(1)  Constant term only     -381.12407  .2943684435D+03    .0000000 |
|(2)  Group effects only      -51.16832  .2235506489D+02    .9240575 |
|(3)  X - variables only      163.14470  .4190132631D+01    .9857657 |
|(4)  X and group effects     354.81332  .9373685874D+00    .9968157 |
+------------------------------------------------------------------+
|                        Hypothesis Tests                            |
|         Likelihood Ratio Test           F Tests                    |
```

```
|          Chi-squared   d.f.  Prob.       F    num. denom.   P value |
|(2) vs (1)   659.911     24  .00000  117.116    24     231   .00000 |
|(3) vs (1)  1088.538      7  .00000 2453.527     7     248   .00000 |
|(4) vs (1)  1471.875     31  .00000 2261.945    31     224   .00000 |
|(4) vs (2)   811.963      7  .00000  731.160     7     224   .00000 |
|(4) vs (3)   383.337     24  .00000   32.388    24     224   .00000 |
+--------------------------------------------------------------------+
+--------------------------------------------------+
| Random Effects Model: v(i,t) = e(i,t) + u(i)     |
| Estimates:  Var[e]              =   .418468D-02  |
|             Var[u]              =   .127110D-01  |
|             Corr[v(i,t),v(i,s)] =   .752323      |
| Lagrange Multiplier Test vs. Model (3) =  479.37 |
| ( 1 df, prob value =  .000000)                   |
| (High values of LM favor FEM/REM over CR model.) |
| Baltagi-Li form of LM Statistic =         174.85 |
| Fixed vs. Random Effects (Hausman)     =   40.99 |
| ( 7 df, prob value =  .000001)                   |
| (High (low) values of H favor FEM (REM).)        |
|             Sum of Squares           .648771D+01  |
|             R-squared                .978056D+00  |
+--------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 LOGQ    |     .79769706       .02494671    31.976   .0000  -1.11237037
 LOGQ2   |     .02011534       .01130089     1.780   .0751   1.45687077
 LOGP1   |    1.11671466       .74579390     1.497   .1343    .37999226
 LOGP2   |     .27128619       .36294718      .747   .4548   -.25308254
 LOGP3   |    -.10761385       .06138583    -1.753   .0796    .66688211
 LOGP4   |     .18385724       .04550246     4.041   .0001  -2.14504306
 LOGP5   |    -.49374865      1.13625272     -.435   .6639  -12.6860637
 Constant|   -4.53328730      14.5229534     -.312   .7549
regress;lhs=logc;rhs=z,one,logq,logq2,cd ; panel ; pds=ti $
+----------------------------------------------------+
| OLS Without Group Dummy Variables                  |
| Ordinary    least squares regression               |
| LHS=LOGC     Mean                 =   .7723984     |
|              Standard deviation   =   1.074424     |
| WTS=none     Number of observs.   =        256     |
| Model size   Parameters           =         11     |
|              Degrees of freedom   =        245     |
| Residuals    Sum of squares       =   2.965806     |
|              Standard error of e  =   .1100242     |
| Fit          R-squared            =   .9899249     |
|              Adjusted R-squared   =   .9895136     |
| Model test   F[ 10,   245] (prob) =2407.23 (.0000) |
+----------------------------------------------------+
+----------------------------------------------------+
| Panel Data Analysis of LOGC       [ONE way]        |
|           Unconditional ANOVA (No regressors)      |
| Source      Variation   Deg. Free.     Mean Square |
| Between       272.013          24.     11.3339     |
| Residual      22.3551         231.     .967752E-01 |
| Total         294.368         255.     1.15439     |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 LOADFCTR|    -.94688632       .18441823    -5.134   .0000    .54786115
 STAGE   |    -.00021794     .402227D-04    -5.418   .0000   507.879666
 POINTS  |     .00199712       .00031682     6.304   .0000   72.9843750
 LOGQ    |     .95227889       .01832119    51.977   .0000  -1.11237037
 LOGQ2   |     .06568531       .01060839     6.192   .0000   1.45687077
 LOGP1   |    -.32662033      1.17956418     -.277   .7821    .37999226
 LOGP2   |    -.28619767       .56614753     -.506   .6136   -.25308254
 LOGP3   |     .16012937       .08634095     1.855   .0649    .66688211
```

```
 LOGP4   |    -.00519153       .07328859     -.071   .9436  -2.14504306
 LOGP5   |    1.43718164      1.78896732      .803   .4225  -12.6860637
 Constant|    20.3856181      22.8643723      .892   .3735
+----------------------------------------------------+
| Least Squares with Group Dummy Variables           |
| Ordinary    least squares regression               |
| LHS=LOGC     Mean                 =    .7723984    |
|              Standard deviation   =   1.074424     |
| WTS=none     Number of observs.   =        256     |
| Model size   Parameters           =         35     |
|              Degrees of freedom   =        221     |
| Residuals    Sum of squares       =    .7726037    |
|              Standard error of e  =   .5912651E-01 |
| Fit          R-squared            =    .9973754    |
|              Adjusted R-squared   =    .9969716    |
| Model test   F[ 34,   221] (prob) =2470.05 (.0000) |
+----------------------------------------------------+
+----------------------------------------------------+
| Panel:Groups   Empty       0,   Valid data      25 |
|                Smallest    2,   Largest         15 |
|                Average group size           10.24 |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 LOADFCTR|    -.89457348       .14242570    -6.281   .0000    .54786115
 STAGE   |    -.00022827     .894260D-04    -2.553   .0113   507.879666
 POINTS  |     .00010341       .00041551      .249   .8037   72.9843750
 LOGQ    |     .75278467       .03923479    19.187   .0000  -1.11237037
 LOGQ2   |    -.00324835       .01306645     -.249   .8039   1.45687077
 LOGP1   |    1.38217070       .72421015     1.909   .0575    .37999226
 LOGP2   |     .61609241       .35323609     1.744   .0824   -.25308254
 LOGP3   |     .00706546       .05918620      .119   .9051    .66688211
 LOGP4   |     .14433953       .04404683     3.277   .0012  -2.14504306
 LOGP5   |   -1.25331458      1.10477945    -1.134   .2577  -12.6860637
+--------------------------------------------------------------------+
|             Test Statistics for the Classical Model                |
+--------------------------------------------------------------------+
|       Model            Log-Likelihood  Sum of Squares    R-squared |
|(1)  Constant term only     -381.12407  .2943684435D+03    .0000000 |
|(2)  Group effects only      -51.16832  .2235506489D+02    .9240575 |
|(3)  X - variables only      207.37940  .2965806000D+01    .9899249 |
|(4)  X and group effects     379.55705  .7726036853D+00    .9973754 |
+--------------------------------------------------------------------+
|                        Hypothesis Tests                            |
|         Likelihood Ratio Test           F Tests                    |
|         Chi-squared   d.f.  Prob.       F    num. denom.   P value |
|(2) vs (1)   659.911     24  .00000  117.116    24     231   .00000 |
|(3) vs (1)  1177.007     10  .00000 2407.226    10     245   .00000 |
|(4) vs (1)  1521.362     34  .00000 2470.054    34     221   .00000 |
|(4) vs (2)   861.451     10  .00000  617.357    10     221   .00000 |
|(4) vs (3)   344.355     24  .00000   26.140    24     221   .00000 |
+--------------------------------------------------------------------+

+--------------------------------------------------+
| Random Effects Model: v(i,t) = e(i,t) + u(i)     |
| Estimates:  Var[e]              =   .349594D-02  |
|             Var[u]              =   .860939D-02  |
|             Corr[v(i,t),v(i,s)] =   .711206      |
| Lagrange Multiplier Test vs. Model (3) =  466.36 |
| ( 1 df, prob value =  .000000)                   |
| (High values of LM favor FEM/REM over CR model.) |
| Baltagi-Li form of LM Statistic =         170.10 |
| Fixed vs. Random Effects (Hausman)     =   44.65 |
| (10 df, prob value =  .000003)                   |
| (High (low) values of H favor FEM (REM).)        |
|             Sum of Squares          .451094D+01  |
```

```
|             R-squared              .984812D+00  |
+-------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 LOADFCTR|   -1.07921018       .13264921    -8.136   .0000    .54786115
 STAGE   |    -.00016415     .672354D-04    -2.441   .0146   507.879666
 POINTS  |     .00044792       .00035950     1.246   .2128   72.9843750
 LOGQ    |     .86611837       .02783747    31.113   .0000  -1.11237037
 LOGQ2   |     .02222380       .01102947     2.015   .0439   1.45687077
 LOGP1   |     .92719911       .70150544     1.322   .1863    .37999226
 LOGP2   |     .30782803       .33937387      .907   .3644   -.25308254
 LOGP3   |    -.02581955       .05671735     -.455   .6489    .66688211
 LOGP4   |     .09284095       .04277517     2.170   .0300  -2.14504306
 LOGP5   |    -.36595849      1.06514141     -.344   .7312  -12.6860637
 Constant|   -2.36774378      13.6315073     -.174   .8621
matrix ; List ; bz=b(1:3);vz=varb(1:3,1:3) ; wald = bz'<vz>bz $
Matrix WALD     has  1 rows and  1 columns.
               1
        +--------------
       1|   74.33957
```

Chapter 10

Systems of Regression Equations

1. The model can be written as $\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix} = \begin{bmatrix}\mathbf{i}\\ \mathbf{i}\end{bmatrix}\mu + \begin{bmatrix}\boldsymbol{\varepsilon}_1\\ \boldsymbol{\varepsilon}_2\end{bmatrix}$. Therefore, the OLS estimator is

$$m = (\mathbf{i}'\mathbf{i} + \mathbf{i}'\mathbf{i})^{-1}(\mathbf{i}'\mathbf{y}_1 + \mathbf{i}'\mathbf{y}_2) = (n\bar{y}_1 + n\bar{y}_2)/(n+n) = (\bar{y}_1 + \bar{y}_2)/2 = 1.5.$$

The sampling variance would be $\text{Var}[m] = (1/2)^2\{\text{Var}[\bar{y}_1] + \text{Var}[\bar{y}_2] + 2\text{Cov}[(\bar{y}_{1\,1}, \bar{y}_2)]\}$.

We would estimate the parts with
$\text{Est.Var}[\bar{y}_1] = s_{11}/n = ((150 - 100(1)^2)/99)/100 = .0051$
$\text{Est.Var}[\bar{y}_2] = s_{22}/n = ((550 - 100(2)^2)/99)/100 = .0152$
$\text{Est.Cov}[\bar{y}_1, \bar{y}_2] = s_{12}/n = ((260 - 100(1)(2))/99)/100 = .0061$

Combining terms, $\text{Est.Var}[m] = .0079$.

The GLS estimator would be

$$[(\sigma^{11} + \sigma^{12})\mathbf{i}'\mathbf{y}_1 + (\sigma^{22} + \sigma^{12})\mathbf{i}'\mathbf{y}_2]/[(\sigma^{11} + \sigma^{12})\mathbf{i}'\mathbf{i} + (\sigma^{22} + \sigma^{12})\mathbf{i}'\mathbf{i}] = w\bar{y}_1 + (1-w)\bar{y}_2$$

where $w = (\sigma^{11} + \sigma^{12}) / (\sigma^{11} + \sigma^{22} + 2\sigma^{12})$. Denoting $\boldsymbol{\Sigma} = \begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix}$, $\boldsymbol{\Sigma}^{-1} = \dfrac{1}{\sigma_{11}\sigma_{22} - \sigma_{12}^2}\begin{bmatrix}\sigma_{22} & -\sigma_{12}\\ -\sigma_{12} & \sigma_{11}\end{bmatrix}$.

The weight simplifies a bit as the determinant appears in both the denominator and the numerator. Thus, $w = (\sigma_{22} - \sigma_{12}) / (\sigma_{11} + \sigma_{22} - 2\sigma_{12})$. For our sample data, the two step estimator would be based on the variances computed above and $s_{11} = .5051$, $s_{22} = 1.5152$, $s_{12} = .6061$. Then, $w = 1.1250$. The FGLS estimate is $1.125(1) + (1 - 1.125)(2) = .875$. The sampling variance of this estimator is $w^2\text{Var}[\bar{y}_1] + (1 - w)^2\text{Var}[\bar{y}_2] + 2w(1 - w)\text{Cov}[\bar{y}_1, \bar{y}_2] = .0050$ as compared to .0079 for the OLS estimator.

2. The model is $\mathbf{y} = \begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} = \begin{bmatrix}\mathbf{i} & \mathbf{0}\\ \mathbf{0} & \mathbf{x}\end{bmatrix}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + \begin{bmatrix}\boldsymbol{\varepsilon}_1\\ \boldsymbol{\varepsilon}_2\end{bmatrix}$, $\sigma^2\boldsymbol{\Omega} = \begin{bmatrix}\sigma_{11}\mathbf{I} & \sigma_{12}\mathbf{I}\\ \sigma_{12}\mathbf{I} & \sigma_{22}\mathbf{I}\end{bmatrix}$.

The generalized least squares estimator is

$$\hat{\boldsymbol{\beta}} = [\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X}]^{-1}\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{y} = \begin{bmatrix}\sigma^{11}\mathbf{i}'\mathbf{i} & \sigma^{12}\mathbf{i}'\mathbf{x}\\ \sigma^{12}\mathbf{i}'\mathbf{x} & \sigma^{22}\mathbf{x}'\mathbf{x}\end{bmatrix}^{-1}\begin{pmatrix}\sigma^{11}\mathbf{i}'\mathbf{y}_1 + \sigma^{12}\mathbf{i}'\mathbf{y}_2\\ \sigma^{12}\mathbf{x}'\mathbf{y}_1 + \sigma^{22}\mathbf{x}'\mathbf{y}_2\end{pmatrix}$$

$$= \left[n\begin{pmatrix}\sigma^{11} & \sigma^{12}\bar{x}\\ \sigma^{12}\bar{x} & \sigma^{22}s_{xx}\end{pmatrix}\right]^{-1}\left[n\begin{pmatrix}\sigma^{11}\bar{y}_1 + \sigma^{12}\bar{y}_2\\ \sigma^{12}s_{x1} + \sigma^{22}s_{x2}\end{pmatrix}\right]$$

where $s_{xx} = \mathbf{x}'\mathbf{x}/n$, $s_{x1} = \mathbf{x}'\mathbf{y}_1/n$, $s_{x2} = \mathbf{x}'\mathbf{y}_2/n$

and σ^{ij} = the ijth element of the 2×2 Σ^{-1}.

To obtain the explicit form, note, first, that all terms σ^{ij} are of the form $\sigma_{ji}/(\sigma_{11}\sigma_{22} - \sigma_{12}^2)$ But, the denominator in these ratios will be cancelled as it appears in both the inverse matrix and in the vector. Therefore, in terms of the original parameters, (after cancelling n), we obtain

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix}\sigma_{22} & -\sigma_{12}\bar{x}\\ -\sigma_{12}\bar{x} & \sigma_{11}s_{xx}\end{bmatrix}^{-1}\begin{bmatrix}\sigma_{22}\bar{y}_1 - \sigma_{12}\bar{y}_2\\ -\sigma^{12}s_{x1} + \sigma_{11}s_{x2}\end{bmatrix} = \frac{1}{\sigma_{11}\sigma_{22}s_{xx} - (\sigma_{12}\bar{x})^2}\begin{bmatrix}\sigma_{11}s_{xx} & \sigma_{12}\bar{x}\\ \sigma_{12}\bar{x} & \sigma_{22}\end{bmatrix}\begin{pmatrix}\sigma_{22}\bar{y}_1 - \sigma_{12}\bar{y}_2\\ -\sigma_{12}s_{x1} + \sigma_{11}s_{x2}\end{pmatrix}.$$

The two elements are
$\hat{\beta}_1 = [\sigma_{11}s_{xx}(\sigma_{22}\bar{y}_1 - \sigma_{12}\bar{y}_2) - \sigma_{12}\bar{x}(\sigma_{12}s_{x1} - \sigma_{11}s_{x2})]/[\sigma_{11}\sigma_{22}s_{xx} - (\sigma_{12}\bar{x})^2]$

$\hat{\beta}_2 = [\sigma_{12}\bar{x}(\sigma_{22}\bar{y}_1 - \sigma_{12}\bar{y}_2) - \sigma_{22}(\sigma_{12}s_{x1} - \sigma_{11}s_{x2})]/[\sigma_{11}\sigma_{22}s_{xx} - (\sigma_{12}\bar{x})^2]$

The asymptotic covariance matrix is

$$[\mathbf{X}'\Omega^{-1}\mathbf{X}]^{-1} = \left[n\begin{pmatrix} \sigma^{11} & \sigma^{12}\bar{x} \\ \sigma^{12}\bar{x} & \sigma^{22}s_{xx} \end{pmatrix} \right]^{-1} = \left[\frac{n}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \begin{pmatrix} \sigma_{22} & -\sigma_{12}\bar{x} \\ -\sigma_{12}\bar{x} & \sigma_{11}s_{xx} \end{pmatrix} \right]^{-1}$$

The OLS estimator is $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \begin{pmatrix} \bar{y}_1 \\ \mathbf{x}'\mathbf{y}/\mathbf{x}'\mathbf{x} \end{pmatrix}$. The sampling variance is

$(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Omega\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} n & 0 \\ 0 & ns_{xx} \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{11}n & \sigma_{12}n\bar{x} \\ \sigma_{12}n\bar{x} & \sigma_{22}ns_{xx} \end{bmatrix} \begin{bmatrix} n & 0 \\ 0 & ns_{xx} \end{bmatrix}^{-1}$. The ns are carried outside the product and reduce to $(1/n)$. This leaves Var[**b**] $= \begin{bmatrix} \sigma_{11}/n & \sigma_{12}\bar{x}/(ns_{xx}) \\ \sigma_{12}\bar{x}/(ns_{xx}) & \sigma_{22}/(ns_{xx})^2 \end{bmatrix}$.

Using the results above, the OLS coefficients are $b_1 = \bar{y}_1 = 150/50 = 3$ and $b_2 = \mathbf{x}'\mathbf{y}_2/\mathbf{x}'\mathbf{x} = 50/100 = 1/2$. The estimators of the disturbance (co-)variances are

$$s_{11} = \Sigma_i(y_{i1} - \bar{y}_1)^2/n = (500 - 50(3)2)/50 = 1$$
$$s_{22} = \Sigma_i(y_{i2} - b_2x_i)^2/n = (90 - (1/2)50)/50 = 1.3$$
$$s_{12} = \Sigma_i(y_{i1} - \bar{y}_1)(y_{i2} - b_2x_i)^2/n = [\mathbf{y}_1'\mathbf{y}_2 - n\bar{y}_1\bar{y}_2 - b_2\mathbf{x}'\mathbf{y}_1 + nb_2\bar{y}_1\bar{x}]/n$$
$$= (40 - 50(3)(1) - (1/2)60 + 50(1/2)(3)(2)/50 = .2$$

Therefore, we estimate the asymptotic covariance matrix of the OLS estimates as

$$\text{Est.Var}[\mathbf{b}] = \begin{bmatrix} 1/50 & .2(2)[50(90)] \\ .2(2)[50/90] & 1.3/90 \end{bmatrix} = \begin{bmatrix} .02 & .0000888 \\ .0000888 & .01444 \end{bmatrix}.$$

To compute the FGLS estimates, we use our results from part a. The necessary statistics for the computation are $s_{11} = 1$, $s_{22} = 1.3$, $s_{11} = .2$, $s_{xx} = 100/50 = 2$, $\bar{x} = 100/50 = 2$,
$\bar{y}_1 = 150/50 = 3$, $\bar{y}_2 = 50/50 = 1$
$s_{x1} = 60/50 = 1.2$, $s_{x2} = 50/50 = 1$

Then, $\hat{\beta}_1 = \{1(2)[1.3(3) - .2(1)] - .2(2)[.2(1.2) - 1(1)]\}/\{1(1.3) - [.2(2)]^2\} = 3.157$

$\hat{\beta}_2 = \{2(2)[1.3(3) - .2(1)] - 1.3[.2(1.2) - 1(1)]\}/\{1(1.3) - [.2(2)]^2\} = 1.011$

The estimate of the asymptotic covariance matrix is

$(1/50)[1(1.3) - (.2)^2]/\{1(1.3)2 - [.2(2)]^2\} \begin{bmatrix} 1(2) & .2(2) \\ .2(2) & 1.3 \end{bmatrix} = \begin{bmatrix} .020656 & .004131 \\ .004131 & .007945 \end{bmatrix}$. Notice that the estimated variance of the FGLS estimator of the parameter of the first equation is larger. The result for the *true* GLS estimator based on known values of the disturbance variances and covariance does not guarantee that the *estimated* variances will be smaller in a finite sample. However, the estimated variance of the second parameter is considerably smaller than that for the OLS estimate.

Finally, to test the hypothesis that $\beta_2 = 1$ we use the z-statistic (asymptotically distributed as standard normal), $z = (1.011 - 1)/(.007945)^2 = .123$. The hypothesis cannot be rejected. □

3. The ordinary least squares estimates of the parameters are

$$b_1 = \mathbf{x}_1'\mathbf{y}_1/\mathbf{x}_1'\mathbf{x}_1 = 4/5 = .8 \quad \text{and} \quad b_2 = \mathbf{x}_2'\mathbf{y}_2/\mathbf{x}_2'\mathbf{x}_2 = 6/10 = .6$$

Then, the variances and covariance of the disturbances are

$$s_{11} = (\mathbf{y}_1'\mathbf{y}_1 - b_1\mathbf{x}_1'\mathbf{y}_1)/n = (20 - .8(4))/20 = .84$$
$$s_{22} = (\mathbf{y}_2'\mathbf{y}_2 - b_2\mathbf{x}_2'\mathbf{y}_2)/n = (10 - .6(6))/20 = .32$$
$$s_{12} = (\mathbf{y}_1'\mathbf{y}_2 - b_2\mathbf{x}_2'\mathbf{y}_1 - b_1\mathbf{x}_1'\mathbf{y}_2 + b_1b_2\mathbf{x}_1'\mathbf{x}_2)/n = (6 - .6(3) - .8(3) + .8(.6)(2))/20 = .246$$

We will require $\mathbf{S}^{-1} = \begin{bmatrix} .84 & .246 \\ .246 & .32 \end{bmatrix}^{-1} = \begin{bmatrix} s^{11} & ^{12} \\ s^{12} & s^{11} \end{bmatrix}$. Then, the FGLS estimator is

$$\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{bmatrix} s^{11}\mathbf{x}_1'\mathbf{x}_1 & s^{12}\mathbf{x}_1'\mathbf{x}_2 \\ s^{12}\mathbf{x}_1'\mathbf{x}_2 & s^{22}\mathbf{x}_2'\mathbf{x}_2 \end{bmatrix}^{-1} \begin{bmatrix} s^{11}\mathbf{x}_1'\mathbf{y}_1 + s^{12}\mathbf{x}_1'\mathbf{y}_2 \\ s^{12}\mathbf{x}_2'\mathbf{y}_1 + s^{22}\mathbf{x}_2'\mathbf{y}_2 \end{bmatrix}.$$

Inserting the values given in the problem produces the FGLS estimates, $\hat{\beta}_1 = .505335$, $\hat{\beta}_2 = .541741$ with estimated asymptotic covariance matrix equal to the inverse matrix shown above, $\text{Est.Var}\left[\hat{\boldsymbol{\beta}}\right] = \begin{bmatrix} .132565 & .0077645 \\ .0077645 & .0252505 \end{bmatrix}$. To test the hypothesis, we use the t statistic, $t = (.505335 - .541741)/[.132565 + .0252505 - 2(.0077645)]^2 = -.0965$ which is quite small. We would not reject the hypothesis.

To compute the maximum likelihood estimates, we would begin with the OLS estimates of σ_{11}, σ_{22}, and σ_{12}. Then, we iterate between the following calculations

(1) Compute the 2×2 matrix, $\mathbf{S}^{-1}$

(2) Compute the 2×2 matrix $[\mathbf{X}'(\mathbf{S}^{-1}\otimes\mathbf{I})\mathbf{X}] = \begin{bmatrix} s^{11}\mathbf{x}_1'\mathbf{x}_1 & s^{12}\mathbf{x}_1'\mathbf{x}_2 \\ s^{12}\mathbf{x}_1'\mathbf{x}_2 & s^{22}\mathbf{x}_2'\mathbf{x}_2 \end{bmatrix}$

$$[\mathbf{X}'(\mathbf{S}^{-1}\otimes\mathbf{I})\mathbf{y}] = \begin{bmatrix} s^{11}\mathbf{x}_1'\mathbf{y}_1 + s^{12}\mathbf{x}_1'\mathbf{y}_2 \\ s^{12}\mathbf{x}_2'\mathbf{y}_1 + s^{22}\mathbf{x}_2'\mathbf{y}_2 \end{bmatrix}$$

(3) Compute the coefficient vector $\hat{\boldsymbol{\beta}} = [\mathbf{X}'(\mathbf{S}^{-1}\otimes\mathbf{I})\mathbf{X}]^{-1}[\mathbf{X}'(\mathbf{S}^{-1}\otimes\mathbf{I})\mathbf{y}]$

Compare this estimate to the previous one. If they are similar enough, exit the iterations.

(4) Recompute **S** using $s_{ij} = \mathbf{y}_i'\mathbf{y}_j - \hat{\beta}_i\mathbf{x}_i'\mathbf{y}_j - \hat{\beta}_j\mathbf{x}_j'\mathbf{y}_i + \hat{\beta}_i\hat{\beta}_j\mathbf{x}_i'\mathbf{x}_j$, i,j = 1,2.

(5) Go back to step (1) and continue.

Our iterations produce the two slope estimates

1: .505335 .541741
2: .601889 .564998
3: .614884 .566875
4: .616559 .567186
5: .616775 .567227
6: .616803 .567232
7: .616807 .567232 converged.

At convergence, we find the estimate of the asymptotic covariance matrix of the estimates as

$$[\mathbf{X}N(\mathbf{S}^{-1}\otimes\mathbf{I})\mathbf{X}]^{-1} = \begin{bmatrix} .155355 & .00576887 \\ .00576887 & .029348 \end{bmatrix} \text{ and } \mathbf{S} = \begin{bmatrix} .8483899 & .1573814 \\ .1573814 & .3205369 \end{bmatrix}.$$

To use the likelihood ratio method to test the hypothesis, we will require the restricted maximum likelihood estimate. Under the hypothesis,the model is the one in Section 15.2.2. The restricted estimate is given in (15-12) and the equations which follow. To obtain them, we make a small modification in our algorithm above. We replace step (3) with

(3') $\hat{\beta} = [s^{11}\mathbf{x}_1'\mathbf{y}_1 + s^{22}\mathbf{x}_2'\mathbf{y}_2 + s^{12}(\mathbf{x}_1'\mathbf{y}_2 + \mathbf{x}_2'\mathbf{y}_1)]/[s^{11}\mathbf{x}_1'\mathbf{x}_1 + s^{22}\mathbf{x}_2'\mathbf{x}_2 + 2s^{12}\mathbf{x}_1'\mathbf{x}_2]$.

Step 4 is then computed using this common estimate for both $\hat{\beta}_1$ and $\hat{\beta}_2$. The iterations produce

1: .5372671
2: .5703837
3: .5725274
4: .5726687
5: .5726780
6: .5726786 converged.

At this estimate, the estimate of $\mathbf{\Sigma}$ is $\begin{bmatrix} .8529188 & .1609926 \\ .1609926 & .3203732 \end{bmatrix}$. The likelihood ratio statistic is given in (15-56). Using our unconstrained and constrained estimates, we find $|\mathbf{W}_u|$ = .2471714 and $|\mathbf{W}_r|$ = .2473338. The statistic is λ = 20(ln.2473338 - ln.2471714) = .0131. This is far below the critical value of 3.84, so once again, we do not reject the hypothesis.

4. The GLS estimator is

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \sigma^{11}\mathbf{X'X} & \sigma^{12}\mathbf{X'X} \\ \sigma^{12}\mathbf{X'X} & \sigma^{22}\mathbf{X'X} \end{bmatrix}^{-1} \begin{bmatrix} \sigma^{11}\mathbf{X'y}_1 + \sigma^{12}\mathbf{X'y}_2 \\ \sigma^{12}\mathbf{X'y}_1 + \sigma^{22}\mathbf{X'y}_2 \end{bmatrix}$$

The matrix to be inverted equals $[\mathbf{\Sigma}^{-1} \otimes \mathbf{X'X}]^{-1}$. But, $[\mathbf{\Sigma}^{-1} \otimes \mathbf{X'X}]^{-1} = \mathbf{\Sigma} \otimes (\mathbf{X'X})^{-1}$. (See (2-76).) Therefore,

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \sigma_{11}(\mathbf{X'X})^{-1} & \sigma_{12}(\mathbf{X'X})^{-1} \\ \sigma_{12}(\mathbf{X'X})^{-1} & \sigma_{22}(\mathbf{X'X})^{-1} \end{bmatrix}^{-1} \begin{bmatrix} \sigma^{11}\mathbf{X'y}_1 + \sigma^{12}\mathbf{X'y}_2 \\ \sigma^{12}\mathbf{X'y}_1 + \sigma^{22}\mathbf{X'y}_2 \end{bmatrix}$$

We now make the replacements $\mathbf{X'y}_1 = (\mathbf{X'X})\mathbf{b}_1$ and $\mathbf{X'y}_2 = (\mathbf{X'X})\mathbf{b}_2$. After multiplying out the product, we find that

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \sigma_{11}\sigma^{11}\mathbf{b}_1 + \sigma_{11}\sigma^{12}\mathbf{b}_2 + \sigma_{12}\sigma^{12}\mathbf{b}_1 + \sigma_{12}\sigma^{22}\mathbf{b}_2 \\ \sigma_{12}\sigma^{11}\mathbf{b}_1 + \sigma_{12}\sigma^{12}\mathbf{b}_2 + \sigma_{22}\sigma^{12}\mathbf{b}_1 + \sigma_{22}\sigma^{22}\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} (\sigma_{11}\sigma^{11} + \sigma_{12}\sigma^{12})\mathbf{b}_1 + (\sigma_{11}\sigma^{12} + \sigma_{12}\sigma^{22})\mathbf{b}_2 \\ (\sigma_{12}\sigma^{11} + \sigma_{22}\sigma^{12})\mathbf{b}_1 + (\sigma_{12}\sigma^{12} + \sigma_{22}\sigma^{22})\mathbf{b}_2 \end{bmatrix}$$

The four scalar terms in the matrix product are the corresponding elements of $\mathbf{\Sigma\Sigma}^{-1} = \mathbf{I}$. Therefore, $\hat{\boldsymbol{\beta}} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}$.

5. The algebraic result is a little tedious, but straightforward. The GLS estimator which is computed is

$$\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{bmatrix} \sigma^{11}\mathbf{x}_1'\mathbf{x}_1 & \sigma^{12}\mathbf{x}_1'\mathbf{x}_2 \\ \sigma^{12}\mathbf{x}_2'\mathbf{x}_1 & \sigma^{22}\mathbf{x}_2'\mathbf{x}_2 \end{bmatrix}^{-1} \begin{bmatrix} \sigma^{11}\mathbf{x}_1'\mathbf{y}_1 + \sigma^{12}\mathbf{x}_1'\mathbf{y}_2 \\ \sigma^{12}\mathbf{x}_2'\mathbf{y}_1 + \sigma^{22}\mathbf{x}_2'\mathbf{y}_2 \end{bmatrix}.$$

It helps at this point to make some simplifying substitutions. The elements in the inverse matrix, σ^{ij}, are all equal to elements of the original matrix divided by the determinant. But, the determinant appears in the leading matrix, which is inverted and in the trailing vector (which is not). Therefore, the determinant will cancel out. Making the substitutions, $\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{bmatrix} \sigma_{22}\mathbf{x}_1'\mathbf{x}_1 & -\sigma_{12}\mathbf{x}_1'\mathbf{x}_2 \\ -\sigma_{12}\mathbf{x}_2'\mathbf{x}_1 & \sigma_{11}\mathbf{x}_2'\mathbf{x}_2 \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{22}\mathbf{x}_1'\mathbf{y}_1 - \sigma_{12}\mathbf{x}_1'\mathbf{y}_2 \\ -\sigma_{12}\mathbf{x}_2'\mathbf{y}_1 + \sigma_{22}\mathbf{x}_2'\mathbf{y}_2 \end{bmatrix}$. Now, we are concerned with probability limits. We divide every element of the matrix to be inverted by n, then because of the inversion, divide the vector on the right by n as well. Suppose, for simplicity, that

$\lim_{n\to\infty}\mathbf{x}_i'\mathbf{x}_j/n = q_{ij}$, i,j = 1,2,3. Then, $plim\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{bmatrix} \sigma_{22}q_{11} & -\sigma_{12}q_{12} \\ -\sigma_{12}q_{12} & \sigma_{11}q_{22} \end{bmatrix}^{-1} plim\begin{bmatrix} \sigma_{22}\mathbf{x}_1'\mathbf{y}_1/n - \sigma_{12}\mathbf{x}_1'\mathbf{y}_2/n \\ -\sigma_{12}\mathbf{x}_2'\mathbf{y}_1/n + \sigma_{11}\mathbf{x}_2'\mathbf{y}_2/n \end{bmatrix}$

Then, we will use $plim\,(1/n)\mathbf{x}_1'\mathbf{y}_1 = \beta_1 q_{11} + plim\,(1/n)\mathbf{x}_1\text{N}\varepsilon_1 = \beta_1 q_{11}$

$plim\,(1/n)\mathbf{x}_1'\mathbf{y}_2 = \beta_2 q_{12} + \beta_3 q_{13}$

$plim\,(1/n)\mathbf{x}_2'\mathbf{y}_1 = \beta_1 q_{12}$

$plim\,(1/n)\mathbf{x}_2'\mathbf{y}_2 = \beta_2 q_{22} + \beta_3 q_{23}$.

Therefore, after multiplying out all the terms,

$$plim\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{bmatrix} \sigma_{22}q_{11} & -\sigma_{12}q_{12} \\ -\sigma_{12}q_{12} & \sigma_{11}q_{22} \end{bmatrix}^{-1} \begin{bmatrix} \beta_1\sigma_{22}q_{11} - \beta_2\sigma_{12}q_{12} - \beta_3\sigma_{12}q_{13} \\ -\beta_1\sigma_{12}q_{12} + \beta_2\sigma_{11}q_{22} + \beta_3\sigma_{11}q_{23} \end{bmatrix}.$$

The inverse matrix is $\dfrac{1}{\sigma_{11}\sigma_{22}q_{11}q_{22} - (\sigma_{12}q_{12})^2}\begin{bmatrix} \sigma_{11}q_{22} & \sigma_{12}q_{12} \\ \sigma_{12}q_{12} & \sigma_{22}q_{22} \end{bmatrix}$, so with $\Delta = (\sigma_{11}\text{F}_{22}q_{11}q_{22} - (\text{F}_{12}q_{12})^2)$

$$\text{plim}\begin{pmatrix}\hat\beta_1\\ \hat\beta_2\end{pmatrix} = \left[\frac{1}{\Delta}\begin{pmatrix}\sigma_{11}q_{22} & \sigma_{12}q_{12}\\ \sigma_{12}q_{12} & \sigma_{22}q_{11}\end{pmatrix}\right]^{-1}\begin{bmatrix}\beta_1\sigma_{22}q_{11}-\beta_2\sigma_{12}q_{12}-\beta_3\sigma_{12}q_{13}\\ -\beta_1\sigma_{12}q_{12}+\beta_2\sigma_{11}q_{22}+\beta_3\sigma_{11}q_{23}\end{bmatrix}.$$

Taking the first coefficient separately and collecting terms,

$\text{plim}\,\hat\beta_1 = \beta_1[\sigma_{11}\sigma_{22}q_{11}q_{22}-(\sigma_{12}q_{12})^2]/\Delta + \beta_2[\sigma_{11}q_{22}\sigma_{12}q_{12} + \sigma_{12}q_{12}\sigma_{11}q_{22}]/\Delta + \beta_3[\sigma_{11}q_{22}\sigma_{12}q_{13} + \sigma_{12}q_{12}\sigma_{11}q_{23}]/\Delta$

The first term in brackets equals Δ while the second equals 0. That leaves

$\text{plim}\,\hat\beta_1 = \beta_1 - \beta_3[\sigma_{11}\sigma_{12}(q_{22}q_{13} - q_{12}q_{23})]/\Delta$ which is not equal to β_1. There are two special cases worthy of note, though. The right hand side does equal β_1 if either (1) $\sigma_{12} = 0$; the regressions are actually unrelated, or (2) $q_{12} = q_{13} = 0$; the regressors in the two equations are uncorrelated. The second of these is similar to our finding for omitted variables in the classical regression model. □

6. The model is $\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix} = \begin{bmatrix}\mathbf{i} & \mathbf{x} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{i}\end{bmatrix}\begin{pmatrix}\alpha_1\\ \beta\\ \alpha_2\end{pmatrix} + \begin{bmatrix}\boldsymbol{\varepsilon}_1\\ \boldsymbol{\varepsilon}_2\end{bmatrix}$. The GLS estimator of the full coefficient vector, θ, is

$$\hat{\boldsymbol\theta} = \begin{bmatrix}\sigma^{11}\begin{pmatrix}n & n\bar{x}\\ n\bar{x} & \mathbf{x'x}\end{pmatrix} & \sigma^{12}\begin{pmatrix}n\\ n\bar{x}\end{pmatrix}\\ \sigma^{12}\begin{pmatrix}n & n\bar{x}\end{pmatrix} & \sigma^{22}n\end{bmatrix}^{-1}\begin{bmatrix}\sigma^{11}\begin{pmatrix}n\bar{y}_1\\ \mathbf{x'y}_1\end{pmatrix}+\sigma^{12}\begin{pmatrix}n\bar{y}_2\\ \mathbf{x'y}_2\end{pmatrix}\\ \sigma^{12}n\bar{y}_1+\sigma^{22}n\bar{y}_2\end{bmatrix}.$$

Let q_{xx} equal $\mathbf{x'x}/n$, $q_{x1} = \mathbf{x'y}_1/n$ and, $q_{x2} = \mathbf{x'y}_2/n$. The ns in the inverse and in the vector cancel. Also, as suggested, we assume that $\bar{x} = 0$. As in the previous exercise, we replace elements of the inverse with elements from the original matrix and cancel the determinant which multiplies the matrix (after inversion) and divides the vector. Thus,

$$\hat{\boldsymbol\theta} = \begin{bmatrix}\sigma_{11} & 0 & -\sigma_{12}\\ 0 & \sigma_{22}q_{xx} & 0\\ -\sigma_{12} & 0 & \sigma_{11}\end{bmatrix}^{-1}\begin{bmatrix}\sigma_{22}\bar{y}_1-\sigma_{12}\bar{y}_2\\ \sigma_{11}q_{x1}-\sigma_{12}q_{x2}\\ -\sigma_{12}\bar{y}_1+\sigma_{11}\bar{y}_2\end{bmatrix}.$$

The inverse of the matrix is straightforward. Proceeding directly, we obtain

$$\hat{\boldsymbol\theta} = \frac{1}{\sigma_{22}q_{xx}(\sigma_{11}\sigma_{22}-\sigma_{12}^2)}\begin{bmatrix}\sigma_{11}\sigma_{22}q_{xx} & 0 & \sigma_{12}\sigma_{22}q_{xx}\\ 0 & \sigma_{11}\sigma_{22}-\sigma_{12}^2 & 0\\ \sigma_{12}\sigma_{22}q_{xx} & 0 & \sigma_{22}q_{xx}\end{bmatrix}^{-1}\begin{bmatrix}\sigma_{22}\bar{y}_1-\sigma_{12}\bar{y}_2\\ \sigma_{11}q_{x1}-\sigma_{12}q_{x2}\\ -\sigma_{12}\bar{y}_1+\sigma_{11}\bar{y}_2\end{bmatrix}.$$

It remains only to multiply the matrices and collect terms. The result is

$$\hat\alpha_1 = \bar{y}_1,\ \hat\alpha_2 = \bar{y}_2,\ \hat\beta = [(q_{x1}/q_{xx}) - (\sigma_{12}\sigma_{22})(q_{x2}/q_{xx})] = b_1 - \gamma b_2.\ □$$

7. Once again, nothing is lost by assuming that $\bar{x} = 0$. Now, the OLS estimators are

$$a_1 = \bar{y}_1,\ a_2 = \bar{y}_2,\ a_3 = \bar{y}_3,\ b = \mathbf{x'y}_1/\mathbf{x'x}.$$

The vector of residuals is
$e_{i1} = y_{i1} - \bar{y}_1 - bx_i$
$e_{i2} = y_{i2} - \bar{y}_2$
$e_{i3} = y_{i3} - \bar{y}_3$

Now, if $y_{i2} + y_{i3} = 1$ at every observation, then $(1/n)\Sigma_i(y_{i2} + y_{i3}) = \bar{y}_2 + \bar{y}_3 = 1$ as well. Therefore, by just adding the two equations, we see that $e_{i2} + e_{i3} = 0$ for every observation. Let $\mathbf{e}_i$ be the 3×1 vector of residuals. Then, $\mathbf{e}_i'\mathbf{c} = 0$, where $\mathbf{c} = [0,1,1]'$. The sample covariance matrix of the residuals is

$\mathbf{S} = [(1/n)\Sigma_i\,\mathbf{e}_i\mathbf{e}_i']$. Then, $\mathbf{Sc} = [(1/n)\Sigma_i\,\mathbf{e}_i\mathbf{e}_i']\mathbf{c} = [(1/n)\Sigma_i\,\mathbf{e}_i\mathbf{e}_i'\mathbf{c}] = [(1/n)\Sigma_i\,\mathbf{e}_i\times 0] = \mathbf{0}$, which means, by definition, that $\mathbf{S}$ is singular.

We can proceed simply by dropping the third equation. The adding up condition implies that $\alpha_3 = 1 - \alpha_2$. So, we can treat the first two equations as a seemingly unrelated regression model and estimate a_3 using the estimate of α_2.

Applications

1. By adding the share equations vertically, we find the restrictions

$$\beta_1 + \beta_2 + \beta_3 = 1$$
$$\delta_{11} + \delta_{12} + \delta_{13} = 0$$
$$\delta_{12} + \delta_{22} + \delta_{23} = 0$$
$$\delta_{13} + \delta_{23} + \delta_{33} = 0$$
$$\gamma_{y1} + \gamma_{y2} + \gamma_{y3} = 0.$$

Note that the adding up condition also implies $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$.
We will eliminate the third share equation. The restrictions imply

$$\beta_3 = 1 - \beta_1 - \beta_2$$
$$\delta_{13} = -\delta_{11} - \delta_{12}$$
$$\delta_{23} = -\delta_{12} - \delta_{22}$$
$$\delta_{33} = -\delta_{13} - \delta_{23} = \delta_{11} + \delta_{22} + 2\delta_{12}$$
$$\gamma_{y3} = -\gamma_{y1} - \gamma_{y2}.$$

By inserting these in the three share equations, we find

$$S_1 = \beta_1 + \delta_{11}\ln p_1 + \delta_{12}\ln p_2 - \delta_{11}\ln p_3 - \delta_{12}\ln p_3 + \gamma_{y1}\ln Y + \varepsilon_1$$
$$= \beta_1 + \delta_{11}\ln(p_1/p_3) + \delta_{12}\ln(p_2/p_3) + \gamma_{y1}\ln Y + \varepsilon_1$$
$$S_2 = \beta_2 + \delta_{12}\ln p_1 + \delta_{22}\ln p_2 - \delta_{12}\ln p_3 - \delta_{22}\ln p_3 + \gamma_{y2}\ln Y + \varepsilon_2$$
$$= \beta_2 + \delta_{12}\ln(p_1/p_3) + \delta_{22}\ln(p_2/p_3) + \gamma_{y2}\ln Y + \varepsilon_2$$
$$S_3 = 1 - \beta_1 - \beta_2 - \delta_{11}\ln p_1 - \delta_{12}\ln p_1 - \delta_{12}\ln p_2 - \delta_{22}\ln p_2 + \delta_{11}\ln p_3 + \delta_{12}\ln p_3 + \delta_{12}\ln p_3$$
$$+ \delta_{22}\ln p_3 - \gamma_{y1}\ln p_3 - \gamma_{y2}\ln p_3 - \varepsilon_1 - \varepsilon_2$$
$$= 1 - S_1 - S_2$$

For the cost function, making the substitutions for β_3, δ_{13}, δ_{23}, δ_{33}, and γ_{y3} produces

$$\ln C = \alpha + \beta_1(\ln p_1 - \ln p_3) + \beta_2(\ln p_2 - \ln p_3)$$
$$+ \delta_{11}((\ln^2 p_1)/2 - \ln p_1\ln p_3 + (\ln^2 p_3)/2)$$
$$+ \delta_{22}((\ln^2 p_2)/2 - \ln p_2\ln p_3 + (\ln^2 p_3)/2) + \delta_{12}(\ln p_1\ln p_2 - \ln p_1\ln p_3 - \ln p_2\ln p_3 + (\ln^2 p_3))$$
$$+ \gamma_{y1}\ln Y(\ln p_1 - \ln p_3) + \gamma_{y2}\ln Y(\ln p_2 - \ln p_3) + \beta_y\ln Y + \beta_{yy}(\ln^2 Y)/2 + \varepsilon_c$$
$$= \alpha + \beta_1\ln(p_1/p_3) + \beta_2\ln(p_2/p_3)$$
$$+ \delta_{11}(\ln^2(p_1/p_3))/2 + \delta_{22}(\ln^2(p_2/p_3))/2 + \delta_{12}\ln(p_1/p_3)\ln(p_2/p_3)$$
$$+ \gamma_{y1}\ln Y\ln(p_1/p_3) + \gamma_{y2}\ln Y\ln(p_2/p_3) + \beta_y\ln Y + \beta_{yy}(\ln^2 Y)/2 + \varepsilon_c$$

The system of three equations (cost and two shares) can be estimated as discussed in the text. Invariance is achieved by using a maximum likelihood estimator. The five parameters eliminated by the restrictions can be estimated after the others are obtained just by using the restrictions. The restrictions are linear, so the standard errors are also striaghtforward to obtain.

The least squares estimates are shown below. Estimated standard errors appear in parentheses.

Variable	**Cost Function**	**Capital Share**	**Labor Share**
One	51.32 (45.91)	-.0174 (.4697)	.2172 (.2408)
$\ln(p_k/p_f)$	-21.74 (20.14)	.2380 (.1045)	.0033 (.0534)
$\ln(p_l/p_f)$	32.39 (21.81)	.0065 (.1059)	.0168 (.0542)
$\ln^2(p_k/p_f)/2$	4.596 (4.604)	-.0007 (.0098)	-.0117 (.0050)
$\ln^2(p_l/p_f)/2$	8.216 (5.159)		
$\ln(p_k/p_f)\ln(p_l/p_f)$	-6.238 (4.684)		
$\ln Y$	1.674 (.9297)		
$\ln^2 Y/2$	,006997 (.0313)		
$\ln Y\ln(p_k/p_f)$	-.3223 (.2652)		
$\ln Y\ln(p_l/p_f)$	.08631 (.1981)		

The estimates do not even come close to satisfying the cross equation restrictions. The parameters in the cost function are extremely large, owing primarily to rather severe multicollinearity among the price terms.

The results of estimation of the system by direct maximum likelihood are shown. The convergence criterion is the value of Belsley (discussed near the end of Section 5.5). The value α shown below is $\mathbf{g}'\mathbf{H}^{-1}\mathbf{g}$ where **g** is the gradient and **H** is the Hessian of the log-likelihood.

Iteration 0, F=46.76391, ln***S***= -7.514268, α= 2.054399

```
Iteration  1, F=136.7448, ln*S*= -16.51236, α= .5796486
Iteration  2, F=146.9803, ln*S*= -17.53591, α= .02179947
Iteration  3, F=147.2268, ln*S*= -17.56055, α= .0004222
      Residual covariance matrix
                   Cost        Capital      Labor
      Cost       .0145572
      Capital    .000304768   .00303853
      Labor     -.000317554  -.000887258   .000798128
        Coefficient Estimate  Std. Error
             α        -6.41878     .6637
             βk       -.0546555     .2422
             βl        .250976      .2138
             δkk       .245259      .06904
             δll       .0245770     .04788
             δkl      -.00403448    .04779
             βy        .572452      .1340
             βyy       .0456587     .01908
             γyk      -.00124236    .008409
             γyl      -.0116921     .004442
             βf        .8036795
             δkf      -.2412245
             δlf      -.0205425
             δff       .261767
             γyf       .0129345
```

The means of the variables are: $\overline{Y}$ = 3531.8, $\overline{p}_k$ = 169.35, $\overline{p}_l$ = 2.039, $\overline{p}_f$ = 26.41. The three factor shares computed at these means are S_k = .4182, S_l = .0865, S_f = .4953. (The sample means are .411, .0954, and .4936.) The matrix of elasticities computed according to (15-72) is

	k	l	f	
Σ =	.01115			k
	.8885	-7.2756		l
	-.1646	.5206	.04819	f

(Two of the three diagonals have the `wrong' sign. This may be due to the very small sample size. The cross elasticities however do conform to what one might expect, the primary one being the evident substitution between capital and fuel.

To test the hypothesis that $\gamma_{yi} = 0$, we reestimate the model without the interaction terms between ln$Y$ and the prices in the cost function and without ln$Y$ in the factor share equations. The iterations for this restricted model are shown below.

Iter.= 0, F=46.76391, log|**S**|= -7.514268, α= 1.912223
Iter.= 1, F=123.7521, log|**S**|= -15.21308, α= .5888180
Iter.= 2, F=136.3410, log|**S**|= -16.47198, α= .2771995
Iter.= 3, F=141.3491, log|**S**|= -16.97279, α= .08024513
Iter.= 4, F=142.5591, log|**S**|= -17.09379, α= .01636212
Converged achieved

Since we are interested only in the test statistic, we have not listed the parameter estimates. The test statistic given in (17-26) is $\lambda = T(\ln|\mathbf{S}_r| - \ln|\mathbf{S}_u|) = 20(-17.09379 - (-17.56055)) = 9.3352$. There are two restrictions since only two of the three parameters are free. The critical value from the chi-squared table is 5.99, so we would reject the hypothesis.

```
?===========================================
? Application 10.2
?===========================================
? a.  Separate regressions and aggregation test.
?     This saves the residuals to be used later.
CALC    ; SS1=0 $
MATRIX  ; EOLS = Init(20,10,0) $
PROCEDURE $
Include ; new ; Firm = company $
REGRESS ; Lhs = I ; Rhs = F,C,one ; Res = e$
CALC    ; SS1=SS1 + Sumsqdev $
MATRIX  ; EOLS(*,company) = e $
ENDPROC $
EXECUTE ; Company=1,10 $
SAMPLE  ; 1-200 $
+----------------------------------------------------+
| Residuals    Sum of squares       =   143205.9     |
|              Standard error of e  =   91.78167     |
| Fit          R-squared            =   .9213540     |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 F       |     .11928083       .02583417     4.617   .0002   4333.84500
 C       |     .37144481       .03707282    10.019   .0000   648.435000
 Constant|   -149.782453      105.842125    -1.415   .1751
+----------------------------------------------------+
| Residuals    Sum of squares       =   158093.3     |
|              Standard error of e  =   96.43445     |
| Fit          R-squared            =   .4708624     |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 F       |     .17485602       .07419805     2.357   .0307   1971.82500
 C       |     .38964189       .14236688     2.737   .0140   294.855000
 Constant|   -49.1983219      148.075365     -.332   .7438
+----------------------------------------------------+
| Residuals    Sum of squares       =   13216.59     |
|              Standard error of e  =   27.88272     |
| Fit          R-squared            =   .7053067     |
|              Adjusted R-squared   =   .6706369     |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 F       |     .02655119       .01556610     1.706   .1063   1941.32500
 C       |     .15169387       .02570408     5.902   .0000   400.160000
 Constant|   -9.95630645      31.3742491     -.317   .7548
+----------------------------------------------------+
| Residuals    Sum of squares       =   2997.444     |
|              Standard error of e  =   13.27856     |
| Fit          R-squared            =   .9135784     |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 F       |     .07794782       .01997330     3.903   .0011   693.210000
 C       |     .31571819       .02881317    10.957   .0000   121.245000
 Constant|   -6.18996051      13.5064781     -.458   .6525
+----------------------------------------------------+
| Residuals    Sum of squares       =   1396.836     |
|              Standard error of e  =   9.064592     |
| Fit          R-squared            =   .6804076     |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
```

F	.16237770	.05703645	2.847	.0111	231.470000
C	.00310174	.02196531	.141	.8894	486.765000
Constant	22.7071160	6.87207605	3.304	.0042	

Residuals	Sum of squares =	1110.533
	Standard error of e =	8.082418
Fit	R-squared =	.9521422

Variable	Coefficient	Standard Error	t-ratio	P[|T|>t]	Mean of X
F	.13145484	.03117234	4.217	.0006	419.865000
C	.08537427	.10030597	.851	.4065	104.285000
Constant	-8.68554338	4.54516804	-1.911	.0730	

Residuals	Sum of squares =	1507.403
	Standard error of e =	9.416516
Fit	R-squared =	.7635009

Variable	Coefficient	Standard Error	t-ratio	P[|T|>t]	Mean of X
F	.08752720	.06562593	1.334	.1999	149.790000
C	.12378141	.01706483	7.254	.0000	314.945000
Constant	-4.49953436	11.2893942	-.399	.6952	

Residuals	Sum of squares =	1773.234
	Standard error of e =	10.21312
Fit	R-squared =	.7444461

Variable	Coefficient	Standard Error	t-ratio	P[|T|>t]	Mean of X
F	.05289413	.01570650	3.368	.0037	670.910000
C	.09240649	.05609897	1.647	.1179	85.6400000
Constant	-.50939018	8.01528894	-.064	.9501	

Residuals	Sum of squares =	1407.360
	Standard error of e =	9.098674
Fit	R-squared =	.6655145

Variable	Coefficient	Standard Error	t-ratio	P[|T|>t]	Mean of X
F	.07538794	.03395227	2.220	.0403	333.650000
C	.08210356	.02799168	2.933	.0093	297.900000
Constant	-7.72283708	9.35933952	-.825	.4207	

Residuals	Sum of squares =	20.02673
	Standard error of e =	1.085377
Fit	R-squared =	.6431578

Variable	Coefficient	Standard Error	t-ratio	P[|T|>t]	Mean of X
F	.00457343	.02716079	.168	.8683	70.9210000
C	.43736919	.07958891	5.495	.0000	5.94150000
Constant	.16151857	2.06556414	.078	.9386	

```
+----------------------------------------------------+
| Ordinary    least squares regression               |
| LHS=I        Mean                 =   145.9582     |
|              Standard deviation   =   216.8753     |
| WTS=none     Number of observs.   =        200     |
| Model size   Parameters           =          3     |
|              Degrees of freedom   =        197     |
| Residuals    Sum of squares       =   1755850.     |
|              Standard error of e  =   94.40840     |
| Fit          R-squared            =   .8124080     |
|              Adjusted R-squared   =   .8105035     |
| Model test   F[  2,   197] (prob) = 426.58 (.0000) |
+----------------------------------------------------+
+--------+-------------+---------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+-------------+---------------+--------+--------+----------+
 F       |     .11556216       .00583571    19.803   .0000   1081.68110
 C       |     .23067849       .02547580     9.055   .0000   276.017150
 Constant|   -42.7143694      9.51167603    -4.491   .0000
? b.  Aggregation test
REGRESS ; LHS = I ; RHS = F,C,one $
CALC    ; SS0=Sumsqdev $
CALC    ; List ; Fstat = ((SS0 - SS1)/(9*3)) / (SS0/(n-10*3))
               ; FC    = Ftb(.95,27,170) $
+-------------------------------------+
| Listed Calculator Results           |
+-------------------------------------+
 FSTAT   =      5.131854
 FC      =      1.551534
? c.  SUR model
NAMELIST ; X1=F1,C1,one $
NAMELIST ; X2=F2,C2,one $
NAMELIST ; X3=F3,C3,one $
NAMELIST ; X4=F4,C4,one $
NAMELIST ; X5=F5,C5,one $
NAMELIST ; X6=F6,C6,one $
NAMELIST ; X7=F7,C7,one $
NAMELIST ; X8=F8,C8,one $
NAMELIST ; X9=F9,C9,one $
NAMELIST ; X10=F10,C10,one $
NAMELIST ; Y=I1,I2,I3,I4,I5,I6,I7,I8,I9,I10 $
SAMPLE ; 1 - 20 $
SURE ; Lhs = Y ; Eq1=X1;Eq2=X2;Eq3=X3;Eq4=X4;Eq5=X6;Eq6=X6
               ; Eq7=X7;Eq8=X8;Eq9=X9;Eq10=X10
     ; Maxit=0 ; OLS $
Criterion function for GLS is log-likelihood.
Iteration    0, GLS           =   -737.6463
Iteration    1, GLS           =   -730.1070
+----------------------------------------------------+
| Estimates for equation: I1                         |
+----------------------------------------------------+
+--------+-------------+---------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|
+--------+-------------+---------------+--------+--------+----------+
 F1      |     .12472490       .01490044     8.371   .0000   4333.84500
 C1      |     .37951869       .02912686    13.030   .0000   648.435000
 Constant|   -178.611571      65.7890483    -2.715   .0066
+----------------------------------------------------+
| Estimates for equation: I2                         |
+----------------------------------------------------+
+--------+-------------+---------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|
+--------+-------------+---------------+--------+--------+----------+
 F2      |     .16828512       .04057787     4.147   .0000   1971.82500
 C2      |     .33587688       .10299836     3.261   .0011   294.855000
 Constant|   -20.3887867      83.2537952     -.245   .8065
```

Estimates for equation: I3

Variable	Coefficient	Standard Error	b/St.Er.	P[|Z|>z]	Mean of X
F3	.03425481	.00925706	3.700	.0002	1941.32500
C3	.12538119	.02040101	6.146	.0000	400.160000
Constant	-14.3822597	20.6146424	-.698	.4854	

Estimates for equation: I4

Variable	Coefficient	Standard Error	b/St.Er.	P[|Z|>z]	Mean of X
F4	.06760969	.01597735	4.232	.0000	693.210000
C4	.30752805	.02536245	12.125	.0000	121.245000
Constant	1.96954637	11.0026359	.179	.8579	

Estimates for equation: I5

Variable	Coefficient	Standard Error	b/St.Er.	P[|Z|>z]	Mean of X
F6	.00635232	.02903793	.219	.8268	419.865000
C6	.12737505	.09456013	1.347	.1780	104.285000
Constant	45.8520779	4.86959707	9.416	.0000	

Estimates for equation: I6

Variable	Coefficient	Standard Error	b/St.Er.	P[|Z|>z]	Mean of X
F6	.12891587	.01798607	7.168	.0000	419.865000
C6	.06768693	.06029084	1.123	.2616	104.285000
Constant	-5.77499083	3.44886478	-1.674	.0940	

Estimates for equation: I7

Variable	Coefficient	Standard Error	b/St.Er.	P[|Z|>z]	Mean of X
F7	.09106397	.04535783	2.008	.0447	149.790000
C7	.12913287	.01446995	8.924	.0000	314.945000
Constant	-6.71472214	8.72476796	-.770	.4415	

Estimates for equation: I8

Variable	Coefficient	Standard Error	b/St.Er.	P[|Z|>z]	Mean of X
F8	.05179274	.00835658	6.198	.0000	670.910000
C8	.04729955	.03473521	1.362	.1733	85.6400000
Constant	4.09249729	5.09237714	.804	.4216	

Estimates for equation: I9

Variable	Coefficient	Standard Error	b/St.Er.	P[|Z|>z]	Mean of X
F9	.07275469	.02111017	3.446	.0006	333.650000
C9	.06640816	.02194422	3.026	.0025	297.900000
Constant	-2.16859331	7.30885683	-.297	.7667	

Estimates for equation: I10

```
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 F10     |    -.01695668       .01550963    -1.093   .2743   70.9210000
 C10     |     .37466423       .05739586     6.528   .0000   5.94150000
 Constant|    2.06101718      1.16003699     1.777   .0756?

c.  Aggregation test according to (10-15)
MATRIX ; Z=Init(3,3,0) ; J=Iden(3); L=-1*J $
MATRIX ; R=[j,z,z,z,z,z,z,z,z,l /
            z,j,z,z,z,z,z,z,z,l /
            z,z,j,z,z,z,z,z,z,l /
            z,z,z,j,z,z,z,z,z,l /
            z,z,z,z,j,z,z,z,z,l /
            z,z,z,z,z,j,z,z,z,l /
            z,z,z,z,z,z,j,z,z,l /
            z,z,z,z,z,z,z,j,z,l /
            z,z,z,z,z,z,z,z,j,l ]
       ; d = R*b ; Vd = R*Varb*R'
       ; list ; AggF = 1/27 * d'<vd>d $
Matrix AGGF     has  1 rows and  1 columns.
               1
        +--------------
       1|   98.53777
CALC   ; List ; Ftb(.95,27,(200-10*3)) $
+------------------------------------+
| Listed Calculator Results          |
+------------------------------------+
 Result  =      1.551534
? d.  Using separate OLS regressions, compute LM statistic
? OLS residuals were saved in matrix EOLS earlier.
MATRIX ; VEOLS = 1/20*EOLS'EOLS
       ; VI = Diag(VEOLS) ; SDI = ISQR(VI)
       ; ROLS = SDI*VEOLS*SDI
       ; RR = ROLS' *ROLS $
CALC   ; List ; LMStat = (20/2)*(Trc(RR)-10)
              ; Ctb(.95, (9*10/2))$
+------------------------------------+
| Listed Calculator Results          |
+------------------------------------+
 LMSTAT  =     97.617948
 Result  =     61.656233

? Constrained Sur model with one coefficient vector.
? This is the unconstrained model in (10-19)-(10-21)
SAMPLE ; 1 - 200 $
REGRESS; Lhs = I ; Rhs = F,C,one $
+----------------------------------------------------+
| Ordinary    least squares regression                |
| LHS=I        Mean                 =   145.9582     |
|              Standard deviation   =   216.8753     |
| WTS=none     Number of observs.   =        200     |
| Model size   Parameters           =          3     |
|              Degrees of freedom   =        197     |
| Residuals    Sum of squares       =   1755850.     |
|              Standard error of e  =   94.40840     |
| Fit          R-squared            =   .8124080     |
|              Adjusted R-squared   =   .8105035     |
| Model test   F[  2,   197] (prob) = 426.58 (.0000) |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 F       |     .11556216       .00583571    19.803   .0000   1081.68110
 C       |     .23067849       .02547580     9.055   .0000   276.017150
 Constant|   -42.7143694      9.51167603    -4.491   .0000
TSCS   ; Lhs = I ; Rhs = F,C,one ; Pds=20 ; Model=S2,R0 $
```

```
+----------------------------------------------------+
| Groupwise Regression Models                        |
| Estimator =                 2 Step GLS             |
| Groupwise Het. and Correlated       (S2)           |
| Nonautocorrelated disturbances      (R0)           |
| Test statistics against the correlation            |
| Deg.Fr. =   45 C*(.95) =  61.66 C*(.99) =  69.96 |
| Test statistics against the correlation            |
| Likelihood ratio    statistic      =    320.2052 |
| Log-likelihood function =           -853.084972 |
+----------------------------------------------------+
+--------+--------------+----------------+--------+--------+
|Variable| Coefficient  | Standard Error |b/St.Er.|P[|Z|>z]|
+--------+--------------+----------------+--------+--------+
 F       |     .10806238       .00241169    44.808   .0000
 C       |     .15079551       .00386063    39.060   .0000
 Constant|   -20.1588844       .79950153   -25.214   .0000
CREATE ; WI = (SDI(firm,firm))^2 $
REGRESS; Lhs = I ; Rhs = F,C,one ; Wts = WI $
+-------------------------------------------------------+
| Ordinary    least squares regression                  |
| LHS=I        Mean                 =   6.993136        |
|              Standard deviation   =   18.01824        |
| WTS=WI       Number of observs.   =        200        |
| Model size   Parameters           =          3        |
|              Degrees of freedom   =        197        |
| Residuals    Sum of squares       =   11690.82        |
|              Standard error of e  =   7.703521        |
| Fit          R-squared            =   .8190465        |
|              Adjusted R-squared   =   .8172094        |
+-------------------------------------------------------+
+--------+--------------+----------------+--------+--------+----------+
|Variable| Coefficient  | Standard Error |t-ratio |P[|T|>t]| Mean of X|
+--------+--------------+----------------+--------+--------+----------+
 F       |     .07847124       .00459121    17.092   .0000   96.8424912
 C       |     .09896094       .00761314    12.999   .0000   23.8374846
 Constant|   -2.96519441       .66964256    -4.428   .0000
```