# APPENDIX 3

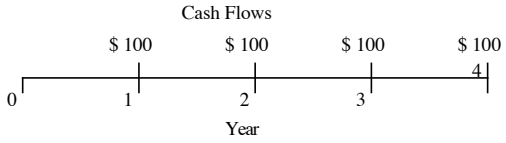
## TIME VALUE OF MONEY

The simplest tools in finance are often the most powerful. *Present value* is a concept that is intuitively appealing, simple to compute, and has a wide range of applications. It is useful in decision making ranging from simple personal decisions—buying a house, saving for a child's education, and estimating income in retirement—to more complex corporate financial decisions—picking projects in which to invest as well as the right financing mix for these projects.

### Time Lines and Notation

Dealing with cash flows that are at different points in time is made easier using a time line that shows both the timing and the amount of each cash flow in a stream. Thus a cash flow stream of $100 at the end of each of the next four years can be depicted on a time line like the one depicted in Figure A3.1.

*Figure A3.1: A Time Line for Cash Flows: $ 100 in Cash Flows Received at the End of Each of Next 4 years*

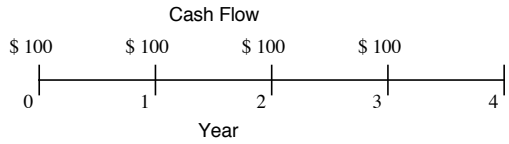



In the figure, 0 refers to right now. A cash flow that occurs at time 0 is therefore already in present value terms and does not need to be adjusted for time value. A distinction must be made here between a *period* of time and a *point* in time. The portion of the time line between 0 and 1 refers to period 1, which in this example is the first year. The cash flow that occurs at the point in time 1 refers to the cash flow that occurs at the end of period 1. Finally, the discount rate, which is 10 percent in this example, is specified for each period on the time line and may be different for each period. Had the cash flows been at the beginning of each year instead of at the end of each year, the time line would have been redrawn as it appears in Figure A3.2.





*Figure A3.2: A Time Line for Cash Flows: $ 100 in Cash Received at the Beginning of Each Year for Next 4 years*




Note that in present value terms, a cash flow that occurs at the beginning of year two is the equivalent of a cash flow that occurs at the end of year one.

Cash flows can be either positive or negative; positive cash flows are called cash inflows and negative cash flows are called cash outflows. For notational purposes, we will assume the following for the chapter that follows:

| *Notation* | *Stands For* |
|---|---|
| PV | Present value |
| FV | Future value |
| $CF_t$ | Cash flow at the end of period *t* |
| A | Annuity: constant cash flows over several periods |
| *r* | Discount rate |
| *g* | Expected growth rate in cash flows |
| *n* | Number of years over which cash flows are received or paid |

### The Intuitive Basis for Present Value

There are three reasons why a cash flow in the future is worth less than a similar cash flow today.

1. ***Individuals prefer present consumption to future consumption.*** People would have to be offered more in the future to give up present consumption. If the preference for current consumption is strong, individuals will have to be offered much more in terms of future consumption to give up current consumption, a trade-off that is captured by a high "real" rate of return or discount rate. Conversely, when the preference for current

consumption is weaker, individuals will settle for much less in terms of future consumption and, by extension, a low real rate of return or discount rate.

2. ***When there is monetary inflation, the value of currency decreases over time.*** The greater the inflation, the greater the difference in value between a nominal cash flow today and the same cash flow in the future.
3. ***A promised cash flow might not be delivered for a number of reasons:*** The promisor might default on the payment, the promisee might not be around to receive payment, or some other contingency might intervene to prevent the promised payment or to reduce it. Any uncertainty (risk) associated with the cash flow in the future reduces the value of the cash flow.

The process by which future cash flows are adjusted to reflect these factors is called discounting, and the magnitude of these factors is reflected in the discount rate. The discount rate can be viewed as a composite of the expected real return (reflecting consumption preferences in the aggregate over the investing population), the expected inflation rate (to capture the deterioration in the purchasing power of the cash flow), and the uncertainty associated with the cash flow.

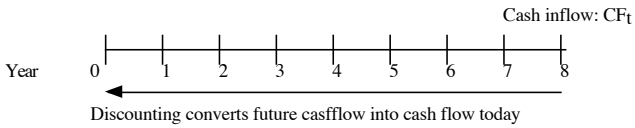
## The Mechanics of Time Value

The process of discounting future cash flows converts them into cash flows in present value terms. Conversely, the process of compounding converts present cash flows into future cash flows. There are five types of cash flows—simple cash flows, annuities, growing annuities, perpetuities, and growing perpetuities—which we discuss next.

### Simple Cash Flows

A simple cash flow is a single cash flow in a specified future time period; it can be depicted on a time line as in Figure A3.3.

*Figure A3.3: Present Value of a Cash Flow*




where $CF_t$ = the cash flow at time *t*.





This cash flow can be discounted back to the present using a discount rate that reflects the uncertainty of the cash flow. Concurrently, cash flows in the present can be compounded to arrive at an expected future cash flow.

***I. Discounting a Simple Cash Flow***

Discounting a cash flow converts it into present value dollars and enables the user to do several things. First, once cash flows are converted into present value dollars, they can be aggregated and compared. Second, if present values are estimated correctly, the user should be indifferent between the future cash flow and the present value of that cash flow. The present value of a cash flow can be written as follows

$$\text{Present Value of Simple Cash Flow} = \frac{CF_t}{(1+r)^t}$$

where *r* = discount rate.

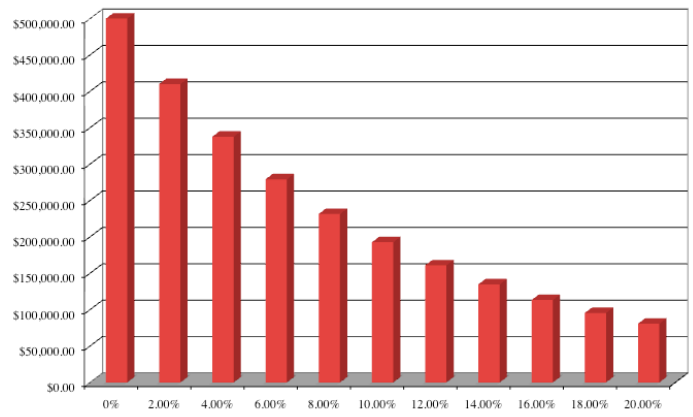
Other things remaining equal, the present value of a cash flow will decrease as the discount rate increases and continue to decrease the further into the future the cash flow occurs.

To illustrate this concept, assume that you own are currently leasing your office space and expect to make a lump-sum payment to the owner of the real estate of $500,000 ten years from now. Assume that an appropriate discount rate for this cash flow is 10 percent. The present value of this cash flow can then be estimated :

$$\text{Present Value of Payment} = \frac{\$500{,}000}{(1.10)^{10}} = \$192{,}772$$

This present value is a decreasing function of the discount rate, as illustrated in Figure A3.4.

*Figure A3.4: Present value of $500,000 in 10 years*




***II. Compounding a Cash Flow***

Current cash flows can be moved to the future by compounding the cash flow at the appropriate discount rate.

$$\text{Future Value of Simple Cash Flow} = CF_0 (1 + r)^t$$

where $CF_0$ = cash flow now, $r$ = discount rate. Again, the compounding effect increases with both the discount rate and the compounding period.

As the length of the holding period is extended, small differences in discount rates can lead to large differences in future value. In a study of returns on stocks and bonds between 1926 and 1997, Ibbotson and Sinquefield found that stocks on the average made 12.4 percent, Treasury bonds made 5.2 percent, and Treasury bills made 3.6 percent. Assuming that these returns continue into the future, Table A3.1 provides the future values of $100 invested in each category at the end of a number of holding periods—one year, five years, ten years, twenty years, thirty years, and forty years.[1]

*Table A3.1 Future Values of Investments—Asset Classes*

1 See Stocks, Bonds, Bills and Inflation Yearbook, 1998, Ibbotson Associates, Chicago.





| *Holding Period (Years)* | *Stocks* | *Treasury Bonds* | *Treasury Bills* |
|---|---|---|---|
| 1 | $112.40 | $105.20 | $103.60 |
| 5 | $179.40 | $128.85 | $119.34 |
| 10 | $321.86 | $166.02 | $142.43 |
| 20 | $1,035.92 | $275.62 | $202.86 |
| 30 | $3,334.18 | $457.59 | $288.93 |
| 40 | $10,731.30 | $759.68 | $411.52 |

The differences in future value from investing at these different rates of return are small for short compounding periods (such as one year) but become larger as the compounding period is extended. For instance, with a forty-year time horizon, the future value of investing in stocks, at an average return of 12.4 percent, is more than twelve times larger than the future value of investing in Treasury bonds at an average return of 5.2 percent and more than twenty-five times the future value of investing in Treasury bills at an average return of 3.6 percent.

*III. The Frequency of Discounting and Compounding*

The frequency of compounding affects both the future and present values of cash flows. In the examples just discussed, the cash flows were assumed to be discounted and compounded annually—that is, interest payments and income were computed at the end of each year, based on the balance at the beginning of the year. In some cases, however, the interest may be computed more frequently, such as on a monthly or semi-annual basis. In these cases, the present and future values may be very different from those computed on an annual basis; the stated interest rate on an annual basis can deviate significantly from the effective or true interest rate. The effective interest rate can be computed as follows:

$$\text{Effective Interest Rate} = \left(1 + \frac{\text{Stated Annual Interest Rate}}{n}\right)^n - 1$$

where $n$ = number of compounding periods during the year (2 = semi-annual; 12 = monthly). For instance, a 10 percent annual interest rate, if there is semi-annual compounding, works out to an effective interest rate of

$$\text{Effective Interest Rate} = 1.05^2 - 1 = 0.10125 \text{ or } 10.125\%$$

As compounding becomes continuous, the effective interest rate can be computed as follows

$$\text{Effective Interest Rate} = \exp^r - 1$$

where exp = exponential function and $r$ = stated annual interest rate. Table A3.2 provides the effective rates as a function of the compounding frequency.

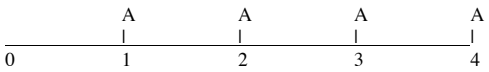
*Table A3.2 Effect of Compounding Frequency on Effective Interest Rates*

| *Frequency* | *Rate* | *t* (Days) | *Formula* | *Effective Annual Rate* |
|---|---|---|---|---|
| Annual | 10% | 1 | 0.10 | 10% |
| Semi-annual | 10% | 2 | $(1 + 0.10/2)^2 - 1$ | 10.25% |
| Monthly | 10% | 12 | $(1 + 0.10/12)^{12} - 1$ | 10.47% |
| Daily | 10% | 365 | $(1 + 0.10/365)^{365} - 1$ | 10.5156% |
| Continuous | 10% | | $\exp^{0.10} - 1$ | 10.5171% |

As you can see, compounding becomes more frequent, the effective rate increases, and the present value of future cash flows decreases.

**Annuities**

An *annuity* is a constant cash flow that occurs at regular intervals for a fixed period of time. Defining A to be the annuity, the time line for an annuity may be drawn as follows:




An annuity can occur at the end of each period, as in this time line, or at the beginning of each period.





***I. Present Value of an End-of-the-Period Annuity***

The present value of an annuity can be calculated by taking each cash flow and discounting it back to the present and then adding up the present values. Alternatively, a formula can be used in the calculation. In the case of annuities that occur at the end of each period, this formula can be written as

$$PV \text{ of an Annuity} = \text{PV(A,r,n)} = A\left[\frac{1 - \frac{1}{(1+r)^n}}{r}\right]$$

where A = annuity, $r$ = discount rate, and $n$ = number of years. Accordingly, the notation we will use in the rest of this book for the present value of an annuity will be PV(A,$r$,$n$).

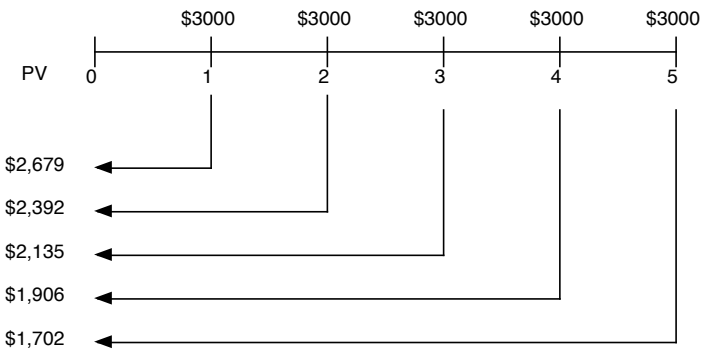
To illustrate, assume again that you are have a choice of buying a copier for $10,000 cash down or paying $3,000 a year, at the end of each year, for five years for the same copier. If the opportunity cost is 12 percent, which would you rather do?

$$PV \text{ of \$3000 each year for next 5 years} = \$3000\left[\frac{1 - \frac{1}{(1.12)^5}}{.12}\right] = \$10{,}814$$

The present value of the installment payments exceeds the cash-down price; therefore, you would want to pay the $10,000 in cash now.

Alternatively, the present value could have been estimated by discounting each of the cash flows back to the present and aggregating the present values as illustrated in Figure A3.5.

*Figure A3.5 :Payment of $ 3000 at the end of each of next 5 years*




*II. Amortization Factors: Annuities Given Present Values*

In some cases, the present value of the cash flows is known and the annuity needs to be estimated. This is often the case with home and automobile loans, for example, where the borrower receives the loan today and pays it back in equal monthly installments over an extended period of time. This process of finding an annuity when the present value is known is examined here:

$$\text{Annuity given Present Value} = A(PV,r,n) = PV\left[\frac{r}{1 - \frac{1}{(1+r)^n}}\right]$$

Suppose you are trying to borrow $200,000 to buy a house with a conventional thirty-year mortgage with monthly payments. The annual percentage rate on the loan is 8 percent. The monthly payments on this loan can be estimated using the annuity due formula:

$$\text{Monthly Interest Rate on Loan} = \text{APR}/12 = 0.08/12 = 0.0067$$

$$\text{Monthly Payment on Mortgage} = \$200{,}000\left[\frac{0.0067}{1 - \frac{1}{(1.0067)^{360}}}\right] = \$1473.11$$





This monthly payment is an increasing function of interest rates. When interest rates drop, homeowners usually have a choice of refinancing, although there is an up-front cost to doing so.

*III. Future Value of End-of-the-Period Annuities*

In some cases, an individual may plan to set aside a fixed annuity each period for a number of periods and will want to know how much he or she will have at the end of the period. The future value of an end-of-the-period annuity can be calculated as follows:

$$FV \text{ of an Annuity} = FV(A,r,n) = A\left[\frac{(1+r)^n - 1}{r}\right]$$

Thus, the notation we will use throughout this book for the future value of an annuity will be FV(A,*r*,*n*).

Individual retirement accounts (IRAs) allow some taxpayers to set aside up to $2,000 a year for retirement and exempts the income earned on these accounts from taxation. If an individual starts setting aside money in an IRA early in his or her working life, the value at retirement can be substantially higher than the nominal amount actually put in. For instance, assume that this individual sets aside $2,000 at the end of every year, starting when she is twenty-five years old, for an expected retirement at the age of sixty-five, and that she expects to make 8 percent a year on her investments. The expected value of the account on her retirement date can be estimated as follows:

$$\text{Expected Value of IRA set - aside at 65} = \$2{,}000\left[\frac{(1.08)^{40} - 1}{.08}\right] = \$518{,}113$$

The tax exemption adds substantially to the value because it allows the investor to keep the pretax return of 8 percent made on the IRA investment. If the income had been taxed at, say, 40 percent, the after-tax return would have dropped to 4.8 percent, resulting in a much lower expected value:

$$\text{Expected Value of IRA set - aside at 65 if taxed} = \$2{,}000\left[\frac{(1.048)^{40} - 1}{.048}\right] = \$230{,}127$$

As you can see, the available funds at retirement drops by more than 55% as a consequence of the loss of the tax exemption.

*IV. Annuity Given Future Value*

Individuals or businesses who have a fixed obligation to meet or a target to meet (in terms of savings) some time in the future need to know how much they should set aside each period to reach this target. If you are given the future value and are looking for an annuity—A(FV,*r*,*n*) in terms of notation:

$$\text{Annuity given Future Value} = A(FV,r,n) = FV\left[\frac{r}{(1+r)^n - 1}\right]$$

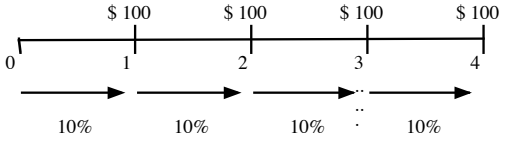
In any *balloon payment loan*, only interest payments are made during the life of the loan, and the principal is paid at the end of the period. Companies that borrow money using balloon payment loans or conventional bonds (which share the same features) often set aside money in sinking funds during the life of the loan to ensure that they have enough at maturity to pay the principal on the loan or the face value of the bonds. Thus, a company with bonds with a face value of $100 million coming due in ten years would need to set aside the following amount each year (assuming an interest rate of 8 percent):

$$\text{Sinking Fund Provision each year} = \$100{,}000{,}000\left[\frac{.08}{(1.08)^{10} - 1}\right] = \$6{,}902{,}950$$

The company would need to set aside $6.9 million at the end of each year to ensure that there are enough funds ($10 million) to retire the bonds at maturity.

*V. Effect of Annuities at the Beginning of Each Year*

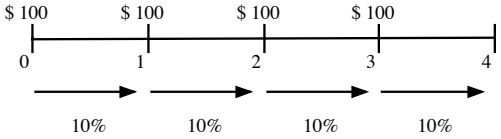
The annuities considered thus far in this appendix are end-of-the-period cash flows. Both the present and future values will be affected if the cash flows occur at the beginning of each period instead of the end. To illustrate this effect, consider an annuity of $100 at the end of each year for the next four years, with a discount rate of 10 percent.




Contrast this with an annuity of $100 at the beginning of each year for the next four years, with the same discount rate.








Because the first of these annuities occurs right now and the remaining cash flows take the form of an end-of-the-period annuity over three years, the present value of this annuity can be written as follows:

$$PV \text{ of \$100 at beginning of each of next 4 years} = \$100 + \$100\left[\frac{1 - \frac{1}{(1.10)^3}}{.10}\right]$$

In general, the present value of a beginning-of-the-period annuity over *n* years can be written as follows:

$$PV \text{ of Beginning of Period Annuities over n years} = A + A\left[\frac{1 - \frac{1}{(1+r)^{n-1}}}{r}\right]$$

This present value will be higher than the present value of an equivalent annuity at the end of each period.

The future value of a beginning-of-the-period annuity typically can be estimated by allowing for one additional period of compounding for each cash flow:

$$FV \text{ of a Beginning-of-the-Period Annuity} = A(1+r)\left[\frac{(1+r)^n - 1}{r}\right]$$

This future value will be higher than the future value of an equivalent annuity at the end of each period.

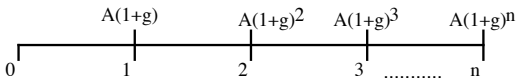
Consider again the example of an individual who sets aside $2,000 at the end of each year for the next forty years in an IRA account at 8 percent. The future value of these deposits amounted to $518,113 at the end of year forty. If the deposits had been made at the beginning of each year instead of the end, the future value would have been higher:

$$\text{Expected Value of IRA (beginning of year)} = \$2{,}000\,(1.08)\left[\frac{(1.08)^{40} - 1}{.08}\right] = \$559{,}562$$

As you can see, the gains from making payments at the beginning of each period can be substantial.

**Growing Annuities**

A *growing annuity* is a cash flow that grows at a constant rate for a specified period of time. If A is the current cash flow, and *g* is the expected growth rate, the time line for a growing annuity appears as follows:




Note that to qualify as a growing annuity, the growth rate in each period has to be the same as the growth rate in the prior period.

In most cases, the present value of a growing annuity can be estimated by using the following formula:

$$PV \text{ of a Growing Annuity} = A(1+g)\left[\frac{1 - \frac{(1+g)^n}{(1+r)^n}}{r-g}\right]$$

The present value of a growing annuity can be estimated in all cases, but one—where the growth rate is equal to the discount rate. In that case, the present value is equal to the nominal sums of the annuities over the period, without the growth effect.

PV of a Growing Annuity for $n$ Years (when $r = g$) = $nA$

Note also that this formulation works even when the growth rate is greater than the discount rate.[2]

To illustrate a growing annuity, suppose you have the rights to a gold mine for the next twenty years, over which time you plan to extract 5,000 ounces of gold every year. The current price per ounce is $300, but it is expected to increase 3 percent a year. The appropriate discount rate is 10 percent. The present value of the gold that will be extracted from this mine can be estimated as follows:

[2]Both the denominator and the numerator in the formula will be negative, yielding a positive present value.

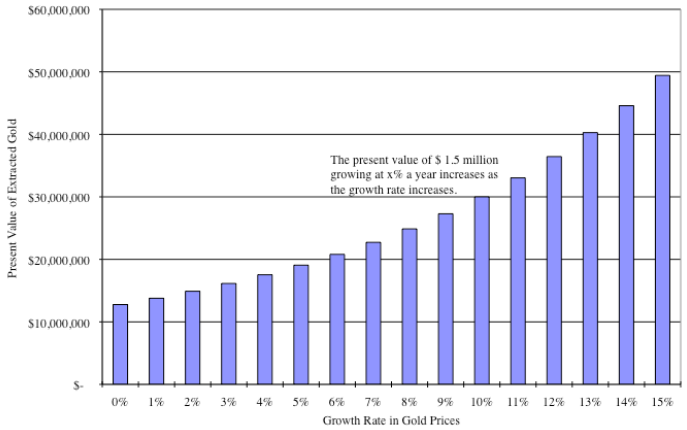




$$PV \text{ of extracted gold} = \$300 * 5000 * (1.03)\left[\frac{1 - \frac{(1.03)^{20}}{(1.10)^{20}}}{.10 - .03}\right] = \$16{,}145{,}980$$

The present value of the gold expected to be extracted from this mine is $16.146 million; it is an increasing function of the expected growth rate in gold prices. Figure A3.6 illustrates the present value as a function of the expected growth rate.

*Figure A3.6: Present Value of Extracted Gold as a function of Growth Rate*




**Perpetuities**

A *perpetuity* is a constant cash flow at regular intervals forever. The present value of a perpetuity can be written as

$$PV \text{ of Perpetuity} = \frac{A}{r}$$

where A is the perpetuity. The most common example offered for a perpetuity is a console bond. A console bond is a bond that has no maturity and pays a fixed coupon. Assume that you have a 6 percent coupon console bond. The value of this bond, if the interest rate is 9 percent, is as follows:

Value of Console Bond = $60/0.09 = $667

The value of a console bond will be equal to its face value (which is usually $1,000) only if the coupon rate is equal to the interest rate.

**Growing Perpetuities**

A *growing perpetuity* is a cash flow that is expected to grow at a constant rate forever. The present value of a growing perpetuity can be written as:

$$PV \text{ of Growing Perpetuity} = \frac{CF_1}{(r - g)}$$

where $CF_1$ is the expected cash flow next year, $g$ is the constant growth rate, and $r$ is the discount rate. Although a growing perpetuity and a growing annuity share several features, the fact that a growing perpetuity lasts forever puts constraints on the growth rate. It has to be less than the discount rate for this formula to work.

Growing perpetuities are especially useful when valuing equity in publicly traded firms, because they could potentially have perpetual lives. Consider a simple example. In 1992, Southwestern Bell paid dividends per share of $2.73. Its earnings and dividends had grown at 6 percent a year between 1988 and 1992 and were expected to grow at the same rate in the long run. The rate of return required by investors on stocks of equivalent risk was 12.23 percent. With these inputs, we can value the stock using a perpetual growth model:

Value of Stock = $2.73 * 1.06/(0.1223 – 0.06) = $46.45

As an aside, the stock was actually trading at $70 per share. This price could be justified by using a higher growth rate. The value of the stock is graphed in Figure A3.7 as a function of the expected growth rate.

*Figure A3.7 Southwestern Bell: Value versus Expected Growth*

The growth rate would have to be approximately 8 percent to justify a price of $70. This growth rate is often referred to as an *implied growth rate*.

## Conclusion

Present value remains one of the simplest and most powerful techniques in finance, providing a wide range of applications in both personal and business decisions. Cash flow can be moved back to present value terms by discounting and moved forward by compounding. The discount rate at which the discounting and compounding are done





reflect three factors: (1) the preference for current consumption, (2) expected inflation, and (3) the uncertainty associated with the cash flows being discounted.

In this appendix, we explored approaches to estimating the present value of five types of cash flows: simple cash flows, annuities, growing annuities, perpetuities, and growing perpetuities.