Lecture 9-10: Portfolio Management - N Risky Assets and a Riskless Asset

I. Reading.
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I. Reading.
   A. BKM, Chapter 8, Sections 8.4 and 8.5 and Appendix 8.A.

II. Standard Deviation of Portfolio Return: N Risky Assets.
   1. Formula.

$$\sigma^2[R_p(t)] = \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_{i,p}\ \omega_{j,p}\ \sigma[R_i(t),\ R_j(t)]$$

where

$\sigma[R_i(t), R_j(t)]$ is the covariance of asset i's return and asset j's return in period t;
$\omega_{i,p}$ is the weight of asset i in the portfolio p;
$\sigma^2[R_p(t)]$ is the variance of return on portfolio p in period t.

   2. The formula says that $\sigma^2[R_p(t)]$ is equal to the sum of the elements in the following N x N matrix.

	1	2	... j ...	N-1	N
1	$\omega_{1,p}\ \omega_{1,p}$ $\sigma[R_1, R_1]$	$\omega_{1,p}\ \omega_{2,p}$ $\sigma[R_1, R_2]$	$\omega_{1,p}\ \omega_{j,p}$ $\sigma[R_1, R_j]$	$\omega_{1,p}\ \omega_{N-1,p}$ $\sigma[R_1, R_{N-1}]$	$\omega_{1,p}\ \omega_{N,p}$ $\sigma[R_1, R_N]$
2	$\omega_{2,p}\ \omega_{1,p}$ $\sigma[R_2, R_1]$	$\omega_{2,p}\ \omega_{2,p}$ $\sigma[R_2, R_2]$	$\omega_{2,p}\ \omega_{j,p}$ $\sigma[R_2, R_j]$	$\omega_{2,p}\ \omega_{N-1,p}$ $\sigma[R_2, R_{N-1}]$	$\omega_{2,p}\ \omega_{N,p}$ $\sigma[R_2, R_N]$
...i...	$\omega_{i,p}\ \omega_{1,p}$ $\sigma[R_i, R_1]$	$\omega_{i,p}\ \omega_{2,p}$ $\sigma[R_i, R_2]$	$\omega_{i,p}\ \omega_{j,p}$ $\sigma[R_i, R_j]$	$\omega_{i,p}\ \omega_{N-1,p}$ $\sigma[R_i, R_{N-1}]$	$\omega_{i,p}\ \omega_{N,p}$ $\sigma[R_i, R_N]$
N-1	$\omega_{N-1,p}\ \omega_{1,p}$ $\sigma[R_{N-1}, R_1]$	$\omega_{N-1,p}\ \omega_{2,p}$ $\sigma[R_{N-1}, R_2]$	$\omega_{N-1,p}\ \omega_{j,p}$ $\sigma[R_{N-1}, R_j]$	$\omega_{N-1,p}\ \omega_{N-1,p}$ $\sigma[R_{N-1},R_{N-1}]$	$\omega_{N-1,p}\ \omega_{N,p}$ $\sigma[R_{N-1}, R_N]$
N	$\omega_{N,p}\ \omega_{1,p}$ $\sigma[R_N, R_1]$	$\omega_{N,p}\ \omega_{2,p}$ $\sigma[R_N, R_2]$	$\omega_{N,p}\ \omega_{j,p}$ $\sigma[R_N, R_j]$	$\omega_{N,p}\ \omega_{N-1,p}$ $\sigma[R_N, R_{N-1}]$	$\omega_{N,p}\ \omega_{N,p}$ $\sigma[R_N, R_N]$

   a. Notice that there are N^2 terms.
   b. The diagonal elements are the variance terms since $\sigma^2[R_i(t)] = \sigma[R_i(t), R_i(t)]$; so there are N variance terms and (N-1)N covariance terms.

c. Notice that this formula specializes to the formula used above for the two asset case:

$$\sigma^2[R_p(t)] = \omega_{1,p}^2 \, \sigma[R_1(t)]^2 + \omega_{2,p}^2 \, \sigma[R_2(t)]^2 + 2 \, \omega_{1,p} \, \omega_{2,p} \, \sigma[R_1(t), \, R_2(t)]$$

	1	2
1	$\omega_{1,p} \; \omega_{1,p}$ $\sigma[R_1, R_1]$	$\omega_{1,p} \; \omega_{2,p}$ $\sigma[R_1, R_2]$
2	$\omega_{2,p} \; \omega_{1,p}$ $\sigma[R_2, R_1]$	$\omega_{2,p} \; \omega_{2,p}$ $\sigma[R_2, R_2]$

III. Effect of Diversification.

A. Consider an equal weighted portfolio (So $\omega_{i,p}$= 1/N for all i.). For example, when N=2, an equal weighted portfolio has 50% in each asset.

B. Suppose all assets have the same $E[R] = \overline{R}$ and $\sigma[R] = \sigma$ and have returns which are uncorrelated. Then, for the equal weighted portfolio:

1. N=2:

$E[R_p(t)] = ½\ E[R_1(t)] + ½\ E[R_2(t)] = \overline{R}$.

$\sigma[R_p(t)]^2 = (½)^2\ \sigma[R_1(t)]^2 + (½)^2\ \sigma[R_2(t)]^2 = ½\ \sigma^2$.

2. N=3:

$E[R_p(t)] = ⅓\ E[R_1(t)] + ⅓\ E[R_2(t)] + ⅓\ E[R_3(t)] = \overline{R}$.

$\sigma[R_p(t)]^2 = (⅓)^2\ \sigma[R_1(t)]^2 + (⅓)^2\ \sigma[R_2(t)]^2 + (⅓)^2\ \sigma[R_3(t)]^2 = ⅓\ \sigma^2$.

3. Arbitrary N:

$E[R_p(t)] = \overline{R}$.

$\sigma[R_p(t)]^2 = \sigma^2/N$.

4. As N increases:
   a. the variance of the portfolio declines to zero.
   b. the portfolio's expected return is unaffected.
5. This is known as the effect of diversification.

C. Suppose all assets have the same $\sigma[R] = \sigma$ and have returns which are correlated.

1. Formulas for expected return and standard deviation of return for the equal weighted portfolio can be written:

$E[R_p(t)]$ = average expected return

$$\sigma^2[R_p(t)] = \sigma^2 \left[\frac{1}{N}\, 1 + \left(1 - \frac{1}{N}\right) \text{average correlation}\right]$$

where

$$\text{average expected return} = \frac{1}{N} \sum_{i=1}^{N} E[R_i(t)]$$

$$\text{average correlation} = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=1,\, j \neq i}^{N} \rho[R_i(t),\, R_j(t)].$$

2. As N increases:
   a. Expected portfolio return is unaffected.
   b. Variance of portfolio return:
      (1) Expressed as a fraction of firm variance, portfolio variance converges to the average pairwise correlation between assets.
3. Shows the benefit of diversification depends on the correlation between the assets.
4. Can see that assets with low correlation maximize the diversification benefits.

D. Suppose assets have non-zero covariances and differing expected returns and standard deviations.

1. Formulas for expected portfolio return and standard deviation can be written:

$E[R_p(t)]$ = average expected return

$\sigma^2[R_p(t)] = \frac{1}{N}$ average variance + $(1 - \frac{1}{N})$ average covariance

where

$$\text{average expected return} = \frac{1}{N}\sum_{i=1}^{N} E[R_i(t)]$$

$$\text{average variance} = \frac{1}{N}\sum_{i=1}^{N} \sigma[R_i(t)]^2$$

$$\text{average covariance} = \frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} \sigma[R_i(t),\ R_j(t)].$$

2. As N increases:
   a. Expected portfolio return is unaffected.
   b. Variance of portfolio return:
      (1) First term (the unique/ firm specific/ diversifiable/ unsystematic risk) goes to zero.
      (2) Second term (the market/ systematic/ undiversifiable risk) remains.
         (a) When the assets are uncorrelated (the case above in III.B), this second term is zero.

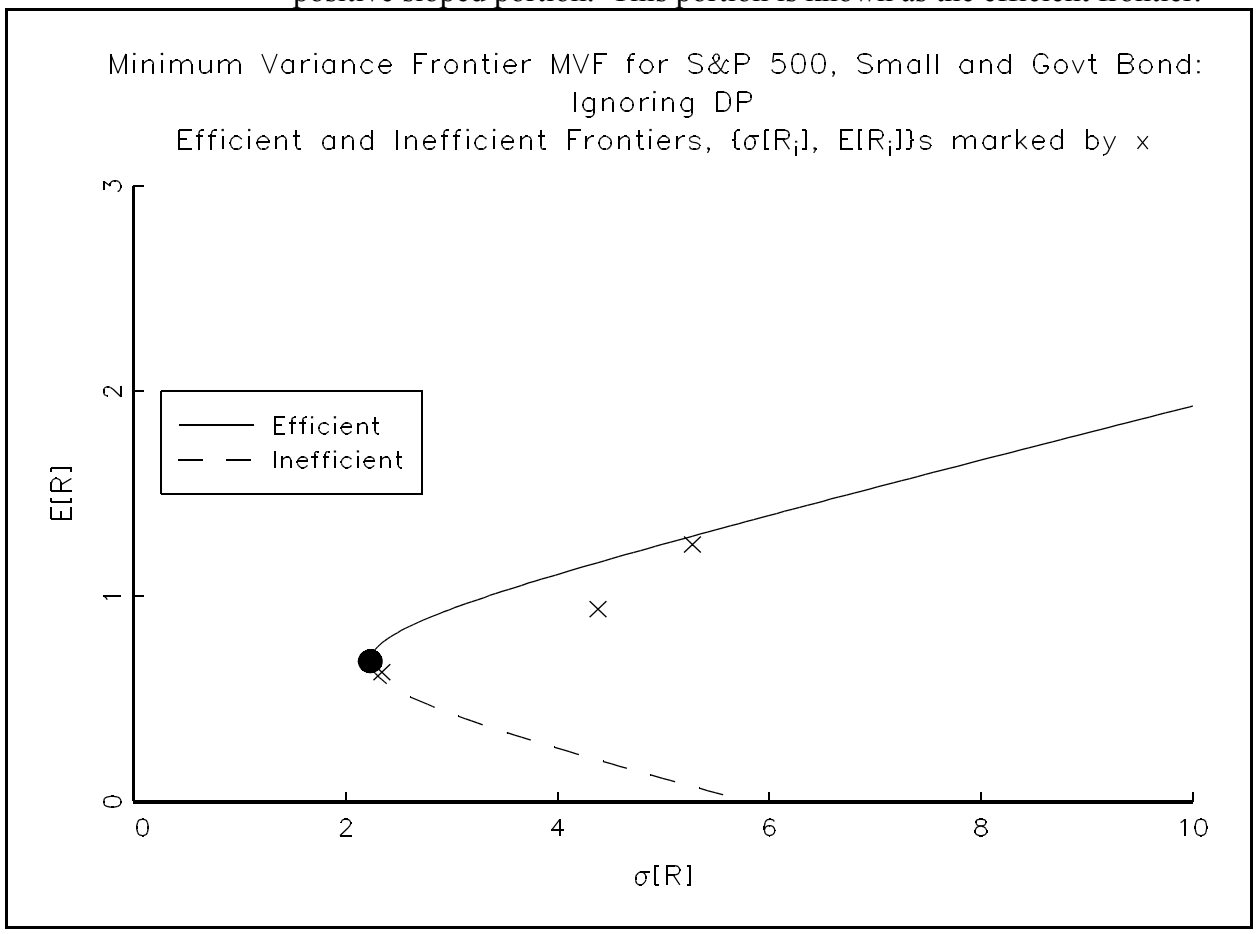
IV. Opportunity Set: N Risky Assets.

A. Set of Possible Portfolios.

1. No longer a curve as in the two asset case.
2. Instead, a set of curves.

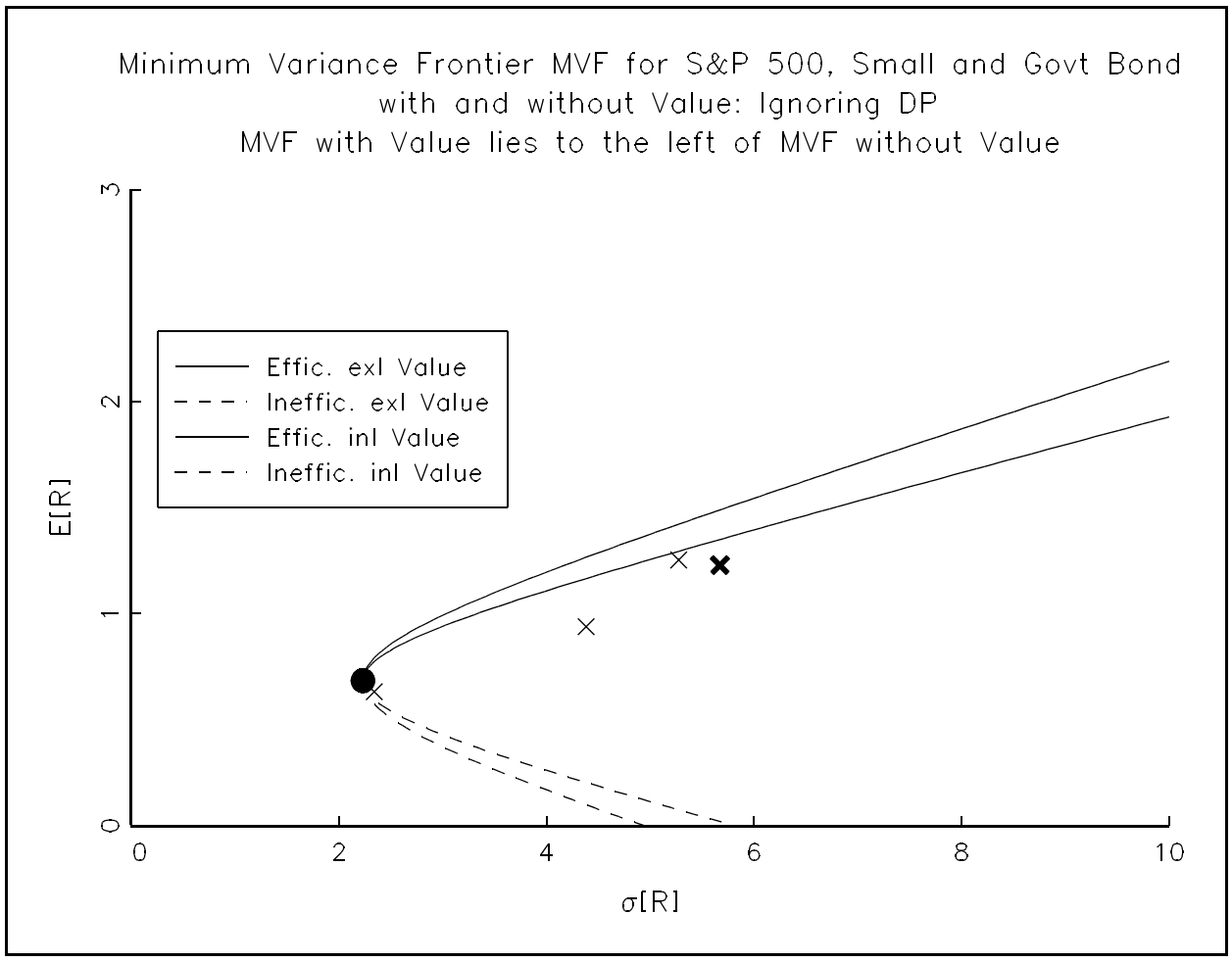
B. Minimum Variance Frontier.

1. Since individuals are risk averse, can restrict attention to the set of portfolios with the lowest variance for a given expected return.
2. This curve is known as the minimum variance frontier (MVF) for the risky assets.
3. Every other possible portfolio is dominated by a portfolio on the MVF (lower variance of return for the same expected return).
4. Example 2 (cont): Ignoring DP. The basic shape of the MVF is the same as that for the MVF for three of funds in this example (S&P 500, small firm fund and govt bond fund) which is graphed below.
5. Further, risk averse individuals would never hold a portfolio on the negative sloped portion of the MVF; so can restrict attention to the positive sloped portion. This portion is known as the efficient frontier.

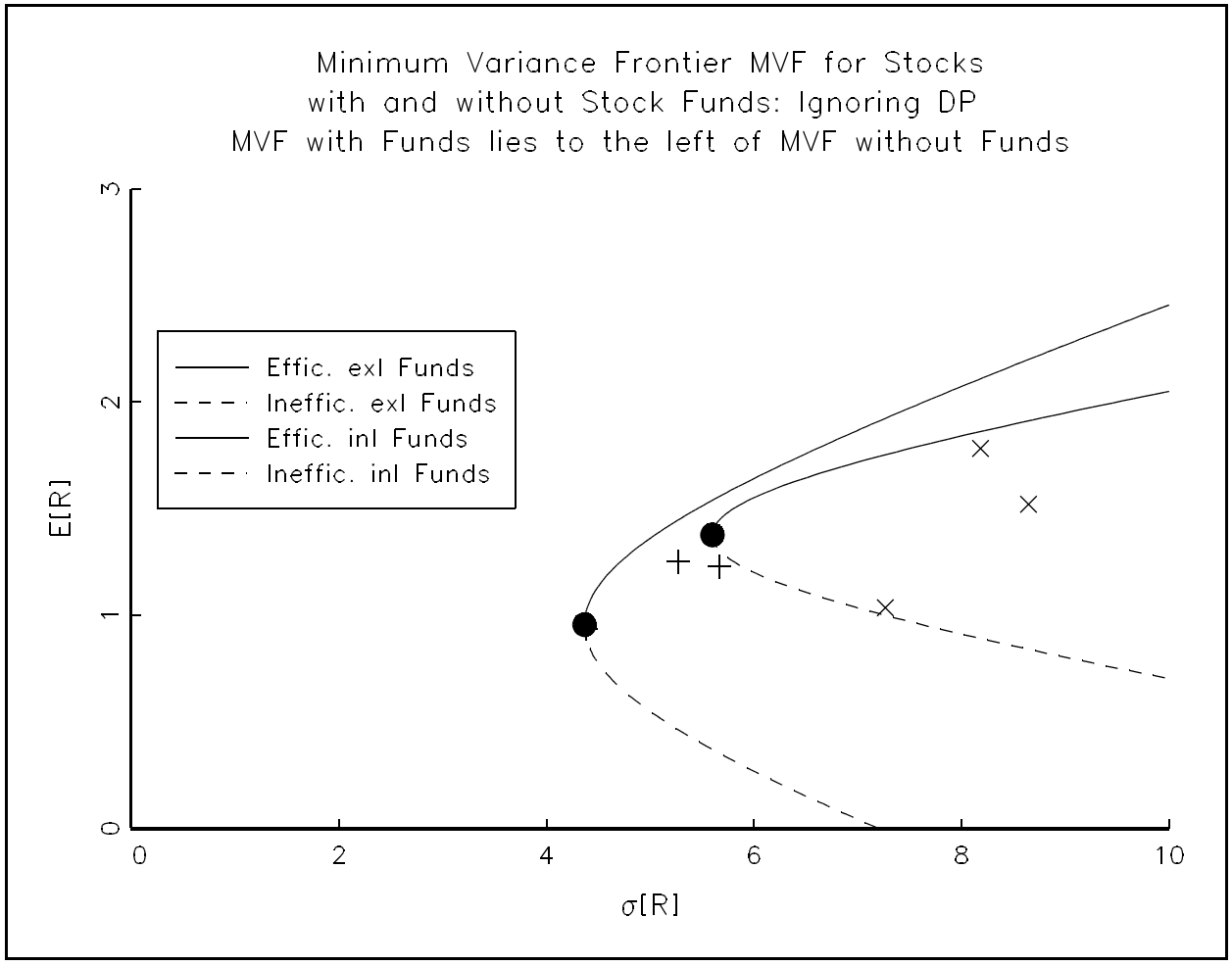


C. Adding risky assets.

1. Adding risky assets to the opportunity set always causes the minimum variance frontier to shift to the left in {σ[R],E[R]} space. Why?
   a. For any given E[R], the portfolio on the MVF for the subset of risky assets is still feasible using the larger set of risky assets.
   b. Further, there may be another portfolio which can be formed from the larger set and which has the same E[R] but an even lower σ[R].
2. Example 2 (cont): Ignoring DP. MVF for the S&P 500, the small firm fund, the value firm fund and the govt bond fund is to the left of the MVF for the S&P 500, the small firm fund and the govt bond fund excluding the value firm fund . This happens even though the value firm fund has an {σ[R],E[R]) denoted by ✖ which lies to the right of the MVF for the 3 funds excluding the value firm fund.



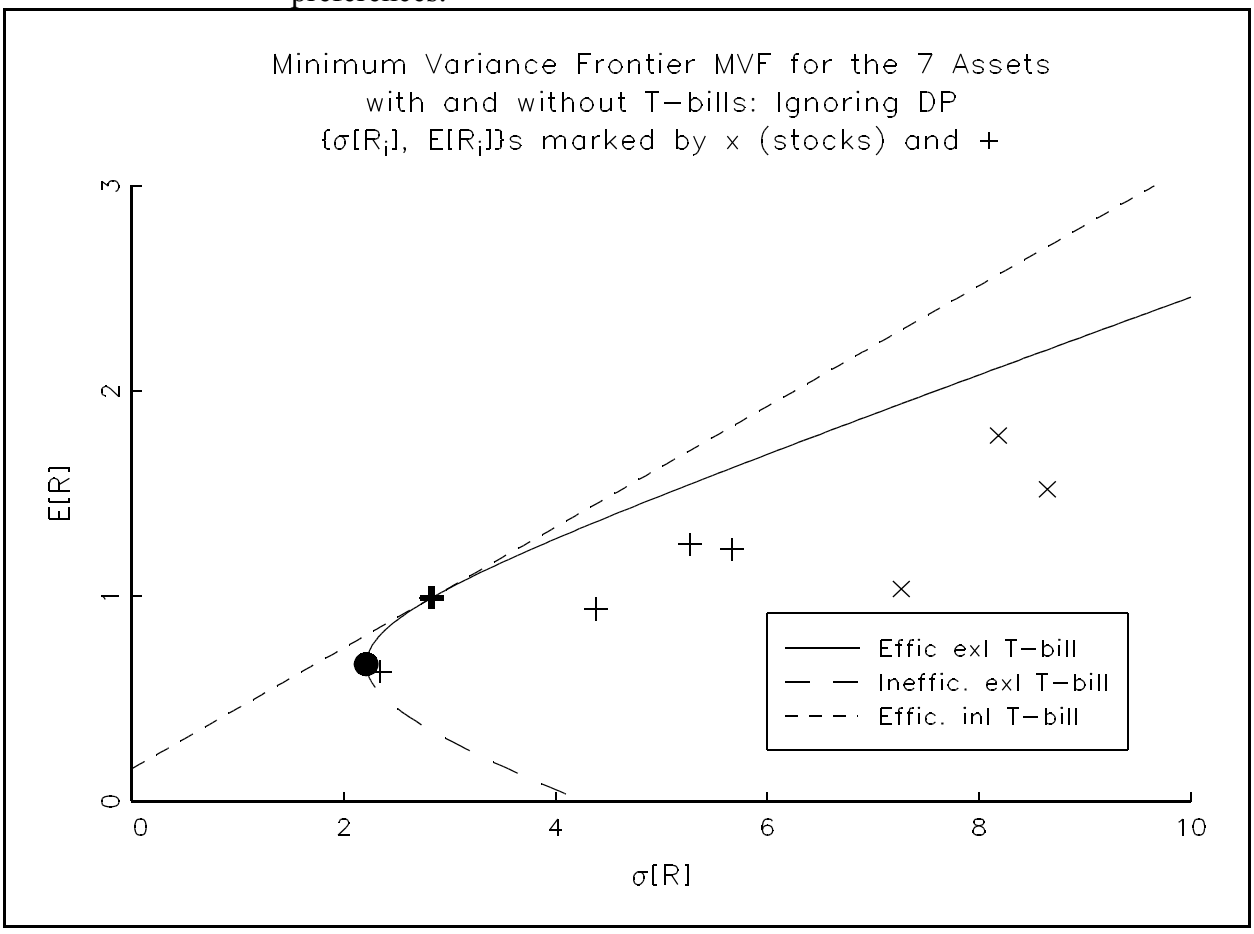
3. Example 2 (cont): Ignoring DP. MVF for all 6 stock assets (including the 3 stock funds or portfolios: S&P 500 fund, small firm fund, and value firm fund) is to the left of the MVF for the 3 individual stocks (ADM, IBM, WAG).



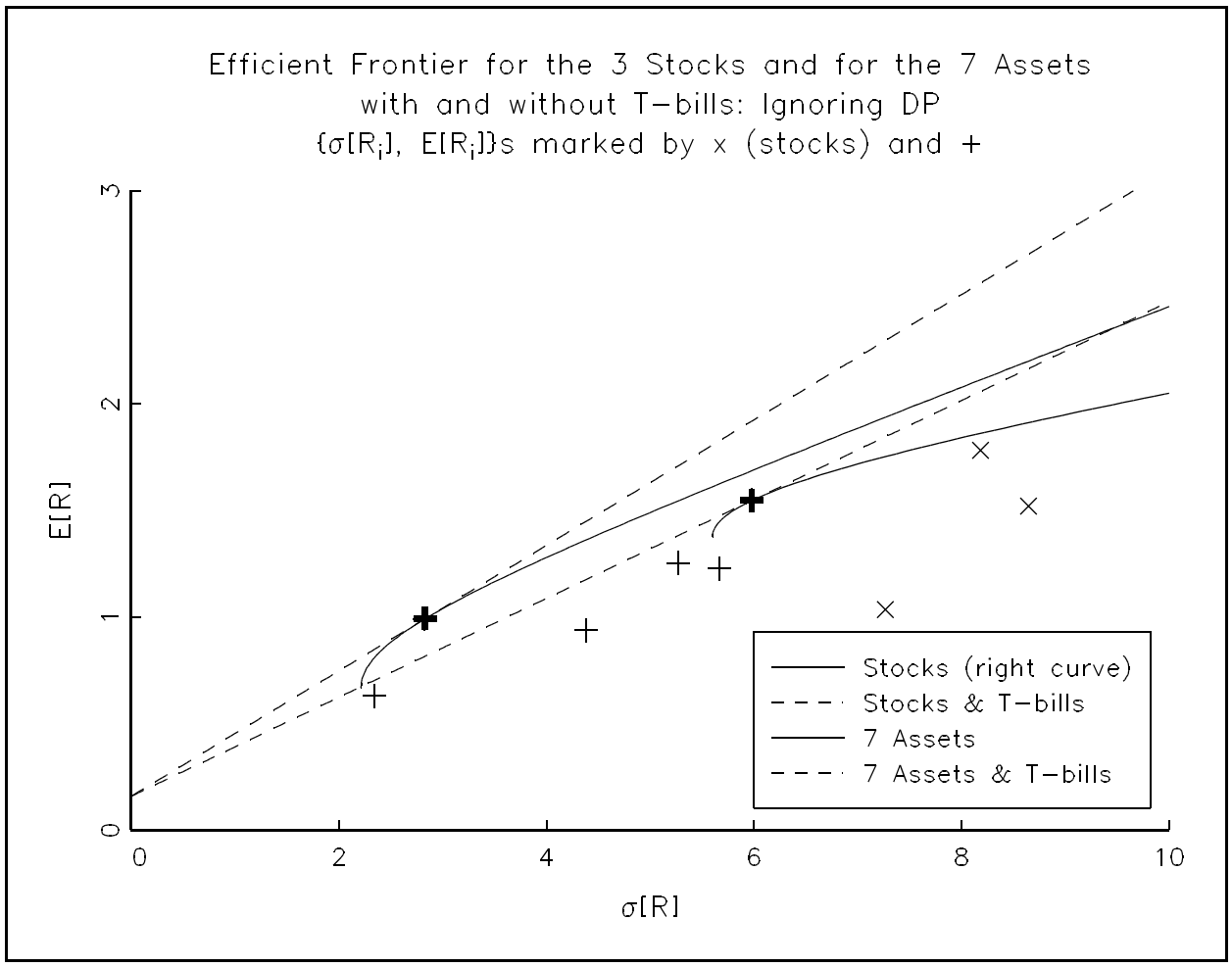
V. Portfolio Choice: N Risky Assets and a Riskless Asset

A. The analysis for the two risky asset and a riskless asset case applies here.

1. Any risk averse individual combines the riskless asset with the risky portfolio whose Capital Allocation Line has the highest slope.
2. That risky portfolio is on the efficient frontier for the N risky assets and is known as the tangency portfolio (✚): calculating the weights of assets in the tangency portfolio can be performed via computer.
3. All risk averse individuals want to hold this tangency portfolio in combination with the riskless asset. The associated Capital Allocation Line is the efficient frontier for the N risky assets and the riskless asset.
4. Only the weights of the tangency portfolio and the riskless asset in an individual's portfolio depend on the individual's tastes and preferences.
5. Example 2 (cont): Ignoring DP. If individuals can form a risky portfolio from the 7 assets and combine that risky portfolio with T-bills, then all individuals will hold ✚ as their risky portfolio. The weights of ✚ and T-bills in an individual's portfolio will depend on that individual's tastes and preferences.

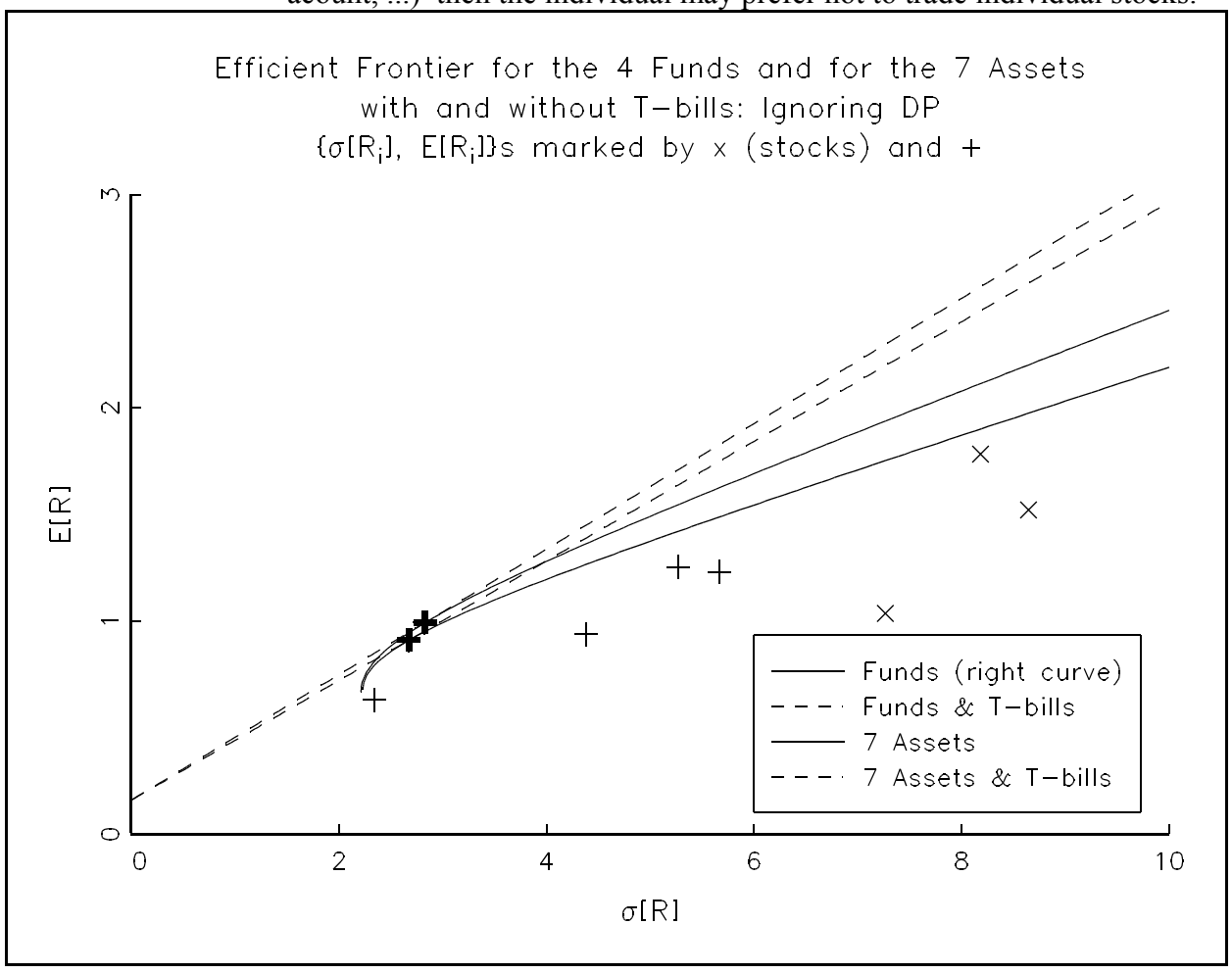


B. Adding risky assets to the set of available risky assets:
   1. shifts the MVF for the risky assets to the left.
   2. allows investors access to a CAL with a higher slope.
   3. increases the utility of any individuals (in the absence of transaction costs).
   4. Example 2 (cont): Ignoring DP. The slope of the CAL available using all 7 risky assets is higher than that for the CAL available using only the 3 individual stocks.



C. Transaction Costs.

1. When the transaction costs associated with forming portfolios increase with the number of assets in the portfolio, there may be some optimal number of assets to have in the portfolio.
2. In this case, assets are added to the portfolio until the benefits from adding one more asset are offset by the associated increase in transactions costs.
3. Example 2 (cont): Ignoring DP. If an investor has used the 4 funds and T-bills to form a portfolio, the benefit from adding the 3 individual stocks appears small (see the graph below). If the investor faces significant fixed costs to start trading individual stocks (find a broker, open a brokerage acount, ...) then the individual may prefer not to trade individual stocks.



Unwise Wisdom:
A 20-stock portfolio gives you all the diversity you need
By Lynn Cowan

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In the 1960s, 30 cents bought a gallon of gas, 20 cents got you a loaf of bread, and a 20-stock portfolio promised shelter from market risk.

Well, the prices have long since risen, but the faith in the 20-stock portfolio hasn't changed much. Some analysts say it should. Americans need a much more diverse investment mix to keep ahead of surging market volatility, they argue. For people without loads of cash, that means mutual funds; but for those able to dabble in individual stocks, it means buying a slew of additional shares. How many more? That's a subject of some debate among money managers and academics.

For instance, a study scheduled for publication in the Journal of Finance finds that investors should hold 50 stocks to match the risk posed by 20 stocks four decades ago. And a University of Nevada study suggests that investors need at least 100 stocks to keep within 5% of average portfolio risk.

The Journal of Finance study, due next month, finds that individual stock volatility more than doubled from 1962 to 1997, increasing the need for diversification . One reason for the change may be the increased market participation of institutional investors, such as pension plans and mutual funds, says Yexiao Xu, one of the study's authors. Such investors tend to react to the same information at the same time, leading to more ups and downs among individual stocks, says Mr. Xu, an assistant professor at the University of Texas school of management.

Although owning 50 stocks may strike the average investor as too many to track successfully, it is chicken feed to two professors from the University of Nevada at Las Vegas. Gerald Newbould and Percy Poon posited in a 1996 study that investors needed to hold more than 100 small-cap or large-cap stocks to remain within 5% of average risk, which they define as the average volatility of the 40,000 simulated portfolios created for the study.

The pair also tracked the returns of their pretend portfolios from 1987 to 1993 before arriving at their conclusions. The scorecard: An investor holding 25 large-cap stocks would end up with a portfolio that could range as far as 20% from an average portfolio's risk. For small-cap stocks, that 20% risk level is the result of holding 40 stocks, since a small-cap diversification of 25 stocks was considered too small to chart. Mr. Poon says the survey's findings are still valid today.

A portfolio with a range of 50 to 100 stocks sounds about right to John Duffy, the head of J.P. Morgan Chase & Co.'s Western region private-banking business, based in Los Angeles. If an investor sticks to U.S. equities, then 50 to 75 stocks will yield a decent level of diversification , he says.

"A lot of clients say, `But I really understand 20 to 25 names.' But that creates a substantial variability in their returns -- it's too small," says Mr. Duffy. A money manager, he adds, should play the dominant role in managing a portfolio.

Mr. Xu at the University of Texas also says his study's findings are at odds with prevailing investing habits. "This is contrary to what a lot of people are doing nowadays," he says.

Investor concerns about tracking so many stocks may be one reason that people stick to fewer shares. But Messrs. Poon and Newbould also found that since 1989, more than a dozen publications have either recommended that investors hold 20 or fewer stocks, or have cited experts touting such a strategy. Among the culprits: textbooks, professional journals and The Wall Street Journal.

What's more, some professional money managers defend the use of the magic number of 20. "You can [diversify] easily with fewer than 20 companies," says Michael Holland, manager of the Holland Balanced Fund in New York. "Where do I get that from? Empirical observation and scars," he says. "I have been in positions where I held a huge number of stocks and it was literally impossible to outperform the market on a long-term basis."

The Holland Balanced Fund fell 0.2% last year, while Standard & Poor's 500-stock index dropped 10.1%. The fund generally holds fewer than 20 stocks, a formula Mr. Holland credits for its performance.

Shapiro Capital Management in Atlanta, which manages $1 billion in assets for individuals and institutions, limits its portfolio to 25 stocks.

"It is awfully hard to pick 20 good stocks that are going to work, much less 30 or 40," says Sam Shapiro, president and chief investment officer. He maintains that a smaller portfolio is less risky for individuals as well as professionals because it is easier to follow each stock more closely through in-depth research.

How many stocks do these academics who are studying the ideal portfolio hold in their own accounts? Mr. Poon at the University of Nevada is invested in mutual funds, and holds few individual stocks. "If I buy stocks, it's just like play money," he says. "If I lose it, I don't care."

Mr. Xu at the University of Texas concentrates his core holdings in index funds and keeps a tiny slice in about 20 individual stocks. He doesn't intend to double his stock holdings to his study's recommended level. "If you're willing to bear more risk," then holding 20 stocks is fine, he says. "Because most of my money is in index funds, I'm OK. I just set aside a small portion of money for my own personal interest."

Ms. Cowan is a reporter for Dow Jones Newswires in Washington.

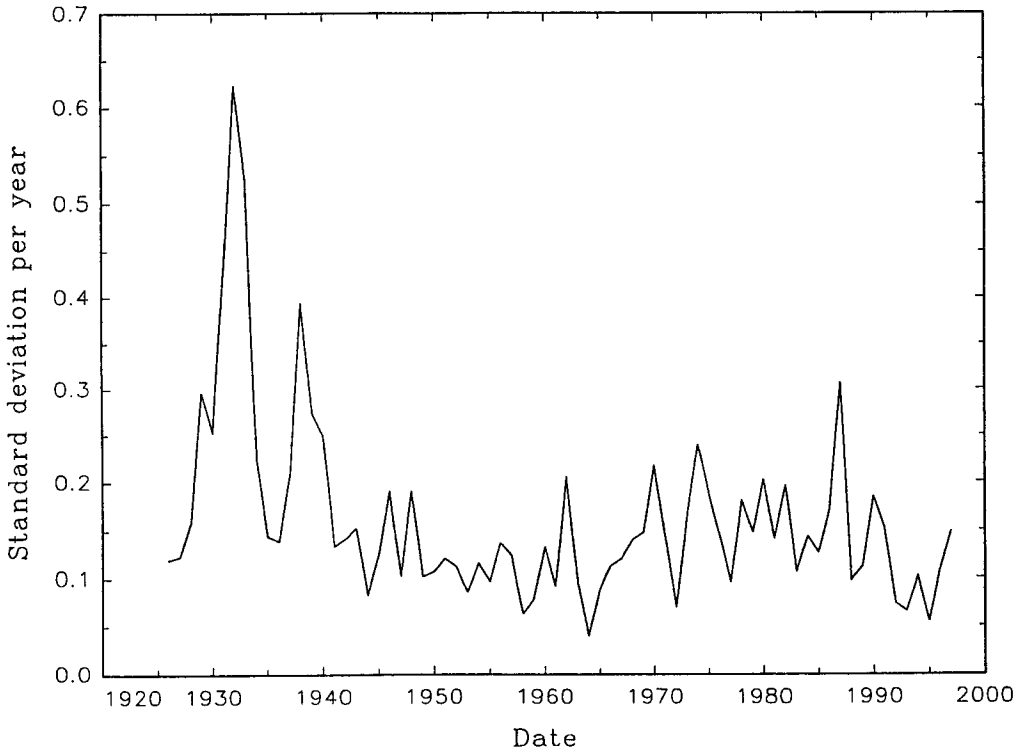


Figure 1. Standard deviation of value-weighted stock index. The standard deviation of monthly returns within each year is shown for the period from 1926 to 1997.

other possibility is that public impressions are formed in part by the behavior of individual stocks rather than the market as a whole. Casual empiricism does suggest increasing volatility for individual stocks. On any specific day, the most volatile individual stocks move by extremely large percentages, often 25 percent or more. The question remains whether such impressions from casual empiricism can be documented rigorously and, if so, whether these patterns of volatility for individual stocks are different from those existing in earlier periods. With this motivation, we now present a graphical summary of the three volatility components described in the previous section.

Figures 2 to 4 plot the three variance components, estimated monthly, using daily data over the period from 1962 to 1997: market volatility MKT, industry-level volatility IND, and firm-level volatility FIRM. All three series are annualized (multiplied by 12). The top panels show the raw monthly time series and the bottom panels plot a lagged moving average of order 12. Note that the vertical scales differ in each figure and cannot be compared with Figure 1 (because we are now plotting variances rather than a standard deviation).

Market volatility shows the well-known patterns that have been studied in countless papers on the time variation of index return variances. Comparing the monthly series with the smoothed version in the bottom panel suggests that market volatility has a slow-moving component along with a

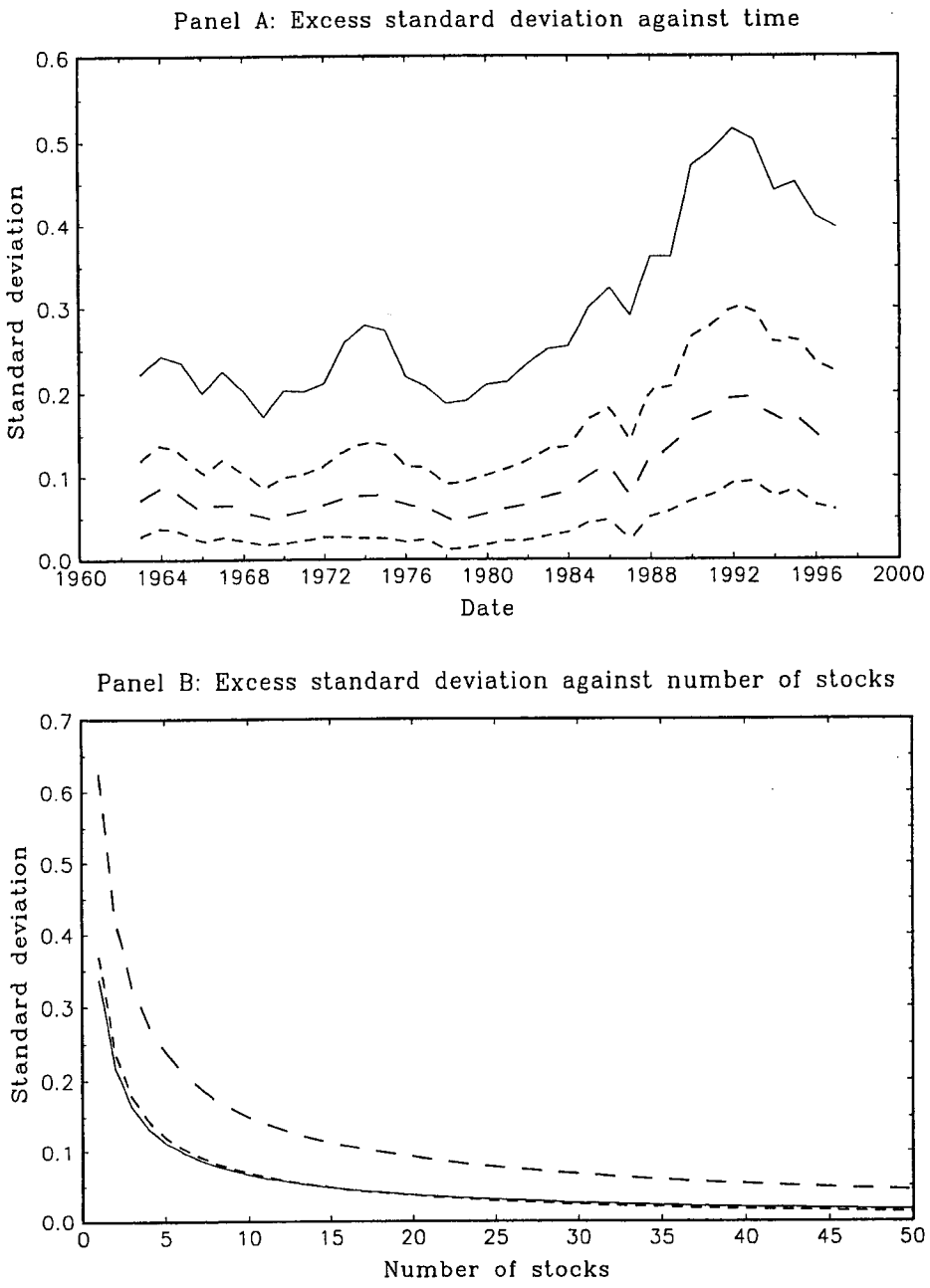


Figure 6. Excess standard deviation against time and number of stocks. The excess standard deviation of a portfolio is the difference between the portfolio's standard deviation and the standard deviation of an equally weighted index. The top panel plots annualized excess standard deviation against time. Excess standard deviation is calculated each year from daily data within the year, for randomly selected portfolios containing two stocks (solid line), five stocks (top dashed line), 20 stocks (long dashed line), and 50 stocks (bottom dashed line). The bottom panel plots annualized excess standard deviation against the number of stocks in the portfolio, for sample periods 1963 to 1973 (solid line), 1974 to 1985 (bottom dashed line), and 1986 to 1997 (top dashed line).