Financial Theory I

Anthony W. Lynch

2 Jan 2011

Contents

List of Figures

CHAPTER 1

Introduction

1.1. Conditioning information: Definitions

Let $\mathcal{F}_t$ be the set of information available to investors at t, let $\mathcal{F}_t^0$ be the set of unconditional information so $\mathcal{F}_t^0 \subset \mathcal{F}_t$, and let $\boldsymbol{r}_{t+1}$ be an $N \times 1$ set of basis returns.

Define

$$X_{t+1} = \{\boldsymbol{\omega}_t' \boldsymbol{r}_{t+1} \,|\, \boldsymbol{\omega}_t \in \mathcal{F}_t\}$$

as the set of payoffs spanned by $\boldsymbol{r}_{t+1}$ using $\mathcal{F}_t$ and

$$X_{t+1}^0 = \{\boldsymbol{\omega}' \boldsymbol{r}_{t+1} \,|\, \boldsymbol{\omega} \in \mathcal{F}_t^0\}$$

as the set of payoffs spanned by $\boldsymbol{r}_{t+1}$ using $\mathcal{F}_t^0$. Notice that $X_{t+1}^0 \subset X_{t+1}$.

X_{t+1} includes managed portfolios, i.e. portfolios whose weights in the N assets depend on the information available at t, so X_{t+1} is *not* spanned by $\boldsymbol{r}_t$ using only $\mathcal{F}_t^0$.

We use different expectations operators, conditional expectations $\mathrm{E}[\cdot \,|\, \mathcal{F}_t] \equiv \mathrm{E}_t[\cdot]$ and unconditional expectations $\mathrm{E}[\cdot \,|\, \mathcal{F}_t^0] \equiv \mathrm{E}[\cdot]$.

1.2. Understanding conditioning information

Example 1. Suppose $\mathcal{F}_t$ is generated by d_t. Then

$$X_{t+1} = \{\boldsymbol{\omega}_t' \boldsymbol{r}_{t+1} \,|\, \boldsymbol{\omega}_t = f(d_t),\ f(\cdot) \text{ a well behaved function} \to \mathbb{R}^N\},$$

so given d_t, we have $\boldsymbol{\omega}_t \in \mathbb{R}^N$ for each element of X_{t+1}.

We can think of $\mathcal{F}_t^0$ as being generated by a constant,

$$X_{t+1}^0 = \{\boldsymbol{\omega}' \boldsymbol{r}_{t+1} \,|\, \omega = f,\ f \in \mathbb{R}^N\}.$$

1.3. Understanding expectations operators

Conditional expectation is defined as

$$\mathrm{E}_t[y_{t+1}] = \underset{\mu_t^y \in \mathcal{F}_t, \mathrm{E}[(\mu_t^y)^2] < \infty}{\arg\min} \mathrm{E}[(y_{t+1} - \mu_t^y)^2 \,|\, \mathcal{F}_t^0],$$

so $\mathrm{E}_t[y_{t+1}] \in \mathcal{F}_t$. $\mathrm{E}_t[y_{t+1}]$ is the best guess of y_{t+1} given $\mathcal{F}_t$.

Unconditional expectation satisfies

$$\mathrm{E}[y_{t+1}] = \underset{\mu^y \in \mathbb{R}, \mathrm{E}[(\mu^y)^2] < \infty}{\arg\min} \mathrm{E}[(y_{t+1} - \mu^y)^2 \,|\, \mathcal{F}_t^0],$$

so $\mathrm{E}[y_{t+1}] \in \mathbb{R}$. $\mathrm{E}[y_{t+1}]$ is the best guess of y_{t+1} given $\mathcal{F}_t^0$.

Example 1 continued. Suppose $\mathcal{F}_t$ is generated by d_t. Then

$$\mathrm{E}_t[y_{t+1}] = \mu_t^y(d_t)$$

and $\mu_t^y(\cdot)$ is a well behaved function that maps into $\mathbb{R}$.

Example 2. In addition to example 1, suppose that y_{t+1} and d_t are multivariate normal and $\mathrm{E}[d_t] = 0$. Then

$$\mathrm{E}_t[y_{t+1}] = \mu^y + b d_t,$$

where

$$b = \frac{\mathrm{cov}[d_t, y_{t+1}]}{\mathrm{var}[d_t]}, \quad \mathrm{E}[y_{t+1}] = \mu^y.$$

1.4. Two issues

(1) What is the payoff space?
(2) What is the expectations operator?

We can calculate

- unconditional expectations for X_{t+1},
- conditional expectations for X_{t+1},
- unconditional expectations for X_{t+1}^0, and
- conditional expectations for X_{t+1}^0.

1.5. Useful results

Law of iterated expectations. If $\mathcal{G}_t \subset \mathcal{F}_t$, then

$$\mathrm{E}[y_{t+1} \,|\, \mathcal{G}_t] = \mathrm{E}[\mathrm{E}[y_{t+1} \,|\, F_t] \,|\, \mathcal{G}_t].$$

Example 2 continued. $\mathcal{F}_t^0 \subset \mathcal{F}_t$, so

$$\begin{aligned} \mathrm{E}_t[y_{t+1}] &= \mu^y + b d_t, \\ \mathrm{E}[y_{t+1}] &= \mu^y, \\ \mathrm{E}[\mathrm{E}_t[y_{t+1}]] &= \mathrm{E}[\mu^y + b d_t] = \mu^y + b\,\mathrm{E}[d_t] = \mu^y. \end{aligned}$$

Note. The converse does *not* hold:

$$\mu^y = \mathrm{E}_t[\mathrm{E}[y_{t+1}]] \neq \mathrm{E}_t[y_{t+1}] = \mu^y + b d_t.$$

Linearity. If $a_t, b_t \in \mathcal{F}_t$, then

$$\mathrm{E}[a_t + b_t y_{t+1} \,|\, \mathcal{F}_t] = a_t + b_t\,\mathrm{E}[y_{t+1} \,|\, \mathcal{F}_t].$$

Conditionally mean zero condition. If $\mathrm{E}[y_{t+1} z_t] = 0$ for all $z_t \in \mathcal{F}_t$, then $\mathrm{E}[y_{t+1} \,|\, \mathcal{F}_t] = 0$.

Example 1 continued. $\mathcal{F}_t$ is generated by d_t. If $\mathrm{E}[y_{t+1} f(d_t)] = 0$ for all well behaved functions $f(\cdot)$, then $\mathrm{E}[y_{t+1} \,|\, \mathcal{F}_t] = 0$.

Conditional covariance. Since $\mathcal{G}_t \subset \mathcal{F}_t$,

$$\begin{aligned}
&\operatorname{cov}[x_{t+1}, y_{t+1} \,|\, \mathcal{G}_t] \\
&\qquad = \mathrm{E}[\operatorname{cov}_t[x_{t+1}, y_{t+1}] \,|\, \mathcal{G}_t] + \operatorname{cov}[\mathrm{E}_t[x_{t+1}], \mathrm{E}_t[y_{t+1}] \,|\, \mathcal{G}_t].
\end{aligned}$$

PROOF FOR $\mathcal{G}_t = \mathcal{F}_t^0$.

$$\begin{aligned}
\operatorname{cov}[x_{t+1}, y_{t+1}] &= \mathrm{E}[x_{t+1} y_{t+1}] - \mathrm{E}[x_{t+1}]\,\mathrm{E}[y_{t+1}] \\
&= \mathrm{E}[\mathrm{E}_t[x_{t+1} y_{t+1}] - \mathrm{E}_t[x_{t+1}]\,\mathrm{E}_t[y_{t+1}]] \\
&\quad + \mathrm{E}[\mathrm{E}_t[x_{t+1}]\,\mathrm{E}_t[y_{t+1}]] - \mathrm{E}[\mathrm{E}_t[x_{t+1}]]\,\mathrm{E}[\mathrm{E}_t[y_{t+1}]].
\end{aligned}$$

□

Projection and the Reisz representation theorem. Let

$$L^2_{t+1} = \{y_{t+1} : y_{t+1} \in \mathcal{F}_{t+1}, \mathrm{E}[y^2_{t+1} \,|\, \mathcal{F}_t] < \infty\}.$$

Theorem 1. *Suppose H_{t+1} is a closed linear subspace of L^2_{t+1}. For $y_{t+1} \in L^2_{t+1}$, the projection* $\operatorname{proj}_t(y_{t+1}|H_{t+1})$ *exists and satisfies*

$$\underset{h_{t+1} \in H_{t+1}}{\arg\min}\ \mathrm{E}_t[(y_{t+1} - h_{t+1})^2] = h^0_{t+1},$$

and $h^0_{t+1} \in H_{t+1}$ also satisfies

$$\mathrm{E}_t[(y_{t+1} - h^0_{t+1})h_{t+1}] = 0 \quad \textit{for all } h_{t+1} \in H_{t+1}.$$

Theorem 2 (Riesz representation theorem)**.** *Let H_{t+1} be a closed linear subspace of L^2_{t+1} and π_t a continuous linear functional on H_{t+1}. Then there exists a unique element*

$$h^{\pi_t}_{t+1} \in H_{t+1} \ni \mathrm{E}_t[h_{t+1} h^{\pi_t}_{t+1}] = \pi_t(h_{t+1}) \quad \textit{for all } h_{t+1} \in H_{t+1}.$$

CHAPTER 2

Law of One Price, No Arbitrage and Stochastic Discount Factors

2.1. Definitions

Payoffs. Let $\tilde{X}_{t+1}$ be the set of asset payoffs at time $t+1$.

Prices. Let $p_t(x_{t+1})$ be the time-t price associated with payoff $x_{t+1} \in \tilde{X}_{t+1}$, so $p_t(\cdot) \in \mathcal{F}_t$. p_t is not uniquely determined without further assumptions, so p_t is a correspondence and not necessarily a functional.

Definition (No arbitrage PNA)**.** A payoff space $\tilde{X}_{t+1}$ and a pricing correspondence $p_t(\cdot)$ have no arbitrage opportunities if for every payoff $x_{t+1} \in \tilde{X}_{t+1}$ such that $x_{t+1} \geq 0$ almost surely with respect to $\mathcal{F}_t$ and $(\mathrm{E}[(x_{t+1})^2 \mid \mathcal{F}_t])^{1/2} > 0$, it is true that $p > 0$ for all $p \in p_t(x_{t+1})$.

Definition (Law of one price LOOP)**.** The law of one price holds if $p_t(\cdot)$ is a linear functional so that if $x^1 = x^2$ for $x^1, x^2 \in \tilde{X}_{t+1}$, then $p_t(x^1) = p_t(x^2)$, and if $\omega^1, \omega^2 \in \mathcal{F}_t$, then

$$p_t(\omega^1 x^1 + \omega^2 x^2) = \omega^1 p_t(x^1) + \omega^2 p_t(x^2).$$

2.2. Setting

Let

$$L^2_{t+1} = \{y_{t+1} : y_{t+1} \in \mathcal{F}_{t+1}, \mathrm{E}[y^2_{t+1} \mid \mathcal{F}_t] < \infty\}.$$

We can show that L^2_{t+1} is a Hilbert space. L^2_{t+1} can be infinite dimensional.

Let $\tilde{X}_{t+1}$ be

$$X_{t+1} \equiv \left\{ x_{t+1} \mid x_{t+1} = \sum_{i \in I} \omega^i_t x^i_{t+1}, \omega^i_t \in \mathcal{F}_t \right\},$$

which is the linear span of x^i_{t+1} for $i \in I$ and $x^i_{t+1} \in L^2_{t+1}$. If I does not have a finite number of elements, we may need to take the closure of the span.

The closure of X_{t+1} is a closed linear subspace of L^2_{t+1}.

Let $p_t(\cdot)$ be the price correspondence for X_{t+1}. Again, if I does not have a finite number of elements, we may need to augment the set of prices in a natural way.

Suppose there exists $x^0_{t+1} \in X_{t+1}$ so that $x^0_{t+1} \geq 0$ and

$$\|x^0_{t+1}\|_t \equiv (\mathrm{E}[(x^0_{t+1})^2 \mid \mathcal{F}_t])^{1/2} > 0$$

that satisfies PNA.

2.3. Relation between PNA, LOOP and the existence stochastic discount factors

Definition (Stochastic discount factor)**.** $m_{t+1} \in L^2_{t+1}$ is a stochastic discount factor (or sdf) for $\tilde{X}_{t+1}$ with respect to $\hat{F}_t \subset F_t$ iff

$$\forall x_{t+1} \in \tilde{X}_{t+1} : \mathrm{E}[m_{t+1}x_{t+1} \mid \hat{F}_t] = \mathrm{E}[p_t(x_{t+1}) \mid \hat{F}_t].$$

The setting described in the previous section is assumed to hold for all the results in this section. First, we can describe the relation between PNA and LOOP.

Proposition 3. *PNA $\Rightarrow$ LOOP.*

PROOF FOR A FINITE NUMBER OF ASSETS. Let N be the number of assets, let $\underset{N\times 1}{\boldsymbol{p}_t}$ be the vector of prices, and let $\underset{N\times 1}{\boldsymbol{x}_{t+1}}$ be the payoffs. Given that PNA holds for x^0_{t+1}, we prove by contradiction by showing that no LOOP $\Rightarrow$ no PNA. Take $\hat{x}_{t+1} \in X_{t+1}$. Suppose $\hat{p}_t, \hat{p}^*_t \in p_t(\hat{x}_{t+1})$ and $\hat{p}_t < \hat{p}^*_t$. Then there exist $\hat{\boldsymbol{\omega}}, \hat{\boldsymbol{\omega}}^* \in \mathcal{F}_t$ where

$$\hat{x}_{t+1} = \hat{\boldsymbol{\omega}}'\boldsymbol{x}_{t+1} = (\hat{\boldsymbol{\omega}}^*)'\boldsymbol{x}_{t+1}, \quad \hat{\boldsymbol{\omega}}' p_t = \hat{p}_t, \quad (\hat{\boldsymbol{\omega}}^*)'\boldsymbol{p}_t = \hat{p}^*_t.$$

Since PNA holds for x^0_{t+1}, then for $\boldsymbol{\omega}^0$,

$$x^0_{t+1} = (\boldsymbol{\omega}^0)'\boldsymbol{x}_{t+1}, \quad p^0_t = (\boldsymbol{\omega}^0)'\boldsymbol{p}_t > 0.$$

Consider a portfolio with weight vector

$$\underbrace{\frac{\hat{p}^*_t - \hat{p}_t}{p^0_t}\boldsymbol{\omega}^0}_{\text{in } x^0_{t+1}} + \underbrace{\hat{\boldsymbol{\omega}}}_{\text{buy cheap } \hat{x}_{t+1}} - \underbrace{\hat{\boldsymbol{\omega}}^*}_{\text{sell expensive } \hat{x}_{t+1}}.$$

The cost of this portfolio is

$$\left(\frac{\hat{p}^*_t - \hat{p}_t}{p^0_t}\right)p^0_t + \hat{p}_t - \hat{p}^*_t = 0.$$

The portfolio's payoff is

$$\frac{\hat{p}^*_t - \hat{p}_t}{p^0_t}x^0_{t+1} + \hat{x}_{t+1} - \hat{x}_{t+1} \geq 0, \quad \text{and} \quad \left\| \frac{\hat{p}^*_t - \hat{p}_t}{p^0_t}x^0_{t+1} \right\| > 0,$$

which contradicts PNA. □

We can also show that PNA implies that the price functional is continuous, since the setting of this chapter and PNA implies that p_t(the zero payoff) must equal 0.

Next, we examine the implications of LOOP for sdfs.

Proposition 4 (LOOP implies existence of an sdf)**.** *If LOOP holds and the price functional is continuous, then there exists a unique payoff $m^*_{t+1} \in X_{t+1}$ which satifies*

$$\forall x_{t+1} \in X_{t+1} : p_t(x_{t+1}) = \mathrm{E}_t[m^*_{t+1}x_{t+1}].$$

PROOF FOR FINITE NUMBER OF ASSETS. Suppose as before that X_{t+1} is generated by an $N \times 1$ vector of basis payoffs $\boldsymbol{x}_{t+1}$ with prices $\boldsymbol{p}_t$. Choose

$$\underset{1\times 1}{m^*_{t+1}} = \underset{1\times N}{\boldsymbol{p}'_t} \underset{N\times N}{\left(\mathrm{E}_t[\boldsymbol{x}_{t+1}\boldsymbol{x}'_{t+1}]\right)^{-1}} \underset{N\times 1}{\boldsymbol{x}_{t+1}}.$$

This m^*_{t+1} satisfies

$$\boldsymbol{p}'_t = \mathrm{E}_t[m^*_{t+1}\boldsymbol{x}'_{t+1}]$$

and

$$\forall \boldsymbol{c} \in \mathcal{F}_t : \boldsymbol{p}'_t\boldsymbol{c} = \mathrm{E}_t[m^*_{t+1}(\boldsymbol{x}'_{t+1}\boldsymbol{c})].$$

□

PROOF FOR INFINITE NUMBER OF ASSETS. Apply the Riesz representation theorem. □

Next, we examine the implications of the existence of an sdf for LOOP.

Proposition 5. *If an sdf exists for X_{t+1}, then LOOP holds.*

PROOF. The proof is immediate. □

Next, we examine the implications of PNA for sdfs.

Proposition 6 (PNA implies existence of a strictly positive sdf)**.** *If PNA holds, then there exists $m_{t+1} \in L^2_{t+1}$ with $m_{t+1} > 0$ almost surely and*

$$\forall x_{t+1} \in X_{t+1} : p_t(x_{t+1}) = \mathrm{E}_t[m_{t+1}x_{t+1}].$$

The proof is omitted.

Finally, we examine the implications of the existence of a strictly positive sdf for PNA.

Proposition 7 (Implication of existence of strictly positive sdf)**.** *If $m_{t+1} \in L^2_{t+1}$ is strictly positive almost surely and is an sdf for X_{t+1}, then PNA holds.*

HEURISTIC PROOF. Consider $x_{t+1} \in X_{t+1}$ such that $x \geq 0$ almost surely and $(\mathrm{E}_t[x^2_{t+1}])^{1/2} > 0$. Now

$$p_t(x_{t+1}) = \mathrm{E}_t[\underset{>0}{m_{t+1}}\underset{\geq 0 \text{ a.s.}}{x_{t+1}}] > 0,$$

so PNA holds. □

2.4. Complete vs incomplete markets

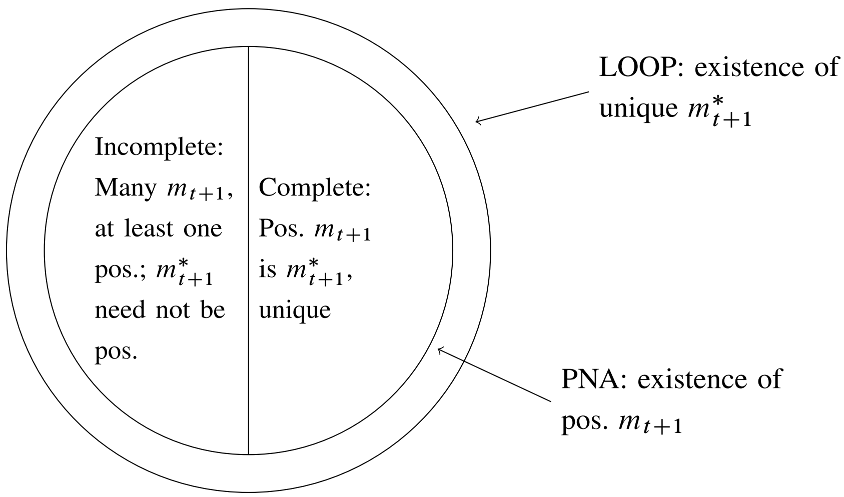
Consider a one period setting with a finite number of states S and $\mathcal{F}_t = \mathcal{F}^0_t$. If N, the number of assets, is less than S, the number of states, then the market is incomplete. If $N = S$, the market is complete.

In particular, let $\mathbf{x}_{t+1}$ be the $N \times S$ vector of basis assets,

$$\mathbf{x}_{t+1} = \begin{bmatrix} \boldsymbol{x}^1_{t+1} \\ \vdots \\ \boldsymbol{x}^N_{t+1} \end{bmatrix}, \quad \boldsymbol{x}^i_{t+1} \in \mathbb{R}^S \text{ for } i = 1, 2, \ldots, N.$$

FIGURE 1. Implications of PNA and LOOP

$$\begin{array}{ccc}
\text{PNA} & \Longrightarrow & \text{LOOP} \\
\Updownarrow & & \Updownarrow \\
\text{existence of a positive sdf} & \Longrightarrow & \text{existence of sdf}
\end{array}$$



If $N < S$, the payoff space

$$X_{t+1} = \{\boldsymbol{\omega}'\boldsymbol{x}_{t+1} : \boldsymbol{\omega} \in \mathbb{R}^N\} \subsetneq \mathbb{R}^S.$$

If $N = S$, then $X_{t+1} = \mathbb{R}^S$ and the market is complete.

2.5. Summary of implications of PNA

Complete markets.

(1) *One* m_{t+1} that satifies the Euler equation and that is m^*_{t+1}.
(2) The *one* m_{t+1} is strictly positive.
(3) The *one* m_{t+1} is m^*_{t+1}, which must be strictly positive.

Incomplete markets.

(1) There are many m_{t+1} which satisfy the Euler equation.
(2) At least *one* is strictly positive.
(3) m^*_{t+1} need not be strictly positive.
(4) There is a relation between m_{t+1} and m^*_{t+1}. We can decompose any m_{t+1} as:

$$m_{t+1} = m^*_{t+1} + e_{t+1}$$

where $\mathrm{E}_t[e_{t+1}x_{t+1}] = 0$ for all $x_{t+1} \in X_{t+1}$. Then:

$$\mathrm{E}_t[m_{t+1}x_{t+1}] = \mathrm{E}_t[m^*_{t+1}x_{t+1}] + \mathrm{E}_t[e_{t+1}x_{t+1}] = \mathrm{E}_t[m^*_{t+1}x_{t+1}].$$

So we can obtain m^*_{t+1} by regressing m_{t+1} on $\boldsymbol{x}$ where $\boldsymbol{x}$ is a basis for X_{t+1}.

2.6. Implications of finding an sdf with respect to $\mathcal{F}_t$

Suppose m_{t+1} constitutes an sdf for $\tilde{X}_{t+1}$ with respect to $\mathcal{F}_t$, i.e.

$$\forall x_{t+1} \in \tilde{X}_{t+1} : \mathrm{E}_t[m_{t+1}x_{t+1}] = p_t(x_{t+1}),$$

and let

$$\tilde{R}_{t+1} = \{r_{t+1} \in \tilde{X}_{t+1} \mid p_t(r_{t+1}) = 1\}.$$

Then

(1)

$$1 = \mathrm{E}_t[m_{t+1}r^f_{t+1}] \Rightarrow r^f_{t+1} = \frac{1}{E_t[m_{t+1}]}.$$

(2) Suppose that $r^f_{t+1} \in \tilde{X}_{t+1}$. Then

$$\forall r^i_{t+1} \in \tilde{R}_{t+1} : 1 = \mathrm{E}_t[m_{t+1}r^i_{t+1}],$$

$$\begin{aligned}
1 &= \mathrm{cov}_t[m_{t+1}, r^i_{t+1}] + \mathrm{E}_t[m_{t+1}]\,\mathrm{E}_t[r^i_{t+1}],\\
\mathrm{E}_t[r^i_{t+1}] &= \frac{1}{\mathrm{E}_t[m_{t+1}]} - \frac{1}{\mathrm{E}_t[m_{t+1}]}\,\mathrm{cov}_t[m_{t+1}, r^i_{t+1}]\\
&= r^f_{t+1} + \underbrace{\left\{-r^f_{t+1}\,\mathrm{var}_t[m_{t+1}]\right\}}_{\lambda_m}\underbrace{\left\{\frac{\mathrm{cov}_t[m_{t+1}, r^i_{t+1}]}{\mathrm{var}_t[m_{t+1}]}\right\}}_{\beta_{i,m}}.
\end{aligned}$$

(3) Suppose again that $r^f_{t+1} \in \tilde{X}_{t+1}$. For any $r^i_{t+1} \in \tilde{R}_{t+1}$,

$$\begin{aligned}
1 &= \mathrm{E}_t[m_{t+1}r^i_{t+1}] \quad \text{and} \quad 1 = \mathrm{E}_t[m_{t+1}r^f_{t+1}],\\
0 &= \mathrm{E}_t[m_{t+1}\{r^i_{t+1} - r^f_{t+1}\}],\\
0 &= \mathrm{cov}_t[m_{t+1}, r^i_{t+1} - r^f_{t+1}] + \mathrm{E}_t[m_{t+1}]\,\mathrm{E}_t[r^i_{t+1} - r^f_{t+1}],\\
\frac{E_t r^i_{t+1} - r^f_{t+1}}{r^f_{t+1}} &= -\rho_t[m_{t+1}, r^i_{t+1}]\sigma_t[m_{t+1}]\sigma_t[r^i_{t+1}]\\
&\Rightarrow \left|\left\{\frac{\mathrm{E}_t\, r^i_{t+1} - r^f_{t+1}}{\sigma_t[r^i_{t+1}]}\right\}\frac{1}{r^f_{t+1}}\right| \le \sigma_t[m_{t+1}]
\end{aligned}$$

holds for any $r^i_{t+1} \in \tilde{R}_{t+1}$.

So if $\boldsymbol{r}$ forms a return basis for $\tilde{X}_{t+1}$ with respect to $\mathcal{F}_t$, $\mathrm{E}_t[\boldsymbol{r}] = \boldsymbol{\mu}_t$, and $\mathrm{cov}_t[\boldsymbol{r}] = V_t$,

$$\underbrace{\left|\left((\boldsymbol{\mu}_t - r^f_{t+1}\mathbf{i})'V_t^{-1}(\boldsymbol{\mu}_t - r^f_{t+1}\mathbf{i})\right)^{1/2}\frac{1}{r^f_{t+1}}\right|}_{\text{max Sharpe measure, H–J bound}} \le \sigma_t[m_{t+1}].$$

2.7. Consumption-based pricing model

Consider the problem

$$\max \mathrm{E}\left[\sum_{\tau=0}^{\infty} \beta^{\tau} u(c_{\tau})\right] \quad \text{wrt. } \{c_{\tau}\}_{\tau=0}^{\infty} \text{ and } \{\boldsymbol{\alpha}_{\tau}\}_{\tau=0}^{\infty}$$

subject to W_0,

$$W_{\tau+1} = (W_{\tau} - c_{\tau} + y_{\tau})(\boldsymbol{\alpha}'_{\tau}(\boldsymbol{r}_{\tau+1} - r^{f}_{\tau+1}\mathbf{i}) + r^{f}_{\tau+1}), \quad \tau = 0, 1, \ldots,$$

and $\{c_{\tau}, \boldsymbol{\alpha}_{\tau}\} \in \mathcal{F}_{\tau}$ for $\tau = 0, 1, \ldots$ where $\boldsymbol{r}_{\tau+1}$ is an $N \times 1$ vector of returns on the N basis assets from time$t$ to time $t+1$ with ith element r^{i}_{t+1}.

Suppose $\{c^{*}_{\tau}, \boldsymbol{\alpha}^{*}_{\tau}\}_{\tau=0}^{\infty}$ is the solution. Consider the following perturbation:

- Reduce c^{*}_{t} by δz_{t} with $z_{t} \in \mathcal{F}_{t}$.
- Invest δz_{t} in asset i.
- Increase c^{*}_{t+1} by $\delta z_{t} r^{i}_{t+1}$.
- Leave all other c^{*}_{τ}s the same.

Along the optimal path, $\delta^{*} = 0$ solves the problem:

$$\max_{\delta} \mathrm{E}[\beta^{t} u(c^{*}_{t} - \delta z_{t}) + \beta^{t+1} u(c^{*}_{t+1} + \delta z_{t} r^{i}_{t+1})].$$

The first-order condition is

$$\mathrm{E}[-\beta^{t} u'(c^{*}_{t} - \delta^{*} z_{t}) z_{t} + \beta^{t+1} u'(c^{*}_{t+1} + \delta^{*} z_{t} r^{i}_{t+1}) z_{t} r^{i}_{t+1}] = 0.$$

Set $\delta^{*} = 0$ and obtain

$$\mathrm{E}[z_{t}\{u'(c^{*}_{t}) - \beta u'(c^{*}_{t+1}) r^{i}_{t+1}\}] = 0.$$

This holds for any $z_{t} \in \mathcal{F}_{t}$, and so

$$\mathrm{E}[u'(c^{*}_{t}) - \beta u'(c^{*}_{t+1}) r^{i}_{t+1} \,|\, \mathcal{F}_{t}] = 0$$

and

$$u'(c^{*}_{t}) = \beta\, \mathrm{E}_{t}[u'(c^{*}_{t+1}) r^{i}_{t+1}].$$

using the result in section 1.5 that provides a sufficient condition for a variable to be conditionally mean zero.

So letting $\boldsymbol{x}_{t+1}$ be an $N \times 1$ vector of payoffs on the N basis assets at time $t+1$ with ith element x^i_{t+1}:

$$1 = \mathrm{E}_t\left[\left\{\beta\frac{u'(c^*_{t+1})}{u'(c^*_t)}\right\}r^i_{t+1}\right], i = 1, 2, ..., N \quad \text{returns,}$$

$$p_t(x^i_{t+1}) = \mathrm{E}_t\left[\left\{\beta\frac{u'(c^*_{t+1})}{u'(c^*_t)}\right\}x^i_{t+1}\right], i = 1, 2, ..., N \quad \text{payoffs,}$$

$$\forall \boldsymbol{z}_t \in \mathcal{F}_t : p_t(z_t x^i_{t+1}) = \mathrm{E}_t\left[\left\{\beta\frac{u'(c^*_{t+1})}{u'(c^*_t)}\right\}z_t x^i_{t+1}\right], i = 1, 2, ..., N, \quad \text{scaled payoffs,}$$

$$0 = \mathrm{E}_t\left[\left\{\beta\frac{u'(c^*_{t+1})}{u'(c^*_t)}\right\}(r^i_{t+1} - r^j_{t+1})\right], i = 1, 2, ..., N \quad \text{zero investment portfolios.}$$

More generally,

$$\forall \boldsymbol{z}_t, z^f_t \in \mathcal{F}_t : z^f_t + \boldsymbol{z}'_t\mathbf{i} = \mathrm{E}_t\left[\left\{\beta\frac{u'(c^*_{t+1})}{u'(c^*_t)}\right\}(z^f_t r^f_{t+1} + \boldsymbol{z}'_t\boldsymbol{r}_{t+1})\right].$$

2.8. Using the consumption-based model to find an sdf with respect to $\mathcal{F}_t$

Can see that

$$m_{t+1} = \beta\frac{u'(c_{t+1})}{u'(c_t)}.$$

satisfies the definition of an sdf with respect to $\mathcal{F}_t$.

So

(1)

$$\frac{1}{r^f_{t+1}} = \mathrm{E}_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}\right]$$

(2)

$$\mathrm{E}_t\, r^i_{t+1} = r^f_{t+1} - r^f_{t+1}\,\mathrm{cov}_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}, r^i_{t+1}\right].$$

and so

$$\mathrm{cov}_t[u'(c_{t+1}), r^i_{t+1}] \uparrow \quad \mathrm{E}_t\, r^i_{t+1} \downarrow$$
$$\mathrm{cov}_t[c_{t+1}, r^i_{t+1}] \downarrow \quad \mathrm{E}_t\, r^i_{t+1} \downarrow$$

When $\mathrm{cov}_t[c_{t+1}, r^i_{t+1}]$ is lower, the asset is more likely to pay off in bad states, and so the agent requires a lower expected return on the asset.

(3) In addition,

$$\left|\left((\mu_t - r^f_{t+1}\mathbf{i})' V_t^{-1}(\mu_t - r^f_{t+1}\mathbf{i})\right)^{1/2}\frac{1}{r^f_{t+1}}\right| \leq \sigma_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}\right].$$

CHAPTER 3

Stochastic Discount Factors and the $\mathcal{F}_t$-Conditional Minimum Variance Frontier

3.1. Setting and definitions

The setting is the same as in chapter 2 plus PNA. X_{t+1} is defined as in chapter 2. Define

$$R_{t+1} \equiv \{r_{t+1} \in X_{t+1} \mid p_t(r_{t+1}) = 1\}.$$

Now we can define the $\mathcal{F}_t$-conditional MVF for R_{t+1}.

Definition ($\mathcal{F}_t$-conditional MVF for R_{t+1})**.** Return $r^{mv}_{t+1} \in R_{t+1}$ is on the $\mathcal{F}_t$-conditional MVF for R_{t+1} iff

$$\text{for} \quad \mu_t \in \mathcal{F}_t : \quad r^{mv}_{t+1} = \underset{r^i_{t+1} \in R_{t+1}, \mathrm{E}_t[r^i_{t+1}] = \mu_t}{\arg\min} \left\{ \sigma^2_t[r^i_{t+1}] \right\}.$$

3.2. Relation between m^* and the $\mathcal{F}_t$-conditional minimum variance frontier

We know that there exists a unique $m^*_{t+1} \in X_{t+1}$ satisfying

$$\forall x_{t+1} \in X_{t+1} : \mathrm{E}_t[m^*_{t+1} x_{t+1}] = p_t(x_{t+1}).$$

So

$$\mathrm{E}_t[m^*_{t+1} m^*_{t+1}] = p_t(m^*_{t+1})$$

since $m^*_{t+1} \in X_{t+1}$. Defining

$$r^*_{t+1} = \frac{m^*_{t+1}}{E_t[m^{*2}_{t+1}]},$$

we know that $r^*_{t+1} \in R_{t+1}$ since $\mathrm{E}_t[m^*_{t+1} r^*_{t+1}] = \mathrm{E}_t[m^*_{t+1} \frac{m^*_{t+1}}{\mathrm{E}_t[m^{*2}_{t+1}]}] = 1$.

Theorem 8. *r^*_{t+1} is the minimum second moment return with respect to $\mathcal{F}_t$.*

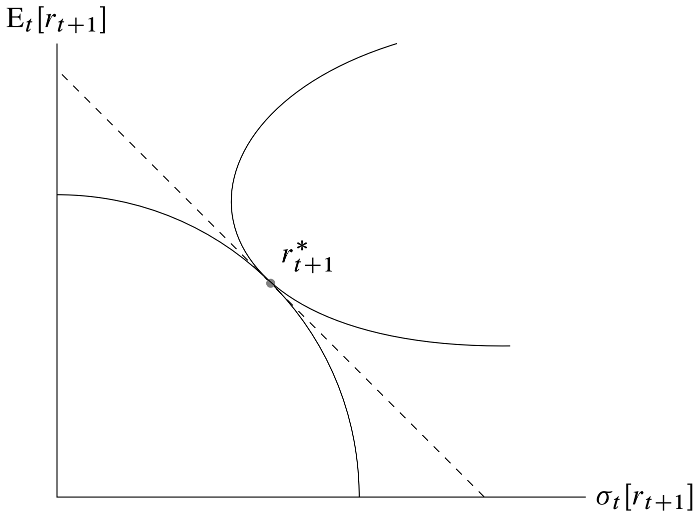
PROOF. Uncentered second moment for $x_{t+1} \in X_{t+1}$ is $\mathrm{E}_t[x^2_{t+1}]$. We know that $r^*_{t+1} = m^*_{t+1} / \mathrm{E}_t[m^{*2}_{t+1}]$. So

$$\mathrm{E}_t[r^{*2}_{t+1}] = \frac{\mathrm{E}_t[m^{*2}_{t+1}]}{(\mathrm{E}_t[m^{*2}_{t+1}])^2} = \frac{1}{\mathrm{E}_t[m^{*2}_{t+1}]} \quad \text{and} \quad m^*_{t+1} = \frac{r^*_{t+1}}{\mathrm{E}_t[r^{*2}_{t+1}]}.$$

For $r_{t+1} \in R_{t+1}$, we can write

$$r_{t+1} = r^*_{t+1} + z_{t+1},$$

FIGURE 2. r^*_{t+1} is the minimum second moment return



where $z_{t+1} = r_{t+1} - r^*_{t+1}$. Now $\mathrm{E}_t[m^*_{t+1} r_{t+1}] = 1$, so $\mathrm{E}_t[r^*_{t+1} r_{t+1}] = \mathrm{E}_t[r^{*2}_{t+1}]$, since $m^*_{t+1} = r^*_{t+1} / \mathrm{E}_t[r^{*2}_{t+1}]$. Thus $\mathrm{E}_t\, r^*_{t+1} r_{t+1} = \mathrm{E}_t[r^*_{t+1}(r^*_{t+1} + z_{t+1})]$, which implies that $\mathrm{E}_t[r^*_{t+1} z_{t+1}] = 0$. So

$$\forall r_{t+1} \in R_{t+1} : \mathrm{E}_t[r^2_{t+1}] = \mathrm{E}_t[r^{*2}_{t+1}] + \mathrm{E}_t[z^2_{t+1}] \geq \mathrm{E}_t[r^{*2}_{t+1}],$$

and so r^*_{t+1} is the minimum second moment return. □

Corollary 9. *Since $\sigma^2_t[r_{t+1}] + (\mathrm{E}_t[r_{t+1}])^2 = \mathrm{E}_t[r^2_{t+1}]$, it follows that r^*_{t+1} is on the $\mathcal{F}_t$-conditional minimum variance frontier (the $\mathcal{F}_t$-conditional MVF).*

3.3. Relation between returns on the $\mathcal{F}_t$-cond. MVF, sdfs wrt $\mathcal{F}_t$ and $\mathcal{F}_t$-cond. Beta pricing

Theorem 10. *$r^{mv}_{t+1} \in R_{t+1}$ is on the $\mathcal{F}_t$-conditional MVF for R_{t+1} $\overset{[1]}{\Leftrightarrow}$ $m_{t+1} = a + b r^{mv}_{t+1}$ with $a, b \in \mathcal{F}_t$ is an sdf with respect to $\mathcal{F}_t$ for X_{t+1} $\overset{[2]}{\Leftrightarrow}$*

$$\forall r^i_{t+1} \in R_{t+1} : \mathrm{E}_t[r^i_{t+1}] - \alpha_{mv} = \beta^{i,mv}_t \lambda_{mv},$$

where

$$\beta^{i,mv}_t = \frac{\mathrm{cov}_t[r^i_{t+1}, r^{mv}_{t+1}]}{\mathrm{var}_t[r^{mv}_{t+1}]}, \quad \lambda_{mv} = \mathrm{E}_t[r^{mv}_{t+1}] - \alpha_{mv} \quad \textit{and} \quad \alpha_{mv} = \frac{1}{\mathrm{E}_t[m_{t+1}]}.$$

Define $Z_{t+1} \equiv \{z_{t+1} \in X_{t+1} : p_t(z_{t+1}) = 0\}$.

Lemma. *$r^i_{t+1} \in R_{t+1}$ can be expressed as*

$$r^i_{t+1} = r^*_{t+1} + w^i_t z^*_{t+1} + n^i_{t+1},$$

where $\mathrm{E}_t[z^*_{t+1} z^i_{t+1}] = \mathrm{E}_t[z^i_{t+1}]$ for all $z^i_{t+1} \in Z_{t+1}$ and $w^i_t \in \mathcal{F}_t$ is chosen so that $\mathrm{E}_t[z^*_{t+1} n^i_{t+1}] =$ 0. Moreover,

(i) $z^*_{t+1} = \text{proj}_t(1|Z_{t+1})$.
(ii) $\mathrm{E}_t[r^*_{t+1} z^*_{t+1}] = \mathrm{E}_t[r^*_{t+1} n^i_{t+1}] = 0$.
(iii) $\mathrm{E}_t[n^i_{t+1}] = 0$.

PROOF OF LEMMA. (1) Existence of z^*_{t+1}: Use the Riesz representation theorem applied to the linear functional $\mathrm{E}_t[\cdot]$ to establish the existence of z^*_{t+1}.

(2) Existence of w^i_t: z^*_{t+1} generates a closed linear subspace in L^2_{t+1} which we denote Z^*_{t+1}. Define $z^i_{t+1} = r^i_{t+1} - r^*_{t+1}$. Since we know $\mathrm{E}_t[(z^i_{t+1} - \text{proj}_t(z^i_{t+1}|Z^*_{t+1}))z^*_{t+1}] = 0$ by construction, we obtain w^i_t by setting $w^i_t z^*_{t+1}$ equal to $\text{proj}_t(z^i_{t+1}|Z^*_{t+1})$ (which we know exists from a result in chapter 1). Using regression theory, we know $w^i_t = \mathrm{E}_t[z^i_{t+1} z^*_{t+1}] / \mathrm{E}_t[(z^*_{t+1})^2] = \mathrm{E}_t[z^i_{t+1}] / \mathrm{E}_t[(z^*_{t+1})^2]$.

(3) Result (ii): Know that for all $z^i_{t+1} \in Z_{t+1}$:

$$\begin{aligned}\mathrm{E}_t[r^{*2}_{t+1}] &= \mathrm{E}_t[r^*_{t+1} r^i_{t+1}] \\ &= \mathrm{E}_t[r^*_{t+1}(r^*_{t+1} + w^i_t z^*_{t+1} + n^i_{t+1})]\end{aligned}$$

$$\Rightarrow \text{for all } z^i_{t+1} \in Z_{t+1} : \mathrm{E}_t[r^*_{t+1} z^*_{t+1}] = \mathrm{E}_t[r^*_{t+1} n^i_{t+1}] = 0$$

since $z^*_{t+1} \in Z_{t+1}$ implies $\mathrm{E}_t[r^*_{t+1} z^*_{t+1}] = 0$.

(4) Result (i): Now for all $z^i_{t+1} \in Z_{t+1}$:

$$\begin{aligned}\mathrm{E}_t[1 z^i_{t+1}] &= \mathrm{E}_t[(\text{proj}_t(1|Z_{t+1}) + e) z^i_{t+1}] \\ &= \mathrm{E}_t[\text{proj}_t(1|Z_{t+1}) z^i_{t+1}] \\ &= \mathrm{E}_t[z^*_{t+1} z^i_{t+1}].\end{aligned}$$

(5) Result (iii): For all $z^i_{t+1} \in Z_{t+1}$:

$$\begin{aligned}\mathrm{E}_t[z^*_{t+1} z^i_{t+1}] &= \mathrm{E}_t[z^*_{t+1}(w^i_t z^*_{t+1} + n^i_{t+1})] \\ &= w^i_t \,\mathrm{E}_t[z^*_{t+1}]. \\ \mathrm{E}_t[z^i_{t+1}] &= \mathrm{E}_t[w^i_t z^*_{t+1} + n^i_{t+1}] \\ &= w^i_t \,\mathrm{E}_t[z^*_{t+1}] + \mathrm{E}_t[n^i_{t+1}].\end{aligned}$$

So $\mathrm{E}_t[n^i_{t+1}] = 0$.

□

Lemma. *$r^{mv}_{t+1} \in R_{t+1}$ is on the $\mathcal{F}_t$-conditional MVF for $R_{t+1} \Leftrightarrow r^{mv}_{t+1} = r^*_{t+1} + w_t z^*_{t+1}$ for some $w_t \in \mathcal{F}_t$.*

PROOF OF LEMMA. For any $r^i_{t+1} \in R_{t+1}$,

$$r^i_{t+1} = r^*_{t+1} + w^i_t z^*_{t+1} + n^i_{t+1}.$$

Now

$$\begin{aligned}&\mathrm{E}_t[r^i_{t+1}] = \mathrm{E}_t[r^*_{t+1}] + w^i_t \,\mathrm{E}_t\, z^*_{t+1}, \\ &\sigma^2_t[r^i_{t+1}] = \sigma^2_t[r^*_{t+1} + w^i_t z^*_{t+1}] + \sigma^2_t[n^i_{t+1}].\end{aligned}$$

For each mean, there is a unique w_t^i. Returns with $n_{t+1}^i = 0$ are minimum variance for each mean. □

Example. Suppose $r_{t+1}^f \in R_{t+1}$. We know that r_{t+1}^f is on the minimum variance frontier, so $r_{t+1}^f = r_{t+1}^* + wz_{t+1}^*$ for some $w \in \mathcal{F}_t$. It follows that $z_{t+1}^* = (r_{t+1}^f - r_{t+1}^*)\frac{1}{w}$. We know that

$$\begin{aligned} \mathrm{E}_t[z_{t+1}^* z_{t+1}^i] &= \mathrm{E}_t[(r_{t+1}^f - r_{t+1}^*)\tfrac{1}{w} z_{t+1}^i] = \mathrm{E}_t[z_{t+1}^i] \\ \Rightarrow \mathrm{E}_t[r_{t+1}^f \tfrac{1}{w} z_{t+1}^i] &= \mathrm{E}_t[z_{t+1}^i] \quad \text{since } \tfrac{1}{w}\,\mathrm{E}_t[r_{t+1}^* z_{t+1}^i] = 0 \\ \Rightarrow (r_{t+1}^f \tfrac{1}{w})\,\mathrm{E}_t[z_{t+1}^i] &= \mathrm{E}_t[z_{t+1}^i] \quad \text{for all } z_{t+1}^i \in Z_{t+1} \\ \Rightarrow w &= r_{t+1}^f. \end{aligned}$$

PROOF OF THEOREM. $\overset{[1]}{\Leftrightarrow}$ Take an arbitrary $r_{t+1} \in R_{t+1}$ and set

$$m_{t+1} = a + br_{t+1} = a + b(r_{t+1}^* + wz_{t+1}^* + n_{t+1})$$

where $a, b, w \in \mathcal{F}_t$ and w is chosen so $\mathrm{E}_t[z_{t+1}^* n_{t+1}] = 0$. We will show that m_{t+1} is a valid sdf for X_{t+1} iff $n_{t+1} = 0$. Then using the previous lemma, the result follows.

Now m_{t+1} must at least price r_{t+1}^* and z_{t+1}^*, so

$$\begin{aligned} 1 &= \mathrm{E}_t[m_{t+1} r_{t+1}^*] = a\,\mathrm{E}_t[r_{t+1}^*] + b\,\mathrm{E}_t[r_{t+1}^{*2}], \\ 0 &= \mathrm{E}_t[m_{t+1} z_{t+1}^*] = a\,\mathrm{E}_t[z_{t+1}^*] + bw\,\mathrm{E}_t[z_{t+1}^{*2}] \\ &= a\,\mathrm{E}_t[z_{t+1}^*] + bw\,\mathrm{E}_t[z_{t+1}^*] \\ &\Rightarrow 0 = a + bw. \end{aligned}$$

Solving for a and b,

$$a = \frac{w}{w\,\mathrm{E}_t[r_{t+1}^*] - \mathrm{E}_t[r_{t+1}^{*2}]}, \quad b = -\frac{a}{w}.$$

So if this m_{t+1} works it must be

$$m_{t+1} = \frac{w - (r_{t+1}^* + wz_{t+1}^* + n_{t+1})}{w\,\mathrm{E}_t[r_{t+1}^*] - \mathrm{E}_t[r_{t+1}^{*2}]}.$$

Now to price an arbitrary $x_{t+1}^i \in X_{t+1}$, note that

$$x_{t+1}^i = p_t(x_{t+1}^i) r_{t+1}^* + w_t^i z_{t+1}^* + n_{t+1}^i,$$

so

$$\begin{aligned} &\mathrm{E}_t[m_{t+1} x_{t+1}^i] = \\ &\qquad \mathrm{E}_t\left[\frac{(w - (r_{t+1}^* + wz_{t+1}^* + n_{t+1}))(p_t(x_{t+1}^i) r_{t+1}^* + w_t^i z_{t+1}^* + n_{t+1}^i)}{w\,\mathrm{E}_t[r_{t+1}^*] - \mathrm{E}_t[r_{t+1}^{*2}]}\right] \\ &\Rightarrow \mathrm{E}_t[m_{t+1} x_{t+1}^i] = p_t(x_{t+1}^i) - \frac{\mathrm{E}_t[n_{t+1} n_{t+1}^i]}{w\,\mathrm{E}_t[r_{t+1}^*] - \mathrm{E}_t[r_{t+1}^{*2}]}. \end{aligned}$$

Using the definition of an sdf, know

$$\forall x_{t+1}^i \in X_{t+1} : \mathrm{E}_t[m_{t+1} x_{t+1}^i] = p_t(x_{t+1}^i).$$

This can be true for all $x^i_{t+1} \in X_{t+1}$ iff $n_{t+1} = 0$, which completes the proof.

So the m_{t+1} linear in r^{mv}_{t+1} that works as a sdf with respect to $\mathcal{F}_t$ for X_{t+1} when r^{mv}_{t+1} is on the on the $\mathcal{F}_t$-conditional MVF for R_{t+1} is given by:

$$m_{t+1} = \frac{w - (r^*_{t+1} + wz^*_{t+1})}{w\,\mathrm{E}_t[r^*_{t+1}] - \mathrm{E}_t[r^{*2}_{t+1}]}.$$

Note that m_{t+1} is not defined when

$$w = \frac{\mathrm{E}_t[r^{*2}_{t+1}]}{\mathrm{E}_t[r^*_{t+1}]},$$

and $\mathrm{E}_t[m_{t+1}] = 0$ when

$$w = \frac{\mathrm{E}_t[r^*_{t+1}]}{1 - \mathrm{E}_t[z^*_{t+1}]}.$$

To prove $\overset{[2]}{\Leftrightarrow}$, note that

$$x^i_{t+1} \in X_{t+1}, \quad p_t(x^i_{t+1}) \neq 0 \Leftrightarrow r^i_{t+1} = \frac{x^i_{t+1}}{p_t(x^i_{t+1})} \in X_{t+1}$$

and

$x^i_{t+1} \in X_{t+1}, \quad p_t(x^i_{t+1}) = 0 \Leftrightarrow$ there exists $x^1_{t+1}, x^2_{t+1} \in X_{t+1}, p_t(x^1_{t+1}) \neq 0$ and $p_t(x^2_{t+1}) \neq 0$ such that $x^i_{t+1} = x^1_{t+1} + x^2_{t+1}$.

It follows that:

$$\begin{aligned}
&\forall x^i_{t+1} \in X_{t+1} : \mathrm{E}_t[(a + br^{mv}_{t+1})x^i_{t+1}] = p_t(x^i_{t+1}),\\
\Leftrightarrow\ &\forall r^i_{t+1} \in R_{t+1} : \mathrm{E}_t[(a + br^{mv}_{t+1})r^i_{t+1}] = 1,\\
\Leftrightarrow\ &\forall r^i_{t+1} \in R_{t+1} : \mathrm{E}_t[(a + br^{mv}_{t+1})]\,\mathrm{E}_t[r^i_{t+1}] + \mathrm{cov}_t[br^{mv}_{t+1}, r^i_{t+1}] = 1\\
\Leftrightarrow\ &\forall r^i_{t+1} \in R_{t+1} : \mathrm{E}_t\, r^i_{t+1} = \underbrace{\frac{1}{\mathrm{E}_t[m_t]}}_{\alpha_{mv}} - \frac{b}{\mathrm{E}_t[m_t]}\,\mathrm{cov}_t[r^{mv}_{t+1}, r^i_{t+1}].
\end{aligned}$$

□

Intuition: We know that

$$\mathrm{E}_t\, r^i_{t+1} = \alpha_{mv} + \lambda_{mv}\beta^{i,mv}_t.$$

See figure 3 on page 23 for the case with $r^f_{t+1} \notin R_{t+1}$.

If $r^f_{t+1} \in R_{t+1}$, then

(1) $m_{t+1} = a + br^{mv}_{t+1} = m^*_{t+1}$ for all r^{mv}_{t+1} on the MVF except r^f_{t+1} (see figure 4 on page 23).

(2) $\dfrac{1}{\mathrm{E}_t[m^*_{t+1}]} = \dfrac{\mathrm{E}_t[r^{*2}_{t+1}]}{\mathrm{E}_t[r^*_{t+1}]} = r^f_{t+1} = \mathrm{E}_t[r^{gmv}_{t+1}] = \dfrac{\mathrm{E}_t[r^*_{t+1}]}{1 - \mathrm{E}_t[z^*_{t+1}]}$ where r^{gmv}_{t+1} is the return on the global minimum-variance portfolio.

(3) $w = \dfrac{\mathrm{E}_t[r^{*2}_{t+1}]}{\mathrm{E}_t[r_{t+1}*]} = r^f_{t+1} = w^f = w^{gmv}$ corresponds to m_{t+1} being a constant, which is *not* a valid sdf, and $r^{gmv}_{t+1} = r^f_{t+1}$ which is *not* a valid beta pricing model.

If $r^f_{t+1} \notin R_{t+1}$, then

(1) Each r^{mv}_{t+1} gives the m^*_{t+1} that corresponds to an augmented space (see figure 5 on page 24):

$$X^a_{t+1} = X_{t+1} + \{\alpha_{mv}\}.$$

The m_{t+1} linear in r^*_{t+1} that works as an sdf is the m^*_{t+1} for X_{t+1}. So the α^* associated with r^*_{t+1} as a 1-factor βpm equals $\dfrac{1}{\mathrm{E}_t[m^*_{t+1}]} = \dfrac{\mathrm{E}_t[r^{*2}_{t+1}]}{\mathrm{E}_t[r^*_{t+1}]}$.

(2) $\alpha^* = \dfrac{1}{\mathrm{E}_t[m^*_{t+1}]} = \dfrac{\mathrm{E}_t[r^{*2}_{t+1}]}{\mathrm{E}_t[r^*_{t+1}]} \neq \mathrm{E}_t[r^{gmv}_{t+1}] = \dfrac{\mathrm{E}_t[r^*_{t+1}]}{1-\mathrm{E}_t[z^*_{t+1}]}$ (see figure 6 on page 24).

(3) $w = \dfrac{\mathrm{E}_t[r^{*2}_{t+1}]}{\mathrm{E}_t[r^*_{t+1}]}$ corresponds to $\mathrm{E}_t[m_{t+1}] = \infty$, so m_{t+1} is not a valid sdf, and $\alpha_{mv} = 0$. $w = \dfrac{\mathrm{E}_t[r^*_{t+1}]}{1-\mathrm{E}_t[z^*_{t+1}]} = w^{gmv}$ corresponds to $\mathrm{E}_t[m_{t+1}] = 0$, so $\alpha_{mv} = \infty$ or $-\infty$, and r^{mv}_{t+1} is not a valid beta pricing model (see figure 7 on page 25).

3.4. Obtaining m^*_{t+1} from m_{t+1}

Suppose that

$$X_{t+1} = \{\boldsymbol{\omega}'_t \boldsymbol{x}_{t+1} \,|\, \boldsymbol{\omega}_t \in \mathcal{F}_t\},$$

so $\underset{N\times 1}{\boldsymbol{x}_{t+1}}$ forms a basis for X_{t+1} with respect to $\mathcal{F}_t$. We know that $m_{t+1} = m^*_{t+1} + e_{t+1}$ where $\mathrm{E}_t[e_{t+1}\boldsymbol{x}'_{t+1}] = \boldsymbol{0}'$ and we can write $m^*_{t+1} = \boldsymbol{c}'\boldsymbol{x}_{t+1}$. So

$$\mathrm{E}_t[m_{t+1}\boldsymbol{x}'_{t+1}] = \mathrm{E}_t[\boldsymbol{c}'\boldsymbol{x}_{t+1}\boldsymbol{x}'_{t+1}] + \underbrace{\mathrm{E}_t[e_{t+1}\boldsymbol{x}']}_{\boldsymbol{0}}$$

$$\Rightarrow \boldsymbol{c}' = \underset{1\times N}{\mathrm{E}_t[m_{t+1}\boldsymbol{x}'_{t+1}]}(\underset{N\times N}{\mathrm{E}_t[\boldsymbol{x}_{t+1}\boldsymbol{x}_{t+1}]})^{-1}.$$

FIGURE 3. Intuition for Theorem 10 when $r^f_{t+1} \notin R_{t+1}$

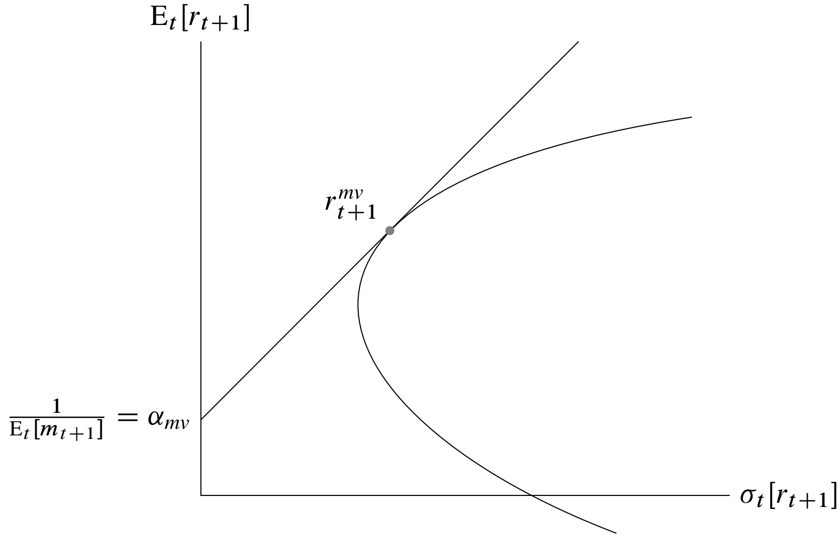


FIGURE 4. $r^f_{t+1} \in R_{t+1}$: $1/\,\mathrm{E}_t[m_{t+1}]$ is the risk free return

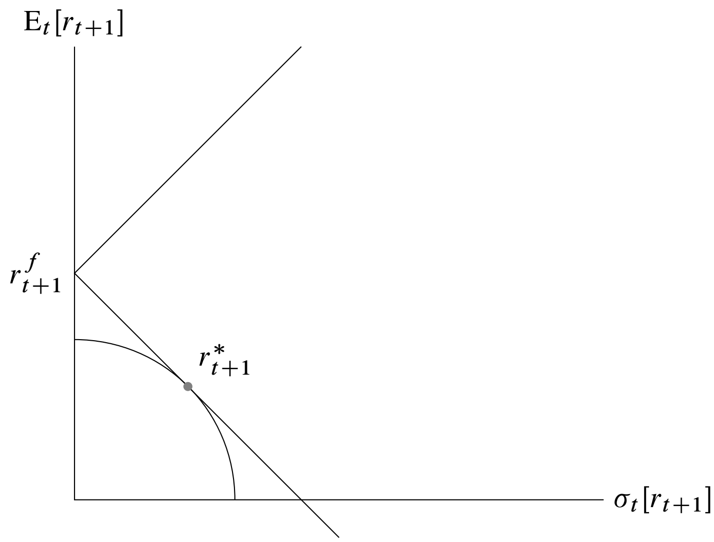


FIGURE 5. $r^f_{t+1} \notin R_{t+1}$: Each r^{mv}_{t+1} gives m^*_{t+1} for an augmented asset space

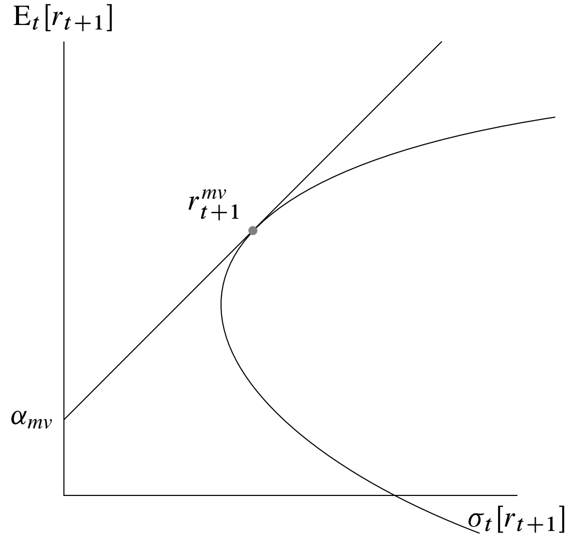


FIGURE 6. $r^f_{t+1} \notin R_{t+1}$: $\alpha^{r^*} \neq E_t[r^{gmv}_{t+1}]$

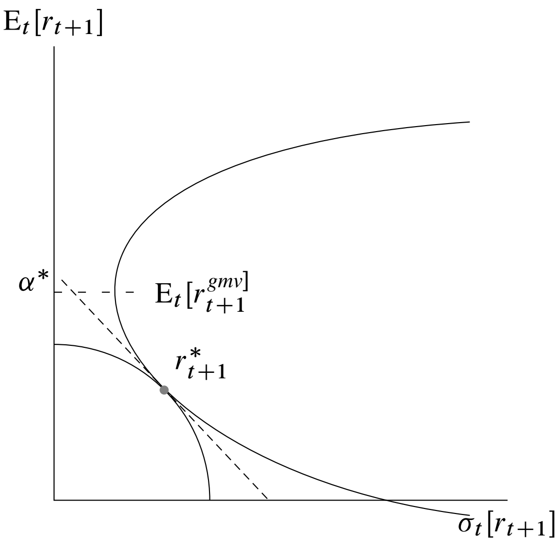
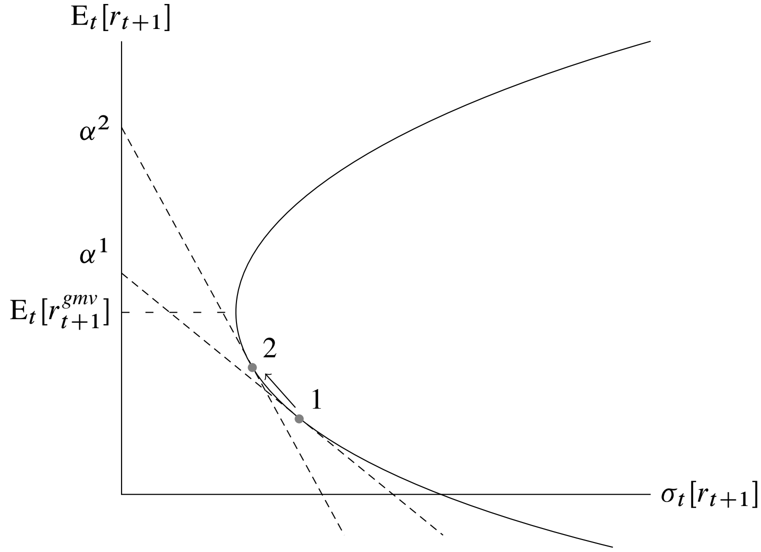


FIGURE 7. $r^f_{t+1} \notin R_{t+1}$: r^{gmv}_{t+1} corresponds to $\alpha_{mv} = \pm\infty$ and so is not a valid beta pricing model



CHAPTER 4

Beta Pricing Models

4.1. Definitions

Let $\mathrm{E}_{\hat{t}}[\cdot] \equiv \mathrm{E}[\cdot|\hat{\mathcal{F}}_t]$. For an arbitrary payoff space $\tilde{X}_{t+1}$, define the return space

$$\tilde{R}_{t+1} \equiv \{r_{t+1} \in \tilde{X}_{t+1} \mid p_t(r_{t+1}) = 1\}.$$

Here is the definition of a Beta pricing model (βpm):

Definition (Beta pricing model)**.** Let $\hat{\mathcal{F}}_t \subset \mathcal{F}_t$. A set of factors $\boldsymbol{h}_{t+1}$ constitutes a beta pricing model (βpm) with respect to $\hat{\mathcal{F}}_t$ for $\tilde{R}_{t+1}$ iff

$$\forall r^i_{t+1} \in \tilde{R}_{t+1} : \mathrm{E}_{\hat{t}}[r^i_{t+1}] = \hat{\alpha} + \hat{\boldsymbol{\lambda}}'\boldsymbol{\beta}^{i,h}_{\hat{t}},$$

where

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\boldsymbol{\lambda}} \end{bmatrix} \in \hat{\mathcal{F}}_t, \quad \boldsymbol{\beta}^{i,h}_{\hat{t}} = \left(\mathrm{E}_{\hat{t}}[\tilde{\boldsymbol{h}}_{t+1}, \tilde{\boldsymbol{h}}'_{t+1}]\right)^{-1} \mathrm{E}_{\hat{t}}[\tilde{\boldsymbol{h}}_{t+1}, r^i_{t+1}], \quad \tilde{\boldsymbol{h}}_{t+1} = \boldsymbol{h}_{t+1} - \mathrm{E}_{\hat{t}}[\boldsymbol{h}_{t+1}].$$

Let $\boldsymbol{r}_{t+1}$ be an $N \times 1$ return vector: so $p_t(\boldsymbol{r}_{t+1}) = \boldsymbol{i}_N$ and $\boldsymbol{r}_{t+1} \in R_{t+1}$. Define X_{t+1} and R_{t+1} as before, and assume that $\boldsymbol{r}_{t+1}$ constitutes a basis set for X_{t+1} with respect to $\mathcal{F}_t$. Let $\hat{\mathcal{F}}_t \subset \mathcal{F}_t$. Define

$$\hat{X}_{t+1} = \{\boldsymbol{\omega}'_{\hat{t}}\boldsymbol{r}_{t+1} \mid \boldsymbol{\omega}_{\hat{t}} \in \hat{\mathcal{F}}_t\},$$
$$\hat{R}_{t+1} = \{\boldsymbol{\omega}'_{\hat{t}}\boldsymbol{r}_{t+1} \mid \boldsymbol{\omega}_{\hat{t}} \in \hat{\mathcal{F}}_t, \boldsymbol{\omega}'_{\hat{t}}\boldsymbol{i}_N = 1\}.$$

Define $\bar{X}_{t+1}$ and $\bar{R}_{t+1}$ analogously for $\bar{\mathcal{F}}_t$

4.2. Equivalence of a βpm and a linear sdf: Basic results

Theorem 11 (Equivalence of a βpm and a linear sdf: Part 1)**.** *Let $\hat{\mathcal{F}}_t \subsetneq \bar{\mathcal{F}}_t \subset \mathcal{F}_t$.*

$\boldsymbol{h}_{t+1}$, a $K \times 1$ vector, constitutes a βpm with respect to $\hat{\mathcal{F}}_t$ for $\bar{R}_{t+1}$ $\Leftarrow$ there exist $\hat{a}, \boldsymbol{b}' \in \hat{\mathcal{F}}_t$ such that

$$m_{t+1} = \hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1}$$

is a valid sdf with respect to $\hat{\mathcal{F}}_t$ for $\bar{X}_{t+1}$.

PROOF. Let

$$\mathrm{E}_{\hat{t}}[m_{t+1}] = \hat{a} + \boldsymbol{b}'\,\mathrm{E}_{\hat{t}}[h_{t+1}] = a,$$

so

$$m_{t+1} = \hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1} = a + \boldsymbol{b}'\tilde{\boldsymbol{h}}_{t+1}.$$

Now

$$m_{t+1} = a + \boldsymbol{b}'\tilde{\boldsymbol{h}}_{t+1}, \text{ an sdf with respect to } \hat{\mathcal{F}}_t \text{ for } \bar{X}_{t+1}$$

$$\begin{aligned}
&\Leftrightarrow \forall x^i_{t+1} \in \bar{X}_{t+1} : \mathrm{E}_{\hat{t}}[m_{t+1}x^i_{t+1}] = \mathrm{E}_{\hat{t}}[p_t(x^i_{t+1})] \\
&\qquad \Leftrightarrow \forall \boldsymbol{\omega}^i_{\bar{t}} \in \bar{\mathcal{F}}_t : \mathrm{E}_{\hat{t}}[m_{t+1}(\boldsymbol{\omega}^i_{\bar{t}})'\boldsymbol{r}_{t+1}] = \mathrm{E}_{\hat{t}}[p_t((\boldsymbol{\omega}^i_{t+1})'\boldsymbol{r}_{t+1})] \\
&\qquad \Leftrightarrow \forall \boldsymbol{\omega}^i_{\bar{t}} \in \bar{\mathcal{F}}_t : \mathrm{E}_{\hat{t}}[(\boldsymbol{\omega}^i_{\bar{t}})'(m_{t+1}\boldsymbol{r}_{t+1} - \boldsymbol{i}_N)] = \boldsymbol{0} \\
&\Rightarrow \forall r^i_{t+1} \in \bar{R}_{t+1} : \mathrm{E}_{\hat{t}}[m_{t+1}r^i_{t+1}] = 1 \\
&\Leftrightarrow \forall r^i_{t+1} \in \bar{R}_{t+1} : 1 = \mathrm{E}_{\hat{t}}[m_{t+1}]\,\mathrm{E}_{\hat{t}}[r^i_{t+1}] + \mathrm{cov}_{\hat{t}}[m_{t+1}, r^i_{t+1}]
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \forall r^i_{t+1} \in \bar{R}_{t+1} : \mathrm{E}_{\hat{t}}[r^i_{t+1}] &= \frac{1}{\mathrm{E}_{\hat{t}}[m_{t+1}]} - \frac{\mathrm{E}_{\hat{t}}[\boldsymbol{b}'\tilde{\boldsymbol{h}}_{t+1}r^i_{t+1}]}{\mathrm{E}_{\hat{t}}[m_{t+1}]} \\
&= \underbrace{\frac{1}{a}}_{\hat{\alpha}} + \underbrace{-\frac{1}{a}\boldsymbol{b}'\,\mathrm{E}_{\hat{t}}[\tilde{\boldsymbol{h}}_{t+1}, \tilde{\boldsymbol{h}}'_{t+1}]}_{\hat{\boldsymbol{\lambda}}'} \underbrace{(\mathrm{E}_{\hat{t}}[\tilde{\boldsymbol{h}}_{t+1}\tilde{\boldsymbol{h}}'_{t+1}])^{-1}(\mathrm{E}_{\hat{t}}[\tilde{\boldsymbol{h}}_{t+1}r^i_{t+1}])}_{\boldsymbol{\beta}^{i,h}_{\hat{t}}}\ .
\end{aligned}$$

□

Demeaning the factors is unimportant. Note that the theorem does not say: $\boldsymbol{h}_{t+1}$ constitutes a βpm with respect to $\hat{\mathcal{F}}_t$ for $\bar{R}_{t+1} \Rightarrow$ there exist $\hat{a}, \boldsymbol{b}' \in \hat{\mathcal{F}}_t$ such that $m_t = \hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1}$ is a valid sdf with respect to $\hat{\mathcal{F}}_t$ for $\bar{X}_{t+1}$. To obtain this result, need to set $\bar{\mathcal{F}}_t \subset \hat{\mathcal{F}}_t$.

Theorem 12 (Equivalence of a βpm and a linear sdf: Part 2)**.** *Let $\bar{\mathcal{F}}_t \subset \hat{\mathcal{F}}_t \subset \mathcal{F}_t$.*

$\boldsymbol{h}_{t+1}$, a $K \times 1$ vector, constitutes a βpm with respect to $\hat{\mathcal{F}}_t$ for $\bar{R}_{t+1} \Leftrightarrow$ there exist $\hat{a}, \boldsymbol{b}' \in \hat{\mathcal{F}}_t$ such that

$$m_{t+1} = \hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1}$$

is a valid sdf with respect to $\hat{\mathcal{F}}_t$ for $\bar{X}_{t+1}$.

PROOF.

$\Leftarrow$: As before.

$\Rightarrow$: Know $\boldsymbol{h}_{t+1}$, a $K \times 1$ vector, constitutes a βpm with respect to $\hat{\mathcal{F}}_t$ for $\bar{R}_{t+1} \Leftrightarrow$

(1) there exists $\hat{a}, \boldsymbol{b}' \in \hat{\mathcal{F}}_t$ such that $\forall r^i_{t+1} \in \bar{R}_{t+1} : \mathrm{E}_{\hat{t}}[(\hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1})r^i_{t+1}] = 1$

Consider $x^i_{t+1} \in \bar{X}_{t+1}$. Can write $x^i_{t+1} = \boldsymbol{\omega}'_{\bar{t}}\boldsymbol{r}_{t+1}$ for $\boldsymbol{\omega}_{\bar{t}} \in \bar{\mathcal{F}}_t$. So know $p_t(x^i_{t+1}) \in \bar{\mathcal{F}}_t \subset \hat{\mathcal{F}}_t$. There are 2 cases.

Case 1. $p_t(x^i_{t+1}) \neq 0$

$$p_t(x^i_{t+1}) \neq 0 \Rightarrow r^i_{t+1} = \frac{x^i_{t+1}}{p_t(x^i_{t+1})} \in \bar{R}_{t+1}.$$

And so

$$(1) \Rightarrow \mathrm{E}_{\hat{t}}[(\hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1})\frac{x^i_{t+1}}{p_t(x^i_{t+1})} = 1$$

$$\Rightarrow \mathrm{E}_{\hat{t}}[(\hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1})x^i_{t+1}] = p_t(x^i_{t+1})$$

as required.

Case 2. $p_t(x^i_{t+1}) = 0$

Take $r^1_{t+1} \in \bar{R}_{t+1}$ and define $r^2_{t+1} = x^i_{t+1} + r^1_{t+1}$. Know $p_t(r^1_{t+1}) = 1$ and so know $r^2_{t+1} \in \bar{R}_{t+1}$ since $p_t(r^2_{t+1}) = p_t(x^i_{t+1}) + p_t(r^1_{t+1}) = 1$.

So

$$(1) \Rightarrow \mathrm{E}_{\hat{t}}[(\hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1})r^1_{t+1}] = 1$$

and

$$(1) \Rightarrow \mathrm{E}_{\hat{t}}[(\hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1})r^2_{t+1}] = 1$$

Since $x^i_{t+1} = r^2_{t+1} - r^1_{t+1}$, it follows that:

$$\mathrm{E}_{\hat{t}}[(\hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1})x^i_{t+1}] = 1 - 1 = 0$$

as required. □

Corollary 13. *Let $\hat{\mathcal{F}}_t \subset \mathcal{F}_t$.*

$\boldsymbol{h}_{t+1}$*, a $K \times 1$ vector, constitutes a βpm with respect to $\hat{\mathcal{F}}_t$ for $\hat{R}_{t+1}$ $\Leftrightarrow$ there exist $\hat{a}, \boldsymbol{b}' \in \hat{\mathcal{F}}_t$ such that*

$$m_{t+1} = \hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1}$$

is a valid sdf with respect to $\hat{\mathcal{F}}_t$ for $\hat{X}_{t+1}$.

Corollary 14. $\boldsymbol{h}_{t+1}$*, a $K \times 1$ vector, constitutes a βpm with respect to $\mathcal{F}_t$ for R_{t+1} $\Leftrightarrow$ there exist $\hat{a}, \boldsymbol{b}' \in \mathcal{F}_t$ such that*

$$m_{t+1} = \hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1}$$

is a valid sdf with respect to $\mathcal{F}_t$ for X_{t+1}.

4.3. Equivalence of a βpm and a linear sdf: Additional useful results

Theorem 15 (Equivalence of a βpm and a linear sdf: Part 3)**.** *Let $\hat{\mathcal{F}}_t \subset \mathcal{F}_t$ and $\bar{\mathcal{F}}_t \subset \mathcal{F}_t$.*

$\boldsymbol{h}_{t+1}$*, a $K \times 1$ vector, constitutes a βpm with respect to $\hat{\mathcal{F}}_t$ for $\bar{R}_{t+1}$ $\Leftrightarrow$ there exists $\hat{a}, \boldsymbol{b}' \in \hat{\mathcal{F}}_t$ such that*

$$m_{t+1} = \hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1}$$

is a valid sdf with respect to $\hat{\mathcal{F}}_t$ for $\bar{R}_{t+1}$.

PROOF. Again let $\mathrm{E}_{\hat{t}}[m_{t+1}] = a$. There exists $\hat{a}, \boldsymbol{b}' \in \hat{\mathcal{F}}_t$ such that

$$m_{t+1} = \hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1}$$

is a valid sdf with respect to $\hat{\mathcal{F}}_t$ for $\bar{R}_{t+1}$

$$\Leftrightarrow \forall r^i_{t+1} \in \bar{R}_{t+1} : \mathrm{E}_{\hat{t}}[m_{t+1} r^i_{t+1}] = 1$$

$$\Leftrightarrow \forall r^i_{t+1} \in \bar{R}_{t+1} : 1 = \mathrm{E}_{\hat{t}}[m_{t+1}]\,\mathrm{E}_{\hat{t}}[r^i_{t+1}] + \mathrm{cov}_{\hat{t}}[m_{t+1}, r^i_{t+1}]$$

$$\begin{aligned}\Leftrightarrow \forall r^i_{t+1} \in \bar{R}_{t+1} : \mathrm{E}_{\hat{t}}[r^i_{t+1}] &= \frac{1}{\mathrm{E}_{\hat{t}}[m_{t+1}]} - \frac{\mathrm{E}_{\hat{t}}[\boldsymbol{b}'\tilde{\boldsymbol{h}}_{t+1} r^i_{t+1}]}{\mathrm{E}_{\hat{t}}[m_{t+1}]} \\ &= \underbrace{\frac{1}{a}}_{\hat{\alpha}} + \underbrace{-\frac{1}{a}\boldsymbol{b}'\,\mathrm{E}_{\hat{t}}[\tilde{\boldsymbol{h}}_{t+1}, \tilde{\boldsymbol{h}}'_{t+1}]}_{\hat{\boldsymbol{\lambda}}'} \underbrace{(\mathrm{E}_{\hat{t}}[\tilde{\boldsymbol{h}}_{t+1}\tilde{\boldsymbol{h}}'_{t+1}])^{-1}(\mathrm{E}_{\hat{t}}[\tilde{\boldsymbol{h}}_{t+1} r^i_{t+1}])}_{\boldsymbol{\beta}^{i,h}_{\hat{t}}}.\end{aligned}$$

□

Let $\hat{\mathcal{F}}_t \subset \bar{\mathcal{F}}_t \subset \mathcal{F}_t$. For the payoff space $\bar{X}_{t+1}$ with respect to $\hat{\mathcal{F}}_t$, define the augmented return space

$$\bar{R}^{\hat{A}}_{t+1} \equiv \{r_{t+1} \in \bar{X}_{t+1} \mid E_{\hat{t}}[p_t(r_{t+1})] = 1\}.$$

Theorem 16 (Equivalence of a βpm and a linear sdf: Part 4)**.** *Let $\hat{\mathcal{F}}_t \subsetneq \bar{\mathcal{F}}_t \subset \mathcal{F}_t$.*

$\boldsymbol{h}_{t+1}$*, a $K \times 1$ vector, constitutes a βpm with respect to $\hat{\mathcal{F}}_t$ for $\bar{R}^{\hat{A}}_{t+1}$ $\Leftrightarrow$ there exist $\hat{a}, \boldsymbol{b}' \in \hat{\mathcal{F}}_t$ such that*

$$m_{t+1} = \hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1}$$

is a valid sdf with respect to $\hat{\mathcal{F}}_t$ for $\bar{X}_{t+1}$.

PROOF.

$\Leftarrow$: As before.

$\Rightarrow$: Know $\boldsymbol{h}_{t+1}$, a $K \times 1$ vector, constitutes a βpm with respect to $\hat{\mathcal{F}}_t$ for $\bar{R}^{\hat{A}}_{t+1}$ $\Leftrightarrow$

(2) there exists $\hat{a}, \boldsymbol{b}' \in \hat{\mathcal{F}}_t$ such that $\forall r^i_{t+1} \in \bar{R}^{\hat{A}}_{t+1} : \mathrm{E}_{\hat{t}}[(\hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1}) r^i_{t+1}] = 1$

Consider $x^i_{t+1} \in \bar{X}_{t+1}$. There are 2 cases.

Case 1. $\mathrm{E}_{\hat{t}}[p_t(x^i_{t+1})] \neq 0$

$$\mathrm{E}_{\hat{t}}[p_t(x^i_{t+1})] \neq 0 \Rightarrow r^i_{t+1} = \frac{x^i_{t+1}}{\mathrm{E}_{\hat{t}}[p_t(x^i_{t+1})]} \in \bar{R}^{\hat{A}}_{t+1}.$$

And so

$$\begin{aligned}(2) \;\Rightarrow \mathrm{E}_{\hat{t}}[(\hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1})\frac{x^i_{t+1}}{\mathrm{E}_{\hat{t}}[p_t(x^i_{t+1})]}] &= 1 \\ \Rightarrow \mathrm{E}_{\hat{t}}[(\hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1}) x^i_{t+1}] &= \mathrm{E}_{\hat{t}}[p_t(x^i_{t+1})]\end{aligned}$$

as required.

Case 2. $\mathrm{E}_{\hat{t}}[p_t(x^i_{t+1})] = 0$

Take $x^1_{t+1} \in \bar{R}_{t+1}$ and define $x^2_{t+1} = x^i_{t+1} + x^1_{t+1}$. Know $\mathrm{E}_{\hat{t}}[p_t(x^1_{t+1})] = 1$ and so $x^2_{t+1} \in \bar{R}_{t+1}$ since $\mathrm{E}_{\hat{t}}[p_t(x^2_{t+1})] = \mathrm{E}_{\hat{t}}[p_t(x^i_{t+1})] + \mathrm{E}_{\hat{t}}[p_t(x^1_{t+1})] = 1$.

So

$$(2) \ \Rightarrow \mathrm{E}_{\hat{t}}[(\hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1})x^1_{t+1}] = 1$$

and

$$(2) \ \Rightarrow \mathrm{E}_{\hat{t}}[(\hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1})x^2_{t+1}] = 1$$

Since $x^i_{t+1} = x^2_{t+1} + x^1_{t+1}$, it follows that:

$$\mathrm{E}_{\hat{t}}[(\hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1})x^i_{t+1}] = 1 - 1 = 0$$

as required. □

For the payoff spaceX_{t+1} with respect to $\mathcal{F}^0_t$, define the augmented return space

$$R^A_{t+1} \equiv \{r_{t+1} \in X_{t+1} \mid E[p_t(r_{t+1})] = 1\}.$$

Corollary 17. *$\boldsymbol{h}_{t+1}$, a $K \times 1$ vector, constitutes a βpm with respect to $\mathcal{F}^0_t$ for $R^A_{t+1} \Leftrightarrow$ there exist $\hat{a}, \boldsymbol{b}' \in \hat{\mathcal{F}}_t$ such that*

$$m_{t+1} = \hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1}$$

is a valid sdf with respect to $\mathcal{F}^0_t$ for X_{t+1}.

4.4. Multi-factor βpm implies single-factor βpm

Proposition 18. *Let $\hat{\mathcal{F}}_t \subset \mathcal{F}_t$ and $\bar{\mathcal{F}}_t \subset \mathcal{F}_t$. $\boldsymbol{h}_{t+1}$, a $K \times 1$ vector, constitutes a K-factor βpm with respect to $\hat{\mathcal{F}}_t$ for $\bar{R}_{t+1} \Rightarrow$ there exists $\boldsymbol{c} \in \hat{\mathcal{F}}_t$, a $K \times 1$ vector, such that $\boldsymbol{c}'\boldsymbol{h}_{t+1}$ constitutes a single factor βpm with respect to $\hat{\mathcal{F}}_t$ for $\bar{R}_{t+1}$.*

PROOF. From theorem 15, if $\boldsymbol{h}_{t+1}$ is a multi-factor βpm for $\bar{R}_{t+1}$, then $m_{t+1} = \hat{a} + \boldsymbol{b}\tilde{\boldsymbol{h}}_{t+1}$ is a valid sdf for $\bar{R}_{t+1}$. Take $g_{t+1} = \boldsymbol{b}'\boldsymbol{h}_{t+1}$ (so $\tilde{g}_{t+1} = \boldsymbol{b}'\tilde{\boldsymbol{h}}_{t+1}$). We know that $m_{t+1} = \hat{a} + g_{t+1}$ is a valid sdf for $\bar{R}_{t+1}$. So from theorem 15, we know that g_{t+1} constitutes a single-factor βpm for $\bar{R}_{t+1}$. □

4.5. Mimicking portfolios

Note that $\underset{N\times 1}{\boldsymbol{r}_{t+1}} \in \hat{R}_{t+1}$ constitutes a basis set for $\hat{X}_{t+1}$ with respect to $\hat{\mathcal{F}}_t$. Suppose that $\underset{K\times 1}{\boldsymbol{h}_{t+1}}$ constitutes a multi-factor βpm for $\hat{R}_{t+1}$ with respect to $\hat{\mathcal{F}}_t$. We can obtain K assets, each an element of $\hat{R}_{t+1}$, that also constitute a multi-factor βpm for $\hat{R}_{t+1}$ with respect to $\hat{\mathcal{F}}_t$. These assets are known as mimicking portfolios. Let's go through the reasoning.

Suppose that $\underset{K\times 1}{\boldsymbol{h}_{t+1}}$ constitutes a multi-factor βpm for $\hat{R}_{t+1}$ with respect to $\hat{\mathcal{F}}_t$. From corollary˙13, we know that there exists $a, \boldsymbol{b}' \in \hat{\mathcal{F}}_t$ such that $a + \boldsymbol{b}'\tilde{\boldsymbol{h}}_{t+1}$ is a valid sdf for $\hat{X}_{t+1}$. Project each element of $\underset{K\times 1}{\tilde{\boldsymbol{h}}_{t+1}}$ on $\underset{N\times 1}{\boldsymbol{r}_{t+1}}$ with respect to $\hat{\mathcal{F}}_t$. Let $\underset{K\times 1}{\boldsymbol{x}^h_{t+1}}$ be the set of assets obtained from the projection, so

$$\underset{K\times 1}{\boldsymbol{x}^h_{t+1}} = \underset{K\times N}{\boldsymbol{Q}} \underset{N\times 1}{\boldsymbol{r}}_{t+1},$$

where

$$\boldsymbol{Q} = \mathrm{E}_{\hat{t}}[\underset{K\times N}{\tilde{\boldsymbol{h}}_{t+1}\boldsymbol{r}'_{t+1}}](\mathrm{E}_{\hat{t}}[\underset{N\times N}{\boldsymbol{r}_{t+1}\boldsymbol{r}'_{t+1}}])^{-1} \in \hat{\mathcal{F}}_t,$$

and $\boldsymbol{x}^h_{t+1} \in \hat{X}_{t+1}$.

We know that

$$\tilde{\boldsymbol{h}}_{t+1} = \boldsymbol{x}^h_{t+1} + \boldsymbol{e}^h_{t+1}, \quad \mathrm{E}_{\hat{t}}[\boldsymbol{e}^h_{t+1}\boldsymbol{r}'_{t+1}] = \boldsymbol{0}.$$

So for all $r^i_{t+1} \in \hat{R}_{t+1}$,

$$\mathrm{E}_{\hat{t}}[(a + \boldsymbol{b}'\tilde{\boldsymbol{h}}_{t+1})r^i_{t+1} = \mathrm{E}_{\hat{t}}[(a + \boldsymbol{b}'\boldsymbol{x}^h_{t+1})r^i_{t+1}] + \underbrace{\mathrm{E}_{\hat{t}}[\boldsymbol{b}'\boldsymbol{e}^h_{t+1}\boldsymbol{r}'_{t+1}]\boldsymbol{\omega}^i}_{\boldsymbol{0}} = 1.$$

where $r^i_{t+1} = \boldsymbol{r}'_{t+1}(\boldsymbol{\omega}^i)$ and $\boldsymbol{\omega}^i \in \hat{\mathcal{F}}_t$.

Since $\boldsymbol{Q} \in \hat{\mathcal{F}}_t$, we can then normalize the vector $\boldsymbol{x}^h_{t+1}$ to be a return vector $\boldsymbol{r}^h_{t+1}$, by scaling the ith element of $\underset{K\times 1}{\boldsymbol{x}^h_{t+1}}$ by the ith element of $\underset{K\times 1}{\boldsymbol{Q}\boldsymbol{i}_N} \in \hat{\mathcal{F}}_t$. Consequently, we know there exists $\tilde{a}, \tilde{\boldsymbol{b}}' \in \hat{\mathcal{F}}_t$ such that

$$\forall x^i \in \hat{X}_{t+1} : \mathrm{E}_{\hat{t}}[(\tilde{a} + \tilde{\boldsymbol{b}}'\boldsymbol{r}^h_{t+1})x^i_{t+1}] = \mathrm{E}_{\hat{t}}[p_t(x^i_{t+1})].$$

From above, we know it follows that $\boldsymbol{r}^h_{t+1}$ constitutes a βpm for $\hat{R}_{t+1}$ with respect to $\hat{\mathcal{F}}_t$. Note that we can take $\hat{\mathcal{F}}_t$ to be $\mathcal{F}_t$ and then the result holds for $\mathcal{F}_t$, X_{t+1} and R_{t+1}.

CHAPTER 5

Conditional vs Unconditional Beta Pricing Models

5.1. Definitions

Let $\mathrm{E}_t[\cdot] \equiv \mathrm{E}[\cdot|\mathcal{F}_t]$. For an arbitrary payoff space $\tilde{X}_{t+1}$, define the return space

$$\tilde{R}_{t+1} \equiv \{r_{t+1} \in \tilde{X}_{t+1} \mid p_t(r_{t+1}) = 1\}.$$

When a set of factors $\boldsymbol{h}_{t+1}$ constitutes a βpm with respect to $\mathcal{F}_t$ for $\tilde{R}_{t+1}$, the set of factors $\boldsymbol{h}_{t+1}$ are said to constitute a conditional βpm with respect to $\mathcal{F}_t$ for $\tilde{R}_{t+1}$. When a set of factors $\boldsymbol{h}_{t+1}$ constitutes a βpm with respect to $\mathcal{F}_t^0$ for $\tilde{R}_{t+1}$, the set of factors $\boldsymbol{h}_{t+1}$ are said to constitute an unconditional βpm for $\tilde{R}_{t+1}$.

Let $\boldsymbol{r}_{t+1}$ be an $N \times 1$ return vector: so $p_t(\boldsymbol{r}_{t+1}) = \boldsymbol{i}_N$ and $\boldsymbol{r}_{t+1} \in R_{t+1}$. Define X_{t+1} and R_{t+1} as before, and assume, as in the previous chapter, that $\boldsymbol{r}_{t+1}$ constitutes a basis set for X_{t+1} with respect to $\mathcal{F}_t$. So

$$\begin{aligned}
X_{t+1} &= \{\boldsymbol{\omega}_t'\boldsymbol{r}_{t+1} \mid \boldsymbol{\omega}_t \in \mathcal{F}_t\},\\
R_{t+1} &= \{\boldsymbol{\omega}_t'\boldsymbol{r}_{t+1} \mid \boldsymbol{\omega}_t \in \mathcal{F}_t, \boldsymbol{\omega}_t'\boldsymbol{i} = 1\},\\
R_{t+1}^A &= \{\boldsymbol{\omega}_t'\boldsymbol{r}_{t+1} \mid \boldsymbol{\omega}_t \in \mathcal{F}_t, \mathrm{E}[\boldsymbol{\omega}_t'\boldsymbol{i}] = 1\}.
\end{aligned}$$

And as before, define X_{t+1}^0 and R_{t+1}^0 as

$$\begin{aligned}
X_{t+1}^0 &= \{\boldsymbol{\omega}'\boldsymbol{r}_{t+1} \mid \boldsymbol{\omega} \in \mathcal{F}_t^0\},\\
R_{t+1}^0 &= \{\boldsymbol{\omega}'\boldsymbol{r}_{t+1} \mid \boldsymbol{\omega} \in \mathcal{F}_t^0, \boldsymbol{\omega}'\boldsymbol{i} = 1\}.
\end{aligned}$$

Now we can define the unconditional MVF for R_{t+1}.

Definition (Unconditional MVF for R_{t+1})**.** Return $r_{t+1}^{mv} \in R_{t+1}$ is on the unconditional MVF for R_{t+1} iff

$$\text{for} \quad \mu \in \mathcal{F}_t^0: \quad r_{t+1}^{mv} = \underset{r_{t+1}^i \in R_{t+1}, \mathrm{E}[r_{t+1}^i] = \mu}{\arg\min} \left\{\sigma^2[r_{t+1}^i]\right\}.$$

5.2. Sdfs and conditional vs unconditional moments

If m_{t+1} is an sdf for X_{t+1} with respect to $\mathcal{F}_t$, then

$$\begin{aligned}
&\forall x_{t+1} \in X_{t+1}: \mathrm{E}[\mathrm{E}_t[m_{t+1}x_{t+1}]] = \mathrm{E}[p_t(x_{t+1})]\\
\Rightarrow\ &\forall x_{t+1} \in X_{t+1}: \mathrm{E}[m_{t+1}x_{t+1}] = \mathrm{E}[p_t(x_{t+1})],
\end{aligned}$$

and so m_{t+1} is an sdf for X_{t+1} with respect to $\mathcal{F}_t^0$.

What about the other direction? Yes.

Theorem 19. *m_{t+1} is an sdf for X_{t+1} with respect to $\mathcal{F}_t^0$ $\Leftrightarrow$ m_{t+1} is an sdf for X_{t+1} with respect to $\mathcal{F}_t$.*

PROOF.
$\Leftarrow$: See above.
$\Rightarrow$: m_{t+1} is an sdf for X_{t+1} using $\mathcal{F}_t^0$ iff

$$\begin{aligned}
&\forall x_{t+1} \in X_{t+1} : \mathrm{E}[(m_{t+1}x_{t+1} - p_t(x_{t+1}))] = 0 \\
&\quad \Leftrightarrow \forall \boldsymbol{z}_t \in \mathcal{F}_t : \mathrm{E}[\boldsymbol{z}_t'(m_{t+1}\boldsymbol{r}_{t+1} - \boldsymbol{i}_N)] = 0 \\
&\qquad\qquad \Leftrightarrow \mathrm{E}_t[m_{t+1}\boldsymbol{r}_{t+1} - \boldsymbol{i}_N] = 0 \text{ using a result from chapter 1} \\
&\qquad\qquad \Leftrightarrow \mathrm{E}_t[m_{t+1}\boldsymbol{r}_{t+1}] = \boldsymbol{i}_N \\
&\qquad\qquad \Leftrightarrow \mathrm{E}_t[m_{t+1}x_{t+1}] = p_t(x_{t+1})
\end{aligned}$$

for all $x_{t+1} \in X_{t+1}$. □

Note that only the $\Leftarrow$ direction is true when the payoff space is X_{t+1}^0 rather than X_{t+1}.

Theorem 20. *m_{t+1} is an sdf for X_{t+1}^0 using $\mathcal{F}_t^0$ $\Leftarrow$ m_{t+1} is an sdf for X_{t+1}^0 using $\mathcal{F}_t$.*

PROOF. If m_{t+1} is an sdf for X_{t+1}^0 with respect to $\mathcal{F}_t$, then

$$\begin{aligned}
&\forall x_{t+1} \in X_{t+1}^0 : \mathrm{E}[\mathrm{E}_t[m_{t+1}x_{t+1}]] = \mathrm{E}[p_t(x_{t+1})] \\
&\Rightarrow \forall x_{t+1} \in X_{t+1}^0 : \mathrm{E}[m_{t+1}x_{t+1}] = \mathrm{E}[p_t(x_{t+1})],
\end{aligned}$$

and so m_{t+1} is an sdf for X_{t+1}^0 with respect to $\mathcal{F}_t^0$. □

Note that theorem 19 implies that any m_{t+1} that works conditionally as an sdf for X_{t+1} works as a βpm for R_{t+1} wrt $\mathcal{F}_t$ and as a βpm for R_{t+1}^A wrt $\mathcal{F}_t^0$. Similarly, theorem 20 implies that any m_{t+1} that works conditionally as an sdf for X_{t+1}^0 works as a βpm for R_{t+1}^0 wrt both $\mathcal{F}_t$ and $\mathcal{F}_t^0$.

5.3. Conditional vs unconditional βpms

Theorem 21. *If $\boldsymbol{h}_{t+1}$ constitutes an unconditional βpm for R_{t+1}^A, then $\boldsymbol{h}_{t+1}$ constitutes a conditional βpm with respect to $\mathcal{F}_t$ for R_{t+1}.*

PROOF. $\boldsymbol{h}$ constitutes an unconditional βpm for R_{t+1}^A

$$\begin{aligned}
&\Leftrightarrow \exists\, \hat{a}, \boldsymbol{b} \in \mathcal{F}_t^0 \ni \forall x_{t+1} \in X_{t+1} : \mathrm{E}[(\hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1})x_{t+1}] = \mathrm{E}[p_t(x_{t+1})] \text{ using corollary 17} \\
&\Leftrightarrow \exists\, \hat{a}, \boldsymbol{b} \in \mathcal{F}_t^0 \ni \forall z_t \in \mathcal{F}_t : \mathrm{E}[((\hat{a} + \boldsymbol{b}'\boldsymbol{h}_{t+1})\boldsymbol{r}_{t+1} - \boldsymbol{i}_N)z_t] = 0 \\
&\Rightarrow \exists\, \hat{a}_t, \boldsymbol{b}_t \in \mathcal{F}_t \ni \mathrm{E}_t[(\hat{a}_t + \boldsymbol{b}_t'\boldsymbol{h}_{t+1})\boldsymbol{r}_{t+1}] = 1 \text{ using a result from chapter 1} \\
&\Leftrightarrow \exists\, \hat{a}_t, \boldsymbol{b}_t \in \mathcal{F}_t \ni \forall x_{t+1} \in X_{t+1} : \mathrm{E}_t[(\hat{a}_t + \boldsymbol{b}_t'\boldsymbol{h}_{t+1})x_{t+1}] = p_t(x_{t+1}),
\end{aligned}$$

so $\boldsymbol{h}_{t+1}$ constitutes a conditional βpm for R_{t+1} wrt$\mathcal{F}_t$. □

It is important that $\boldsymbol{h}_{t+1}$ constitutes an unconditional βpm for R_{t+1}^A and not just R_{t+1} for the proof of the theorem to hold. Note that if $\boldsymbol{h}_{t+1}$ constitutes an unconditional βpm for R_{t+1}^0, then it does not necessarily follow that $\boldsymbol{h}_{t+1}$ constitutes a conditional βpm with respect to $\mathcal{F}_t$ for R_{t+1}^0.

Note. If $\boldsymbol{h}_{t+1}$ constitutes a conditional βpm for R_{t+1} wrt $\mathcal{F}_t$, then $\boldsymbol{h}_{t+1}$ does not necessarily constitute an unconditional βpm for R^A_{t+1}. Here's where the proof to theorem 21 breaks down going in the other direction: $\boldsymbol{h}_{t+1}$ constitutes a conditional βpm for R_{t+1} wrt $\mathcal{F}_t$

$$
\begin{aligned}
&\Leftrightarrow \exists\, \hat{a}_t, \boldsymbol{b}_t \in \mathcal{F}_t \ni \forall x_{t+1} \in X_{t+1} : \mathrm{E}_t[(\hat{a}_t + \boldsymbol{b}'_t \boldsymbol{h}_{t+1}) x_{t+1}] = p_t(x_{t+1})\\
&\Leftrightarrow \exists\, \hat{a}_t, \boldsymbol{b}_t \in \mathcal{F}_t \ni \mathrm{E}_t[(\hat{a}_t + \boldsymbol{b}'_t \boldsymbol{h}_{t+1}) \boldsymbol{r}_{t+1}] = 1\\
&\not\Rightarrow \exists\, \hat{a}, \boldsymbol{b} \in \mathcal{F}^0_t \ni \forall z_t \in \mathcal{F}_t : \mathrm{E}[((\hat{a} + \boldsymbol{b}' \boldsymbol{h}_{t+1}) \boldsymbol{r}_{t+1} - \boldsymbol{i}_N) z_t] = 0\\
&\Leftrightarrow \exists\, \hat{a}, \boldsymbol{b} \in \mathcal{F}^0_t \ni \forall x_{t+1} \in X_{t+1} : \mathrm{E}[(\hat{a} + \boldsymbol{b}' \boldsymbol{h}_{t+1}) x_{t+1}] = \mathrm{E}[p_t(x_{t+1})]
\end{aligned}
$$

$\Leftrightarrow$ $\boldsymbol{h}$ constitutes an unconditional βpm for R^A_{t+1}.

It is useful to ask what conditions are sufficient for $\boldsymbol{h}_{t+1}$ constitutes a βpm for R_{t+1} wrt $\mathcal{F}_t$ to imply $\boldsymbol{h}_{t+1}$ constitutes an unconditional βpm for R^A_{t+1}. First, as shown above, if m_{t+1} constitutes a conditional βpm for R_{t+1} wrt $\mathcal{F}_t$, where that m_{t+1} works as an sdf for X_{t+1} wrt $\mathcal{F}_t$, then that m_{t+1} works as an unconditional βpm for R^A_{t+1}. This direction works when the factor is an sdf for X_{t+1} wrt $\mathcal{F}_t$ because

$$
\begin{aligned}
&\forall x^i_{t+1} \in X_{t+1} : \mathrm{E}_t[(\underbrace{a}_{=0} + \underbrace{b}_{=1} m_{t+1}) x_{t+1}] = p_t(x_{t+1})\\
\Rightarrow\ &\forall x^i_{t+1} \in X_{t+1} : \mathrm{E}[(a + b m_{t+1}) x_{t+1}] = \mathrm{E}[p_t(x_{t+1})], \quad a, b \in \mathcal{F}^0_t.
\end{aligned}
$$

There are three other results.

Theorem 22. *If $\boldsymbol{h}_{t+1}$ constitutes a conditional βpm for R_{t+1} wrt $\mathcal{F}_t$, so that there exists $\alpha, \boldsymbol{\lambda} \in \mathcal{F}_t \ni$*

$$\forall r^i_{t+1} \in R_{t+1} : \mathrm{E}_t\, r^i_{t+1} = \alpha + \boldsymbol{\lambda}' \beta^{i,h}_t$$

and

$$
\begin{aligned}
\hat{a} &= \frac{1}{\alpha}[1 + \boldsymbol{\lambda}'(\mathrm{cov}_t[\boldsymbol{h}_{t+1}, \boldsymbol{h}'_{t+1}])^{-1} \mathrm{E}_t[\boldsymbol{h}_{t+1}]] \in \mathcal{F}^0_t,\\
\boldsymbol{b}' &= -\boldsymbol{\lambda}'(\mathrm{cov}_t[\boldsymbol{h}_{t+1}, \boldsymbol{h}'_{t+1}])^{-1} \frac{1}{\alpha} \in \mathcal{F}^0_t,
\end{aligned}
\tag{3}
$$

then $\boldsymbol{h}_{t+1}$ constitutes an unconditional βpm for R^A_{t+1}.

PROOF.

$$
\begin{aligned}
&\forall r^i_{t+1} \in R_{t+1} : \mathrm{E}_t[r^i_{t+1}] = \alpha + \boldsymbol{\lambda}' \beta_{i,h}\\
\Leftrightarrow\ &\forall x_{t+1} \in X_{t+1} : \mathrm{E}_t[(\hat{a} + \boldsymbol{b}' \boldsymbol{h}_{t+1}) x_{t+1}] = p_t(x_{t+1}) \Leftrightarrow\\
&\forall x_{t+1} \in X_{t+1} : \mathrm{E}[(\hat{a} + \boldsymbol{b}' \boldsymbol{h}_{t+1}) x_{t+1}] = \mathrm{E}[p_t(x_{t+1})].
\end{aligned}
\tag{4}
$$

(3) and (4) $\Rightarrow$ $\boldsymbol{h}_{t+1}$ constitutes an unconditional β-pm for R^A_{t+1}. $\square$

Note.

- (3) can hold when $\alpha, \boldsymbol{\lambda}$ and $\beta^{i,h}_t \notin \mathcal{F}^0_t$.
- (3) holds when $\mathrm{E}_t[\cdot]$ and $\mathrm{E}[\cdot]$ are the same.

Theorem 23. *If $\boldsymbol{h}_{t+1}$ constitutes a conditional βpm for R_{t+1} wrt $\mathcal{F}_t$ and*

$$\begin{aligned} &\mathrm{E}_t[\boldsymbol{h}_{t+1}] = \mathbf{0} \\ &\forall r^i_{t+1} \in R_{t+1} : \mathrm{cov}_t[\boldsymbol{h}_{t+1}, r^i_{t+1}] \in \mathcal{F}_t^0, \end{aligned} \tag{5}$$

then $\boldsymbol{h}_{t+1}$ constitutes an unconditional βpm for R_{t+1}.

PROOF. Note that $\beta_t^{i,h} = (\mathrm{cov}_t[\boldsymbol{h}_{t+1}, \boldsymbol{h}'_{t+1}])^{-1}\,\mathrm{cov}_t[\boldsymbol{h}_{t+1}, r^i_{t+1}]$, and (5) $\Rightarrow \mathrm{cov}_t[\boldsymbol{h}_{t+1}, r^i_{t+1}] = \mathrm{cov}[\boldsymbol{h}_{t+1}, r^i_{t+1}]$.
Since
$\boldsymbol{h}_{t+1}$ constitutes a conditional βpm for R_{t+1} wrt $\mathcal{F}_t$
$\Leftrightarrow$there exists $\alpha_t, \boldsymbol{\lambda}_t \in \hat{\mathcal{F}}_t \ni \forall r^i_{t+1} \in R_{t+1} : \mathrm{E}_t[r^i_{t+1}] = \alpha_t + \boldsymbol{\lambda}'\beta_t^{i,h}$,
it follows that
(5) and $\boldsymbol{h}_{t+1}$ constitutes a conditional βpm for R_{t+1} wrt $\mathcal{F}_t$
$\Rightarrow \forall r^i_{t+1} \in R_{t+1} : \mathrm{E}[r^i_{t+1}] = \mathrm{E}[\alpha_t] + \mathrm{E}[\boldsymbol{\lambda}'(\mathrm{cov}_t[\boldsymbol{h}_{t+1}, \boldsymbol{h}'_{t+1}])^{-1}]\,\mathrm{cov}[\boldsymbol{h}_{t+1}, r^i_{t+1}]$
using the law of iterated expectations. So $\boldsymbol{h}_{t+1}$ constitutes an unconditional βpm for R_{t+1}. □

Theorem 24. *If $\boldsymbol{h}_{t+1}$ constitutes a conditional βpm for R_{t+1} wrt $\mathcal{F}_t$ and*

$$\begin{aligned} &\mathrm{E}_t[\boldsymbol{h}_{t+1}] = \mathbf{0} \\ &\beta_t^{i,h} \in \mathcal{F}_t^0 \end{aligned} \tag{6}$$

then $\boldsymbol{h}_{t+1}$ constitutes an unconditional βpm for R_{t+1}.

PROOF. Note that (6) $\Rightarrow \beta_t^{i,h} = \beta^{i,h}$.
Since
$\boldsymbol{h}_{t+1}$ constitutes a conditional βpm for R_{t+1} wrt $\mathcal{F}_t$
$\Leftrightarrow$ there exists $\alpha_t, \boldsymbol{\lambda}_t \in \hat{\mathcal{F}}_t \ni \forall r^i_{t+1} \in R_{t+1} : \mathrm{E}_t[r^i_{t+1}] = \alpha_t + \boldsymbol{\lambda}'\beta_t^{i,h}$,
it follows that
(6) and $\boldsymbol{h}_{t+1}$ constitutes a conditional βpm for R_{t+1} wrt $\mathcal{F}_t$
$\Rightarrow \forall r^i_{t+1} \in R_{t+1} : \mathrm{E}[r^i_{t+1}] = \mathrm{E}[\alpha_t] + \mathrm{E}[\boldsymbol{\lambda}']\beta^{i,h}$
using the law of iterated expectations. So $\boldsymbol{h}_{t+1}$ constitutes an unconditional βpm for R_{t+1}. □

5.4. Relation between returns on the uncond. MVF, sdfs wrt $\mathcal{F}_t^0$ and uncond. Beta pricing

Recall $Z_{t+1} \equiv \{z_{t+1} \in X_{t+1} : p_t(z_{t+1}) = 0\}$. Define r^*_{t+1} and z^*_{t+1} as in chapter 3.

Lemma. *$r^i_{t+1} \in R_{t+1}$ can be expressed as*

$$r^i_{t+1} = r^*_{t+1} + w^i z^*_{t+1} + \nu^i_{t+1},$$

where $w^i \in \mathcal{F}_t^0$ is chosen so that $\mathrm{E}[z^*_{t+1}\nu^i_{t+1}] = 0$. *Moreover,*

(i) $\mathrm{E}[z^*_{t+1} z^i_{t+1}] = \mathrm{E}[z^i_{t+1}]$ *for all* $z^i_{t+1} \in Z_{t+1}$.
(ii) $z^*_{t+1} = \mathrm{proj}(1|Z_{t+1})$.
(iii) $\mathrm{E}[r^*_{t+1} z^*_{t+1}] = \mathrm{E}[r^*_{t+1} n^i_{t+1}] = 0$.
(iv) $\mathrm{E}[\nu^i_{t+1}] = 0$.

PROOF OF LEMMA. (1) Result (i): From the relevant lemma in chapter 3, know that

$$\mathrm{E}_t[z^*_{t+1} z^i_{t+1}] = \mathrm{E}_t[z^i_{t+1}] \text{ for all } z^i_{t+1} \in Z_{t+1}$$

Using the law of iterated expectations, it follows that

$$\mathrm{E}[z^*_{t+1} z^i_{t+1}] = \mathrm{E}[z^i_{t+1}] \text{ for all } z^i_{t+1} \in Z_{t+1}$$

as required.

(2) Existence of w^i: z^*_{t+1} generates a closed linear subspace in L^2_{t+1} which we denote Z^*_{t+1}. Define $z^i_{t+1} = r^i_{t+1} - r^*_{t+1}$. Since we know $\mathrm{E}[(z^i_{t+1} - \mathrm{proj}(z^i_{t+1}|Z^*_{t+1}))z^*_{t+1}] = 0$ by construction, we obtain w^i by setting $w^i z^*_{t+1}$ equal to $\mathrm{proj}(z^i_{t+1}|Z^*_{t+1})$ (which we know exists from a result in chapter 1). Using regression theory, we know $w^i = \mathrm{E}[z^i_{t+1} z^*_{t+1}]/\,\mathrm{E}_t[(z^*_{t+1})^2] = \mathrm{E}[z^i_{t+1}]/\,\mathrm{E}_t[(z^*_{t+1})^2]$.

(3) Result (iii): Know that

$$\text{for all } z^i_{t+1} \in Z_{t+1} : \mathrm{E}_t[r^{*2}_{t+1}] = \mathrm{E}_t[r^*_{t+1} r^i_{t+1}]$$

Using the law of iterated expectations it follows that

$$\begin{aligned} \mathrm{E}[r^{*2}_{t+1}] &= \mathrm{E}[r^*_{t+1} r^i_{t+1}] \\ &= \mathrm{E}[r^*_{t+1}(r^*_{t+1} + w_t z^*_{t+1} + \nu^i_{t+1})] \end{aligned}$$

$$\Rightarrow \text{for all } z^i_{t+1} \in Z_{t+1} : \mathrm{E}[r^*_{t+1} z^*_{t+1}] = \mathrm{E}[r^*_{t+1} \nu^i_{t+1}] = 0$$

since $z^*_{t+1} \in Z_{t+1}$ implies $\mathrm{E}[r^*_{t+1} z^*_{t+1}] = 0$.

(4) Result (ii): Now for all $z^i_{t+1} \in Z_{t+1}$:

$$\begin{aligned} \mathrm{E}[1 z^i_{t+1}] &= \mathrm{E}[(\mathrm{proj}(1|Z_{t+1}) + e) z^i_{t+1}] \\ &= \mathrm{E}[\mathrm{proj}(1|Z_{t+1}) z^i_{t+1}] \\ &= \mathrm{E}[z^*_{t+1} z^i_{t+1}]. \end{aligned}$$

(5) Result (iv): For all $z^i_{t+1} \in Z_{t+1}$:

$$\begin{aligned} \mathrm{E}[z^*_{t+1} z^i_{t+1}] &= \mathrm{E}[z^*_{t+1}(w^i z^*_{t+1} + \nu^i_{t+1})] \\ &= w^i\, \mathrm{E}_t[z^*_{t+1}]. \\ \mathrm{E}[z^i_{t+1}] &= \mathrm{E}[w^i z^*_{t+1} + \nu^i_{t+1}] \\ &= w^i\, \mathrm{E}[z^*_{t+1}] + \mathrm{E}[\nu^i_{t+1}]. \end{aligned}$$

So $\mathrm{E}[\nu^i_{t+1}] = 0$.

□

Lemma. *$r^{mv}_{t+1} \in R_{t+1}$ is on the unconditional MVF for $R_{t+1} \Leftrightarrow r^{mv}_{t+1} = r^*_{t+1} + w z^*_{t+1}$ for some $w \in \mathcal{F}^0_t$.*

PROOF OF LEMMA. For any $r^i_{t+1} \in R_{t+1}$,

$$r^i_{t+1} = r^*_{t+1} + w^i z^*_{t+1} + \nu^i_{t+1}.$$

Now

$$\begin{aligned}
\mathrm{E}[r_{t+1}^i] &= \mathrm{E}[r_{t+1}^*] + w^i\,\mathrm{E}[z_{t+1}^*],\\
\sigma^2[r_{t+1}^i] &= \sigma^2[r_{t+1}^* + w^i z_{t+1}^*] + \sigma^2[\nu_{t+1}^i].
\end{aligned}$$

For each mean, there is a unique w^i. Returns with $\nu_{t+1}^i = 0$ are minimum variance for each mean. □

However, it is not possible to show that: $r_{t+1}^{mv} \in R_{t+1}$ is on the unconditional MVF for $R_{t+1} \overset{[1]}{\Leftrightarrow}$ $m_{t+1} = a + b r_{t+1}^{mv}$ with $a, b \in \mathcal{F}_t^0$ is an sdf with respect to $\mathcal{F}_t^0$ for $X_{t+1} \overset{[2]}{\Leftrightarrow}$

$$\forall r_{t+1}^i \in R_{t+1} : \mathrm{E}[r_{t+1}^i] - \alpha_{mv} = \beta^{i,mv}\lambda_{mv},$$

where

$$\beta^{i,mv} = \frac{\operatorname{cov}[r_{t+1}^i, r_{t+1}^{mv}]}{\operatorname{var}[r_{t+1}^{mv}]}, \quad \lambda_{mv} = \mathrm{E}[r_{t+1}^{mv}] - \alpha_{mv} \quad \text{and} \quad \alpha_{mv} = \frac{1}{\mathrm{E}[m_{t+1}]}.$$

Let's see where the proof of $\overset{[1]}{\Leftrightarrow}$ breaks down. Take an arbitrary $r_{t+1} \in R_{t+1}$ and set

$$m_{t+1} = a + b r_{t+1} = a + b(r_{t+1}^* + w z_{t+1}^* + \nu_{t+1})$$

where $a, b, w \in \mathcal{F}_t^0$ and w is chosen so $\mathrm{E}[z_{t+1}^* \nu_{t+1}] = 0$. We will not be able to show that m_{t+1} is a valid sdf for X_{t+1} iff $\nu_{t+1} = 0$.

Now m_{t+1} must at least price r_{t+1}^* and z_{t+1}^*, so

$$\begin{aligned}
1 = \mathrm{E}[m_{t+1} r_{t+1}^*] &= a\,\mathrm{E}[r_{t+1}^*] + b\,\mathrm{E}[r_{t+1}^{*2}],\\
0 = \mathrm{E}[m_{t+1} z_{t+1}^*] &= a\,\mathrm{E}[z_{t+1}^*] + bw\,\mathrm{E}[z_{t+1}^{*2}]\\
&= a\,\mathrm{E}[z_{t+1}^*] + bw\,\mathrm{E}[z_{t+1}^*]\\
\Rightarrow 0 &= a + bw.
\end{aligned}$$

Solving for a and b,

$$a = \frac{w}{w\,\mathrm{E}[r_{t+1}^*] - \mathrm{E}[r_{t+1}^{*2}]}, \quad b = -\frac{a}{w}.$$

So if this m_{t+1} works it must be

$$m_{t+1} = \frac{w - (r_{t+1}^* + w z_{t+1}^* + \nu_{t+1})}{w\,\mathrm{E}_[rt + 1^*] - \mathrm{E}[r_{t+1}^{*2}]}.$$

Now to price an arbitrary $x_{t+1}^i \in X_{t+1}$, note that

$$x_{t+1}^i = p_t(x_{t+1}^i)(r_{t+1}^* + w^i z_{t+1}^* + \nu_{t+1}^i),$$

where $p_t(x_{t+1}^i) \in \mathcal{F}_t$ need not be in $\mathcal{F}_t^0$. So

$$\begin{aligned}
&\mathrm{E}[m_{t+1} x_{t+1}^i] =\\
&\qquad \mathrm{E}\left[\frac{(w - (r_{t+1}^* + w z_{t+1}^* + n_{t+1}))\,p_t(x_{t+1}^i)(r_{t+1}^* + w^i z_{t+1}^* + \nu_{t+1}^i))}{w\,\mathrm{E}[r_{t+1}^*] - \mathrm{E}[r_{t+1}^{*2}]}\right]
\end{aligned}$$

$$\Rightarrow \mathrm{E}[m_{t+1}x^i_{t+1}] = \frac{\mathrm{E}[p_t(x^i_{t+1})(wr^*_{t+1} - r^{*2}_{t+1} - \nu^i_{t+1}(w - r^*_{t+1} - wz^*_{t+1})]}{w\,\mathrm{E}[r^*_{t+1}] - \mathrm{E}[r^{*2}_{t+1}]} - \frac{\mathrm{E}[\nu_{t+1}r^i_{t+1}]}{w\,\mathrm{E}[r^*_{t+1}] - \mathrm{E}[r^{*2}_{t+1}]}.$$

Know that, for m_{t+1} to be an sdf for X_{t+1} wrt $\mathcal{F}^0_t$, need

$$\forall x^i_{t+1} \in X_{t+1} : \mathrm{E}[m_{t+1}x^i_{t+1}] = \mathrm{E}[p_t(x^i_{t+1})].$$

Note that $\nu_{t+1} = 0$ does not imply that this definition is satisfied for all $x^i_{t+1} \in X_{t+1}$, which is why the proof of $\overset{[1]}{\Leftrightarrow}$ for theorem 10 breaks down here. Note that we can only prove the $\Rightarrow$ part of $\overset{[2]}{\Leftrightarrow}$ using theorem 11.

Finally, we can use theorem 10 to prove the following result:

Theorem 25. *$r^{mv}_{t+1} \in R_{t+1}$ is on the unconditional MVF for R^0_{t+1} $\overset{[1]}{\Leftrightarrow}$ $m_{t+1} = a + br^{mv}_{t+1}$ with $a, b \in \mathcal{F}^0_t$ is an sdf with respect to $\mathcal{F}^0_t$ for X^0_{t+1} $\overset{[2]}{\Leftrightarrow}$*

$$\forall r^i_{t+1} \in R^0_{t+1} : \mathrm{E}[r^i_{t+1}] - \alpha_{mv} = \beta^{i,mv}\lambda_{mv},$$

where

$$\beta^{i,mv} = \frac{\mathrm{cov}[r^i_{t+1}, r^{mv}_{t+1}]}{\mathrm{var}[r^{mv}_{t+1}]}, \quad \lambda_{mv} = \mathrm{E}[r^{mv}_{t+1}] - \alpha_{mv} \quad \textit{and} \quad \alpha_{mv} = \frac{1}{\mathrm{E}[m_{t+1}]}.$$

PROOF. Same as for theorem 10 with $\mathcal{F}_t$ set equal to $\mathcal{F}^0_t$. Note that X_{t+1} becomes X^0_{t+1} when $\mathcal{F}_t$ is set equal to $\mathcal{F}^0_t$. □

5.5. Conditional vs unconditional MVF

Theorem 26. *r^{mv}_{t+1} lies on the unconditional MVF for R_{t+1} $\not\Leftarrow\Rightarrow$ r^{mv}_{t+1} lies on the conditional MVF for R_{t+1} wrt $\mathcal{F}_t$.*

See figure 8 on page 39.

PROOF.

$\Rightarrow$: We have shown above that any return on the unconditional MVF for R_{t+1} can be written

$$r^{mv}_{t+1} = r^*_{t+1} + wx^*_{t+1}, \quad w \in \mathcal{F}^0_t.$$

In chapter 3, we show that any return on the MVF for R_{t+1} with respect to $\mathcal{F}_t$ can be written

$$r^{mv}_{t+1} = r^*_{t+1} + w_t x^*_{t+1}, \quad w_t \in \mathcal{F}_t.$$

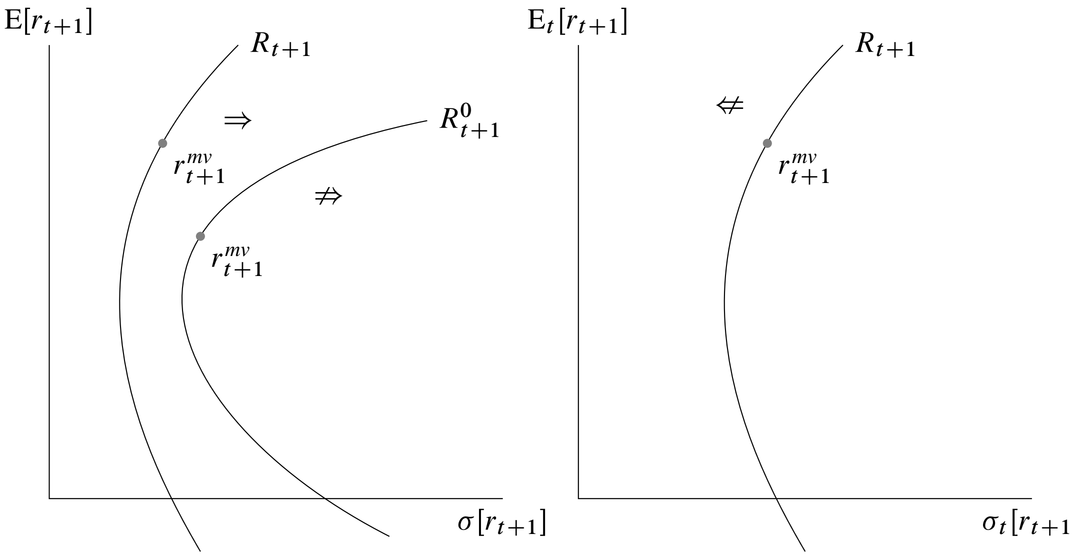
Since $w \in \mathcal{F}^0_t$ implies $w \in \mathcal{F}_t$, this completes the proof.

$\not\Leftarrow$: See counterexample A. □

Note. r^{mv}_{t+1} lies on the unconditional MVF for R^0_{t+1} ($\not\Leftarrow$?) $\not\Rightarrow$ r^{mv}_{t+1} lies on the conditional MVF for R_{t+1} wrt $\mathcal{F}_t$. For $\not\Rightarrow$: see counterexample B below.

Note. r^{mv}_{t+1} lies on the unconditional MVF for R^0_{t+1} $\not\Leftarrow$ ($\not\Rightarrow$?) r^{mv}_{t+1} lies on the conditional MVF for R^0_{t+1} wrt $\mathcal{F}_t$. For $\not\Leftarrow$: see counterexample A.

FIGURE 8. $\mathcal{F}_t$-conditional and unconditional MVFs



BRUTE FORCE PROOF OF $\Rightarrow$. The $\Rightarrow$ part of the theorem can be restated as r_{t+1} *not* on the $\mathcal{F}_t$-conditional MVF for $R_{t+1} \Rightarrow r_{t+1}$ not on the unconditional MVF for R_{t+1}.

Suppose without loss of generality that $\mathcal{F}_t$ is generated by z_t, where Z_t is the set of values taken by z_t, and that $\hat{r}_{t+1}$ is not on the $\mathcal{F}_t$-conditional MVF for R_{t+1}. Letting $\hat{Z}_t$ be the set of z_t values for which $\hat{r}_{t+1}$ is not on the $\mathcal{F}_t$-conditional MVF, $\mathrm{Prob}(z_t \in \hat{Z}_t) > 0$ using $\mathcal{F}^0_t$.

Now there exists $\hat{\hat{w}}(z_t)$ such that defining $\hat{\hat{r}}_{t+1} = \hat{\hat{w}}(z_t)\boldsymbol{r}_{t+1}$

$$\forall z_t \in Z_t : \mathrm{E}_t[\hat{\hat{r}}_{t+1}] = \mathrm{E}_t[\hat{r}_{t+1}], \tag{7}$$

$$\begin{aligned} &\forall z_t \in \hat{Z}_t : \sigma_t[\hat{\hat{r}}_{t+1}] < \sigma_t[\hat{r}_{t+1}], \\ &\forall z_t \notin \hat{Z}_t : \sigma_t[\hat{\hat{r}}_{t+1}] \leq \sigma_t[\hat{r}_{t+1}]. \end{aligned} \tag{8}$$

Statement (7) implies that

$$\mathrm{E}[\hat{\hat{r}}_{t+1}] = \mathrm{E}[\hat{r}_{t+1}]. \tag{9}$$

Statements (7) and (8) imply that

$$\begin{aligned} &\forall z_t \in \hat{Z}_t : \mathrm{E}_t[\hat{\hat{r}}^2_{t+1}] < \mathrm{E}_t[\hat{r}^2_{t+1}], \\ &\forall z_t \notin \hat{Z}_t : \mathrm{E}_t[\hat{\hat{r}}^2_{t+1}] \leq \mathrm{E}_t[\hat{r}^2_{t+1}], \end{aligned}$$

which again imply that

$$\mathrm{E}[\hat{\hat{r}}^2_{t+1}] < \mathrm{E}[\hat{r}^2_{t+1}] \tag{10}$$

since $\mathrm{Prob}(z_t \in \hat{Z}_t) > 0$ using $\mathcal{F}^0_t$. Statements (9) and (10) imply that

$$\sigma[\hat{\hat{r}}_{t+1}] < \sigma[\hat{r}_{t+1}],$$

so $\hat{r}_{t+1}$ does *not* lie on the unconditional MVF for R_{t+1}. $\square$

Why does the $\Rightarrow$ proof break down with R^0_{t+1}? Note that $\hat{\hat{r}}_{t+1}$ is formed using weight vector $\hat{\hat{w}}(z_t) \in \mathcal{F}_t$. Unless $\hat{\hat{w}}(z_t) = \hat{\hat{w}}$, a constant vector for all $z_t \in Z_t$, then $\hat{\hat{r}}_{t+1}$ does not belong in R^0_{t+1}.

Counterexample A. This counterexample illustrates the $\not\Leftarrow$ part of the theorem.

Suppose that $r^f_{t+1} \in \mathcal{F}^0_t$. There are two possible states given $\mathcal{F}_t$, each equally likely. r^p_{t+1} lies on the $\mathcal{F}_t$-conditional MVF for R_{t+1}, so

$$
\begin{aligned}
&\mathrm{E}[r^p_{t+1} \,|\, s_1] = r^f_{t+1} + \delta, \quad \sigma^2[r^p_{t+1} \,|\, s_1] > 0,\\
&\mathrm{E}[r^p_{t+1} \,|\, s_2] = r^f_{t+1} - \delta, \quad \sigma^2[r^p_{t+1} \,|\, s_2] > 0\\
&\Rightarrow \mathrm{E}[r^p_{t+1}] = r^f_{t+1},
\end{aligned}
$$

but

$$
\underset{>0}{\sigma^2[r^p_{t+1}]} = \underbrace{\mathrm{E}[\sigma^2_t[r^p_{t+1}]]}_{>0} + \underbrace{\sigma^2[\mathrm{E}_t[r^p_{t+1}]]}_{\geq 0},
$$

so r^p_{t+1} does *not* lie on the unconditional MVF for R_{t+1}.

This counterexample also works replacing R_{t+1} with R^0_{t+1}. So it also shows that r^{mv}_{t+1} lies on the unconditional MVF for $R^0_{t+1} \not\Leftarrow r^{mv}_{t+1}$ lies on the conditional MVF for R^0_{t+1} wrt $\mathcal{F}_t$.

Counterexample B. This counterexample illustrates r^{mv}_{t+1} lies on the unconditional MVF for $R^0_{t+1} \not\Rightarrow r^{mv}_{t+1}$ lies on the conditional MVF for R_{t+1} wrt $\mathcal{F}_t$. (see figure 9 on page 42).

In this counterexample, there are two states, 1 and 2, each equally likely, so Prob(1) = Prob(2) = 0.5. Turning to the payoff space, there are two risky assets, A and B, and a riskless asset, whose returns satisfy:

$$
r_{f|1} = 1 = r_{f|2}, \quad \mu_{A|1} = 1.1 = \mu_{A|2}, \quad \sigma^2_{A|1} = 0.01 = \sigma^2_{A|2},
$$

$$
\mu_{B|1} = 1.3, \quad \mu_{B|2} = 0.7, \quad \sigma^2_{B|1} = 0.01 = \sigma^2_{B|2}, \quad \sigma_{A,B|1} = \sigma_{A,B|2} = 0,
$$

where $r_{f|s}$ is the riskfree rate in state s, $\mu_{i|s}$ is the conditional expected return on risky asset i in state s, $\sigma_{i|s}$ is the conditional standard deviation of return on risky asset i in state s, and $\sigma_{i,j|s}$ is the conditional covariance of returns on risky assets i and j in state s. We are interested in the conditional MVF for R_{t+1} and the unconditional MVF for R^0_{t+1}. With a risk free asset and 2 uncorrelated assets A and B, the weight vector for the tangency portfolio P is given by

$$
w_P = \frac{V^{-1}(\boldsymbol{\mu} - r_f \boldsymbol{i})}{\boldsymbol{i}' V^{-1}(\boldsymbol{\mu} - r_f \boldsymbol{i})} = \begin{bmatrix} \frac{\mu_A - r_f}{\sigma^2_A} \\ \frac{\mu_B - r_f}{\sigma^2_B} \end{bmatrix} \frac{1}{\left(\dfrac{\mu_A - r_f}{\sigma^2_A} + \dfrac{\mu_B - r_f}{\sigma^2_B} \right)}
$$

since

$$
V = \begin{bmatrix} \sigma^2_A & 0 \\ 0 & \sigma^2_B \end{bmatrix},
$$

implies that

$$V^{-1} = \begin{bmatrix} \frac{1}{\sigma_A^2} & 0 \\ 0 & \frac{1}{\sigma_B^2} \end{bmatrix}.$$

We next apply these formulas to obtain the weight vector for the tangency portfolio in each state.

$$w_{P|1} = \begin{bmatrix} \frac{0.1}{0.01} \\ \frac{0.3}{0.01} \end{bmatrix} \frac{1}{10+30} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}, \quad w_{P|2} = \begin{bmatrix} \frac{0.1}{0.01} \\ \frac{-0.3}{0.01} \end{bmatrix} \frac{1}{10-30} = \begin{bmatrix} -0.5 \\ 1.5 \end{bmatrix}.$$

Thus, any portfolio on the conditional MVF for R_{t+1} is a combination of r^f and a risky-asset portfolio which has these tangency-portfolio weights in A and B in each state. Thus, asset A is not on the conditional MVF for R_{t+1}.

Now we can calculate unconditional moments for returns as follows:

$$\mu_A = 1.1, \quad \sigma_A^2 = 0.01, \quad \mu_B = 1,$$
$$\sigma_B^2 = \mathrm{E}[\sigma_{B|s}^2] + \sigma^2[\mu_{B|s}] = 0.01 + (\tfrac{1}{2}0.3^2 + \tfrac{1}{2}0.3^2) = 0.01 + 0.09 = 0.1,$$

and

$$\sigma_{A,B} = \mathrm{E}[\sigma_{A,B|s}] + \sigma[\mu_{A|s}, \mu_{B|s}] = 0 + 0 = 0.$$

We next calculate the weight vector for the tangency portfolio when using unconditional moments.

$$w_P = \begin{bmatrix} \frac{0.1}{0.01} \\ \frac{0}{0.1} \end{bmatrix} \frac{1}{10} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus when using unconditional moments, $P = A$. So asset A is on the unconditional MVF for $R_{t+1}^{\mathbf{0}}$, but is *not* on the conditional MVF for R_{t+1}.

5.6. Extending the results of this chapter

Let $\hat{\mathcal{F}}_t \subset \mathcal{F}_t$ and $\mathrm{E}_{\hat{t}}[\cdot] \equiv \mathrm{E}[\cdot|\hat{\mathcal{F}}_t]$ as before. All the results in this chapter continue to hold if we replace $\mathcal{F}_t^{\mathbf{0}}$ with $\hat{\mathcal{F}}_t$. So we can replace $\mathcal{F}_t^{\mathbf{0}}$ with $\hat{\mathcal{F}}_t$, $X_{t+1}^{\mathbf{0}}$ and $R_{t+1}^{\mathbf{0}}$ with

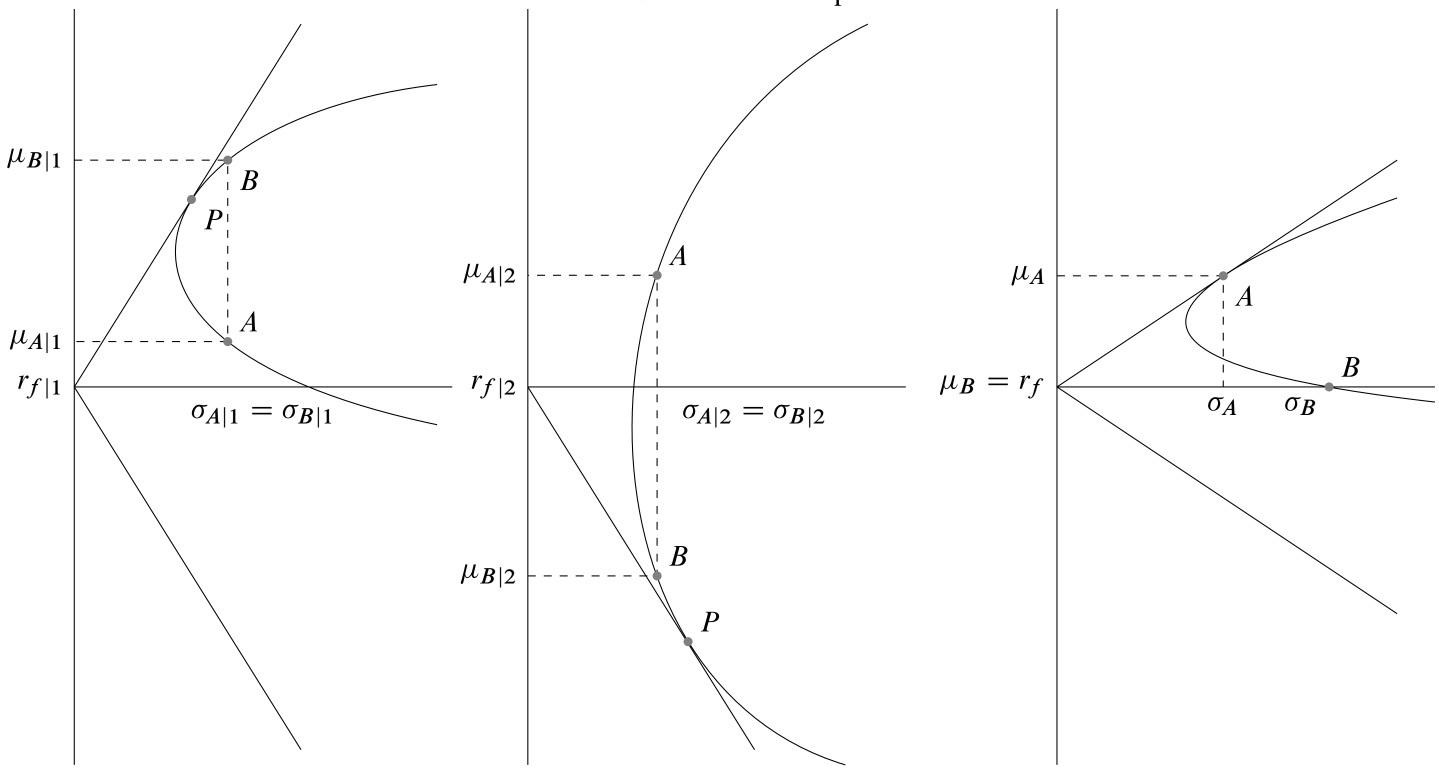
$$\hat{X}_{t+1} = \{\boldsymbol{\omega}_{\hat{t}}' \boldsymbol{r}_{t+1} \,|\, \boldsymbol{\omega}_{\hat{t}} \in \hat{\mathcal{F}}_t\} \text{and}$$
$$\hat{R}_{t+1} = \{\boldsymbol{\omega}_{\hat{t}}' \boldsymbol{r}_{t+1} \,|\, \boldsymbol{\omega}_{\hat{t}} \in \hat{\mathcal{F}}_t, \boldsymbol{\omega}_{\hat{t}}' \boldsymbol{i}_N = 1\},$$

which are defined as before, and R_{t+1}^A with

$$R_{t+1}^{\hat{A}} \equiv \{r_{t+1} \in X_{t+1} \,|\, E_{\hat{t}}[p_t(r_{t+1})] = 1\},$$

and all the results of this chapter continue to hold.

FIGURE 9. Counterexample B



Cond. MVF in State 1 for R_{t+1} Cond. MVF in State 2 for R_{t+1} Uncond. MVF for R^0_{t+1}

5.7. Joint test problem: Market efficiency and candidate sdf

Definition (Market efficiency)**.** A market is efficient if the price $p_t(\cdot)$ is an unbiased estimate of value given all available information at time t $(\mathcal{F}_t)$.

Value depends on the choice of model, i.e. the choice of sdf, so the definition can be quantified as follows:
A market is efficient with respect to $\mathcal{F}_t$ using m_{t+1} if

$$\forall x_{t+1} \in X_{t+1} : \mathrm{E}_t[m_{t+1}x_{t+1}] - p_t(x_{t+1}) = 0. \tag{11}$$

So (11) holding depends on: the choice of m_{t+1} (model) and the $p_t(\cdot)$ functional (market efficiency).

Notice that (11) implies that

$$\forall x_{t+1} \in X_{t+1} : \mathrm{E}[(m_{t+1}x_{t+1} - p_t(x_{t+1}))] = 0. \tag{12}$$

So if (12) does *not* hold, then (11) does *not* hold.

Notice that (12) implies (11), but, letting $\hat{X}_{t+1} \subsetneq X_{t+1}$

$$\forall x_{t+1} \in \hat{X}_{t+1} : \mathrm{E}[(m_{t+1}x_{t+1} - p_t(x_{t+1}))z_t] = 0 \not\Rightarrow (11),$$

so it is difficult to prove market efficiency.

CHAPTER 6

Prices in a Multiperiod Setting

6.1. Basics

We want to price an asset i that pays a stream of dividends d_t^i, d_{t+1}^i. At each time $\tau \geq t$, there exists a set of sdf $M_{\tau+1}$ such that $m_{\tau+1} \in M_{\tau+1}$ implies that

$$\forall x_{\tau+1} \in X_{\tau+1} : \mathrm{E}_\tau[m_{\tau+1} x_{\tau+1}] = p_\tau(x_{\tau+1}).$$

Let p_τ^i be the price at τ of the remaining $d_{\tau+1}^i$. We know that $x_{\tau+1}^i = d_{\tau+1}^i + p_{\tau+1}^i$, and so since $x_{\tau+1}^i \in X_{\tau+1}$,

$$\mathrm{E}_\tau[m_{\tau+1}(d_{\tau+1}^i + p_{\tau+1}^i)] = p_\tau^i \quad \text{for } \tau \geq t. \tag{13}$$

Since this is true for all $\tau \geq t$, we can start with

$$\mathrm{E}_t[m_{t+1}(d_{t+1}^i + p_{t+1}^i)] = p_t^i, \tag{14}$$

$$\mathrm{E}_{t+1}[m_{t+2}(d_{t+2}^i + p_{t+2}^i)] = p_{t+1}^i. \tag{15}$$

Substitute (15) into (14),

$$\mathrm{E}_t[m_{t+1}(d_{t+1}^i + \mathrm{E}_{t+1}[m_{t+2}(d_{t+2}^i + p_{t+2}^i)])] = p_t^i,$$

and using the law of iterated expectations,

$$\mathrm{E}_t[m_{t+1}(d_{t+1}^i + m_{t+2}(d_{t+2}^i + p_{t+2}^i))] = p_t^i. \tag{16}$$

Iterating forward gives

$$\mathrm{E}_t\left[\sum_{\tau=1}^{T}\left(\prod_{j=1}^{\tau} m_{t+j}\right) d_{t+\tau}^i\right] + \mathrm{E}_t\left[\left(\prod_{j=1}^{T} m_{t+j}\right) p_{t+T}^i\right] = p_t^i. \tag{17}$$

Suppose that the transversality condition holds:

$$\lim_{T\to\infty} \mathrm{E}_t\left[\left(\prod_{j=1}^{T} m_{t+j}\right) p_{t+T}^i\right] = 0. \tag{TC}$$

Then

$$p_t^i = \mathrm{E}_t\left[\sum_{\tau=1}^{\infty}\left(\prod_{j=1}^{\tau} m_{t+j}\right) d_{t+\tau}^i\right]. \tag{18}$$

Suppose that $d^i_{t+\tau} = 0$ for $\tau \neq T$. Then

$$p^i_t = \mathrm{E}_t\left[\left(\prod_{j=1}^{T} m_{t+j}\right) d^i_{t+T}\right],$$

so $\left(\prod_{j=1}^{T} m_{t+j}\right)$ is a valid T-period sdf that prices time $t+T$ payoffs at t.

$p_\tau + i$ and d^i_τ are not usually stationary, so rewrite (18) in terms of price-dividend ratio p^i_t/d^i_t and dividend growth rates

$$\eta^i_{t+\tau} = \frac{d^i_{t+\tau}}{d^i_{t+\tau-1}}.$$

Then (18) becomes

$$\mathrm{E}_t\left[\sum_{\tau=1}^{\infty}\left(\prod_{j=1}^{\tau} m_{t+j}\eta^i_{t+j}\right)\right] = \frac{p^i_t}{d^i_t}. \tag{19}$$

6.2. Bubbles

Bubbles occur when (TC) is not satisfied.

Example. Suppose $d^i_\tau = 0$ for $\tau \geq t+1$. First, $p^i_\tau = 0$ for $\tau \geq t$ is the only solution to (13) that satisfies (TC) at all $\tau \geq t$. Now consider $\{p^i_\tau\}_{\tau \geq t}$ satisfying

$$p^i_t = c^i \quad \text{and} \quad p^i_{\tau+1} = \frac{p^i_\tau}{\mathrm{E}_\tau[m_{\tau+1}]}, \quad \tau \geq t.$$

Clearly such a p^i_τ satisfies (0) but need not satisfy (TC) since

$$\mathrm{E}_t\left[\left(\prod_{j=1}^{T} m_{t+j}\right) p^i_{t+T}\right] = p^i_t \quad \text{for all } T.$$

In general, when the (TC) is not imposed, a multiplicity of price processes can support $\{d^i_\tau\}_{\tau>t}$ for a given $\{m_\tau\}_{\tau>t}$.

6.3. Volatility tests

Imposing (TC) implies (18) or (19). Taking the variance of both sides of each gives

$$\mathrm{var}[p^i_t] = \mathrm{var}\left[\mathrm{E}_t\left[\sum_{\tau=1}^{\infty}\left(\prod_{j=1}^{\tau} m_{t+j}\right) d^i_{t+\tau}\right]\right] \leq \mathrm{var}\left[\sum_{\tau=1}^{\infty}\left(\prod_{j=1}^{\tau} m_{t+j}\right) d^i_{t+\tau}\right]$$

and

$$\mathrm{var}\left[\frac{p^i_t}{d^i_t}\right] = \mathrm{var}\left[\mathrm{E}_t\left[\sum_{\tau=1}^{\infty}\left(\prod_{j=1}^{\tau} m_{t+j}\right)\left(\prod_{j=1}^{\tau} \eta^i_{t+j}\right)\right]\right]$$
$$\leq \mathrm{var}\left[\sum_{\tau=1}^{\infty}\left(\prod_{j=1}^{\tau} m_{t+j}\right)\left(\prod_{j=1}^{\tau} \eta^i_{t+j}\right)\right].$$

If $m_\tau = m$ for $\tau \geq t$, then

$$\mathrm{var}[p_t^i] \leq \mathrm{var}\left[\sum_{\tau=1}^{\infty} m^\tau d_{t+\tau}^i\right],$$

and

$$\mathrm{var}\left[\frac{p_t^i}{d_t^i}\right] \leq \mathrm{var}\left[\sum_{\tau=1}^{\infty} m^\tau \left(\prod_{j=1}^{\tau} \eta_{t+j}^i\right)\right].$$

These inequalities may be violated because: m_τ is not constant; or, (TC) is violated.

6.4. Asset specific sdfs

Suppose $\{m_{\tau+1}\}_{\tau\geq t}$ is allowed to be asset specific so $\{m_{\tau+1}^i\}_{\tau\geq t}$ is an sdf for asset i if

$$\mathrm{E}_\tau[m_{\tau+1}^i(d_{\tau+1}^i + p_{\tau+1}^i)] = p_\tau^i \quad \text{for } \tau \geq t.$$

This greatly expands the set of valid sdfs.

A particularly useful asset specific sdf is $\{(r_{\tau+1}^i)^{-1}\}_{\tau\geq t}$. Notice that

$$\begin{aligned} 1 &= (r_{\tau+1}^i)^{-1} r_{\tau+1}^i \\ \Leftrightarrow\ p_\tau^i &= (r_{\tau+1}^i)^{-1}(p_{\tau+1}^i + d_{\tau+1}^i) \\ \Rightarrow\ p_\tau^i &= \mathrm{E}_\tau[(r_{\tau+1}^i)^{-1}(p_{\tau+1}^i + d_{\tau+1}^i)]. \end{aligned} \tag{20}$$

Further using (20), we can get an even stronger result than (17),

$$\sum_{\tau=1}^{T}\left(\prod_{j=1}^{\tau}(r_{t+j}^i)^{-1}\right) d_{t+\tau}^i + \left(\prod_{j=1}^{T}(r_{t+j}^i)^{-1}\right) p_{t+T}^i = p_t^i.$$

Assuming $\lim_{T\to\infty}(\prod_{j=1}^{T}(r_{t+j}^i)^{-1})p_{t+T}^i = 0$, then

$$\sum_{\tau=1}^{\infty}\left(\prod_{j=1}^{\tau}(r_{t+j}^i)^{-1}\right) d_{t+\tau}^i = p_\tau^i. \tag{21}$$

Notice that this result is tautological: it must be true. This is the danger of using an asset specific sdf.

Taking expectations of (21) also gives (17) with $m_{t+\tau} = (r_{t+\tau}^i)^{-1}$. Dividing through by d_t^i gives

$$\mathrm{E}_t\left[\sum_{\tau=1}^{\infty}\left(\prod_{j=1}^{\tau} r_{t+j}^i\right)^{-1}\left(\prod_{j=1}^{\tau}\eta_{t+j}^i\right)\right] = \frac{p_t^i}{d_t^i}. \tag{22}$$

6.5. An approximate pricing relation

We can use a Taylor series expansion to obtain the approximate relation

$$\frac{p_t^i}{d_t^i} \approx c + \frac{1}{1-\Omega}\,\mathrm{E}_t\left[\sum_{j=1}^{\infty}\Omega^j(\hat{\eta}_{t+j}^i - \hat{r}_{t+j}^i)\right],$$

where

$$\hat{\eta}^i_\tau = \ln d^i_\tau - \ln d^i_{\tau-1}, \quad \hat{r}^i_\tau = \ln r^i_\tau, \quad \Omega = \exp(\mathrm{E}[\hat{\eta}^i_\tau] - \mathrm{E}[\hat{r}^i_\tau]).$$

6.6. Another approximate pricing relation

Let

$$\hat{d}^i_\tau \equiv \ln d^i_\tau, \quad \hat{p}^i_\tau \equiv \ln p^i_\tau, \quad \hat{r}^i_\tau \equiv \ln r^i_\tau.$$

We can write

$$\begin{aligned}
\hat{r}^i_{\tau+1} &= \ln(p^i_{\tau+1} + d^i_{\tau+1}) - \ln(p^i_\tau) \\
&= \ln\left(\frac{p^i_{\tau+1} + d^i_{\tau+1}}{p^i_{\tau+1}}\right) + \hat{p}^i_{\tau+1} - \hat{p}^i_\tau \\
&= \hat{p}^i_{\tau+1} - \hat{p}^i_\tau + \ln(1 + \exp(\hat{d}^i_{\tau+1} - \hat{p}^i_{\tau+1})). \qquad (23)
\end{aligned}$$

Any nonlinear function $f(x_{\tau+1})$ can be approximated around the unconditional mean of $x_{\tau+1}$, $\bar{x}$, using a first order Taylor series expansion,

$$f(x_{\tau+1}) \approx f(\bar{x}) + f'(\bar{x})(x_{\tau+1} - \bar{x}).$$

Taking $f(x)$ to be $\ln(1 + \exp(x))$,

$$\ln(1 + \exp(x)) \approx \ln(1 + \exp(\bar{x})) + \frac{\exp(\bar{x})}{1 + \exp(\bar{x})}(x - \bar{x}).$$

Taking $x_{\tau+1}$ to be $(\hat{d}^i_{\tau+1} - \hat{p}^i_{\tau+1})$ with mean $\overline{\hat{d}^i - \hat{p}^i}$ implies that (23) can be written

$$\begin{aligned}
\hat{r}^i_{\tau+1} &\approx \hat{p}^i_{\tau+1} - \hat{p}^i_\tau + \ln(1 + \exp(\overline{\hat{d}^i - \hat{p}^i})) \\
&\qquad + \underbrace{\frac{\exp(\overline{\hat{d}^i - \hat{p}^i})}{1 + \exp(\overline{\hat{d}^i - \hat{p}^i})}}_{1-\rho^i} (\hat{d}^i_{\tau+1} - \hat{p}^i_{\tau+1} - (\overline{\hat{d}^i - \hat{p}^i})) \\
&\approx \rho^i \hat{p}^i_{\tau+1} - \hat{p}^i_\tau + \ln\tfrac{1}{\rho^i} + (1 - \rho^i)\hat{p}^i_{\tau+1} + (1 - \rho^i)(\hat{d}^i_{\tau+1} - \hat{p}^i_{\tau+1}) \\
&\qquad - (1 - \rho^i)\ln(\exp(\underbrace{\overline{\hat{d}^i - \hat{p}^i}}_{\frac{1}{\rho^i}-1})) \\
&\approx \underbrace{-\ln(\rho^i) - (1 - \rho^i)\ln(\tfrac{1}{\rho^i} - 1)}_{k^i} + \rho^i \hat{p}^i_{\tau+1} + (1 - \rho^i)\hat{d}^i_{\tau+1} - \hat{p}^i_\tau, \quad \tau \ge t.
\end{aligned}$$

Thus

$$\hat{p}^i_\tau \approx k^i + (1 - \rho^i)\hat{d}^i_{\tau+1} - \hat{r}^i_{\tau+1} + \rho^i \hat{p}^i_{\tau+1}, \quad \text{for } \tau \ge t. \qquad (24)$$

This is a linear difference equation for $\hat{p}^i$ which can be solved forward to give from t

$$\hat{p}^i_t \approx k^i \underbrace{\sum_{j=0}^{T}(\rho^i)^j}_{\frac{1-\rho^{T+1}}{1-\rho^i}} + \sum_{j=0}^{T}(\rho^i)^j((1-\rho^i)\hat{d}^i_{t+j+1} - \hat{r}^i_{t+j+1}) + (\rho^i)^T \hat{p}^i_T$$

for any $T \geq t$.

So long as

$$\lim_{T\to\infty} \rho^T \hat{p}^i_{t+T} = 0,$$

then

$$\hat{p}^i_t \approx \frac{k^i}{1-\rho^i} + \sum_{j=0}^{\infty}(\rho^i)^j((1-\rho^i)\hat{d}^i_{t+j+1} - \hat{r}^i_{t+j+1}).$$

We can take expectations to get

$$\hat{p}^i_t \approx \frac{k^i}{1-\rho^i} + \mathrm{E}_t\left[\sum_{j=0}^{\infty}(\rho^i)^j((1-\rho^i)\hat{d}^i_{t+j+1} - \hat{r}^i_{t+j+1})\right]. \tag{25}$$

We can also obtain an expression for the log price dividend ratio,

$$\hat{p}^i_t - \hat{d}^i_t \approx \frac{k^i}{1-\rho^i} + \mathrm{E}_t\left[\sum_{j=0}^{\infty}(\rho^i)^j(\underbrace{\hat{\eta}^i_{t+j+1}}_{\hat{d}^i_{t+j+1}-\hat{d}^i_{t+j}} - \hat{r}^i_{t+j+1})\right]$$

since

$$\begin{aligned}
&\sum_{j=0}^{\infty}(\rho^i)^j(1-\rho^i)\hat{d}^i_{t+j+1} - \hat{d}^i_t \\
&= \hat{d}^i_{t+1} - \hat{d}^i_t - \rho^i\hat{d}^i_{t+1} + \rho^i\hat{d}_{t+1+1} - (\rho^i)^2\hat{d}_{t+1+1} + (\rho^i)^2\hat{d}_{t+1+2} \\
&\qquad - (\rho^i)^3\hat{d}_{t+1+2} + \cdots = \sum_{j=0}^{\infty}(\rho^i)^j\hat{\eta}^i_{t+1+j}.
\end{aligned} \tag{26}$$

Define $(\mathrm{E}_{t+1} - \mathrm{E}_t)(.)$ such that $(\mathrm{E}_{t+1} - \mathrm{E}_t)(y_{t+1}) \equiv \mathrm{E}_{t+1}[y_{t+1}] - \mathrm{E}_t[y_{t+1}]$. We can obtain a return decomposition as follows:

$$\begin{aligned}
\hat{r}^i_{t+1} - \mathrm{E}_t[\hat{r}^i_{t+1}] \approx k + \rho^i\hat{p}^i_{t+1} + (1-\rho^i)\hat{d}^i_{t+1} - \hat{p}^i_t - k - \rho^i\,\mathrm{E}_t[\hat{p}^i_{t+1}] \\
- (1-\rho^i)\,\mathrm{E}_t[\hat{d}^i_{t+1}] + \hat{p}^i_t \qquad \text{using (24)}
\end{aligned}$$

$$\approx \rho^i(\hat{p}^i_{t+1} - \mathrm{E}_t[\hat{p}^i_{t+1}]) + (1-\rho^i)(\hat{d}^i_{t+1} - \mathrm{E}_t[\hat{d}^i_{t+1}])$$

$$\approx -\rho^i(\mathrm{E}_{t+1} - \mathrm{E}_t)\left(\sum_{j=0}^{\infty}(\rho^i)^j \hat{r}^i_{t+2+j}\right) \qquad \text{using (25)}$$

$$+\underbrace{\rho^i(\mathrm{E}_{t+1} - \mathrm{E}_t)\left(\sum_{j=0}^{\infty}(\rho^i)^j(1-\rho^i)\hat{d}^i_{t+2+j}\right) + (1-\rho^i)(\mathrm{E}_{t+1}[\hat{d}^i_{t+1}] - \mathrm{E}_t[\hat{d}^i_{t+1}])}$$

$$\left| \begin{aligned} &= \rho^i(\mathrm{E}_{t+1} - \mathrm{E}_t)\underbrace{\left(\sum_{j=0}^{\infty}(\rho^i)^j(1-\rho^i)\hat{d}_{t+2+j} - \hat{d}^i_{t+1}\right)}_{\sum_{j=0}^{\infty}(\rho^i)^j\eta^i_{t+2+j} \text{ using (26)}} + (\mathrm{E}_{t+1} - \mathrm{E}_t)(\hat{d}^i_{t+1}) \\ &= (\mathrm{E}_{t+1} - \mathrm{E}_t)\left(\sum_{j=0}^{\infty}(\rho^i)^{j+1}\hat{\eta}^i_{t+2+j} + \hat{d}^i_{t+1} - \underbrace{\hat{d}^i_t}_{\substack{\uparrow \\ \text{can add since } (\mathrm{E}_{t+1} - \mathrm{E}_t)(\hat{d}^i_{t+1}) = 0}}\right) \end{aligned} \right|$$

$$\approx -(\mathrm{E}_{t+1} - \mathrm{E}_t)\left(\sum_{\hat{j}=1}^{\infty}(\rho^i)^{\hat{j}}\hat{r}^i_{t+1+\hat{j}}\right) + (\mathrm{E}_{t+1} - \mathrm{E}_t)\left(\sum_{\hat{j}=0}^{\infty}(\rho^i)^{\hat{j}}\hat{\eta}^i_{t+1+\hat{j}}\right) \quad \text{setting } \hat{j} = j+1.$$

6.7. Risk neutral probability measure

Suppose there is no arbitrage in the setting of chapter 2. Then for all $\tau \geq 1$, there exists $m_{t+\tau} \in M_{t+\tau}$, a strictly positive sdf for $\tilde{X}_{t+\tau}$ wrt $\mathcal{F}_{t+\tau-1}$. Define $p^\tau_t(.)$ such that $p^\tau_t(x_{t+\tau})$ is the time-t price for the payoff $x_{t+\tau}$ received at time $t+\tau$.

One-period setting. In a one-period setting we know that

$$\begin{aligned} p_t(x_{t+1}) &= \mathrm{E}_t[x_{t+1}m_{t+1}] \quad \text{for all } x_{t+1} \in \tilde{X}_{t+1} \\ &= \int x_{t+1}(s)m_{t+1}(s)f^1_t(s)\,ds. \end{aligned}$$

Let $f^\tau_t(s)$ be the conditional density for the possible states at $t+\tau$ given $\mathcal{F}_t$.

Define

$$f^{1*}_t(s) = \frac{1}{\mathrm{E}_t[m_{t+1}]}m_{t+1}(s)f_t(s)$$

to be the risk neutral measure. Notice that $f^{1*}_t(s) \geq 0$ and $\int f^{1*}_t(s)\,ds = 1$.

So

$$\begin{aligned} p_t(x_{t+1}) &= \int \mathrm{E}_t[m_{t+1}]x_{t+1}(s)f^{1*}_t(s)\,ds \\ &= \mathrm{E}^*_t\left[\frac{1}{r^f_{t+1}}x_{t+1}\right] \quad \text{for all } x_{t+1} \in \tilde{X}_{t+1}, \end{aligned}$$

where $\mathrm{E}^*_t[\cdot]$ is a conditional expectation with respect to the risk neutral measure.

So using the risk neutral measure, prices are obtained by discounting back at r^f_{t+1} and then taking expectations.

Multiperiod setting: Natural extension of single period measure. We know that

$$\begin{aligned} p^T_t(x_{t+T}) &= \mathrm{E}_t\left[\left(\prod_{j=1}^T m_{t+j}\right)x_{t+T}\right] \quad \text{for all } x_{t+T} \in \tilde{X}_{t+T} \\ &= \int x_{t+T}(s)\left(\prod_{j=1}^T m_{t+j}\right)(s) f^T_t(s)\, ds. \end{aligned}$$

Define

$$f^{T+}_t(s) = \frac{1}{\mathrm{E}_t\left[\prod_{j=1}^T m_{t+j}\right]}\left(\prod_{j=1}^T m_{t+j}\right)(s) f^T_t(s)$$

to be the risk neutral measure. Notice that the scaling term is an element of $\mathcal{F}_t$. Notice that $f^{T+}_t(s) \geq 0$ and that

$$\int f^{T+}_t(s) = \int \frac{1}{\mathrm{E}_t\left[\prod_{j=1}^T m_{t+j}\right]}\left(\prod_{j=1}^T m_{t+j}\right)(s) f^T_t(s)\, ds = 1.$$

So

$$\begin{aligned} p^T_t(x_{t+T}) &= \int \mathrm{E}_t\left[\prod_{j=1}^T m_{t+j}\right] x_{t+T}(s) f^{T+}_t(s)\, ds \\ &= \mathrm{E}^+_t\left[\frac{1}{r^{T\text{-period } f}_{t+T}} x_{t+T}\right], \end{aligned}$$

where E^+_t is conditional expectation with respect to this risk neutral measure and $r^{T\text{-period } f}_{t+T}$ is the yield at time t on a T-period discount bond.

Multiperiod setting: More common measure. Define

$$f^{T*}_t(s) = \frac{1}{\left(\prod_{j=1}^T \mathrm{E}_{t+j-1}[m_{t+j}]\right)(s)}\left(\prod_{j=1}^T m_{t+j}\right)(s) f^T_t(s)$$

to be the risk neutral measure. Notice that the scaling term is *not* an element of $\mathcal{F}_t$. Notice that $f^{T*}_t(s) \geq 0$ and that

$$\mathrm{E}_t\left[\frac{\prod_{j=1}^T m_{t+j}}{\prod_{j=1}^T \mathrm{E}_{t+j-1}[m_{t+j}]}\right] = \mathrm{E}_t\left[\frac{\prod_{j=1}^{T-1} m_{t+j}\, \mathrm{E}_{t+T-1}\, m_{t+T}}{\prod_{j=1}^{T-1} \mathrm{E}_{t+j-1}[m_{t+j}]\, \mathrm{E}_{t+T-1}[m_{t+T}]}\right] = \cdots = 1.$$

So

$$\begin{aligned} p^T_t(x_{t+T}) &= \int \left(\prod_{j=1}^T \mathrm{E}_{t+j-1}[m_{t+j}]\right) x_{t+1}(s) f^{T*}_t(s)\, ds \\ &= \mathrm{E}^*_t\left[\frac{1}{\prod_{j=1}^T r^f_{t+1}} x_{t+T}\right], \end{aligned}$$

where E^*_t is conditional expectation with respect to the risk neutral measure.

6.8. Dynamically completing the market

One-period setting. Suppose the number of states S is finite and define the Arrow–Debreu security for state s to be an asset that pays 1 in state s and 0 in all other states. The market is complete if all S Arrow–Debreu securities are elements of the payoff space. The market is complete when $X_{t+1} = \mathbb{R}^S$. If N, the number of assets, is less than S, the market is incomplete. In particular, if $N < S$ and $\boldsymbol{x}_{t+1}$ is the $N \times S$ vector of basis assets,

$$\boldsymbol{x}_{t+1} = \begin{bmatrix} \boldsymbol{x}^1_{t+1} \\ \vdots \\ \boldsymbol{x}^N_{t+1} \end{bmatrix}, \quad \boldsymbol{x}^i_{t+1} \in \mathbb{R}^S, \quad i = 1, \dots, N,$$

then the payoff space

$$X_{t+1} = \left\{\boldsymbol{\omega}'\boldsymbol{x}_{t+1} \mid \boldsymbol{\omega}' \in \mathbb{R}^N\right\} \subsetneq \mathbb{R}^S.$$

Multiperiod setting. If no rebalancing is allowed between t (today) and $t+T$ (the terminal date), then the one period analysis applies. If rebalancing is allowed between t and $t+T$, then N can be less than S, but still markets are complete in the sense that the space of payoffs achievable at $t+T$ (X_{t+T}) equals $\mathbb{R}^S$.

Let $S_{t+\tau}$ be the number of states possible at time $t+\tau$ given the state at $t+\tau-1$ and define

$$S^T \equiv \prod_{\tau=1}^{\tau} S_{t+\tau}.$$

So without loss of generality, we are assuming that the number of states possible at time $t+\tau$ given the state at $t+\tau-1$ are the same, irrespective of the state at $t+\tau-1$. Consider N assets that pay off $\boldsymbol{x}_{t+T}$ at time $t+T$:

$$\boldsymbol{x}_{t+T} = \begin{bmatrix} \boldsymbol{x}^1_{t+T} \\ \vdots \\ \boldsymbol{x}^N_{t+T} \end{bmatrix},$$

where $\underset{1\times S^T}{\boldsymbol{x}^i_{t+T}} \in \mathbb{R}^{S^T}$ for $i = 1, \dots, N$. Define $\{m_{t+\tau}\}_{\tau=1}^T$ such that $m_{t+\tau}$ is the 1-period sdf for pricing assets at $t+\tau-1$ wrt $\mathcal{F}_{t+\tau-1}$ using payoffs at $t+\tau$.

One-period analysis with no rebalancing. In this setting, the payoff space is given by:

$$X^T_{t+T} = \{\boldsymbol{\omega}'_t \boldsymbol{x}_{t+T} : \boldsymbol{\omega}_t \in \mathcal{F}_t\}$$

and

$$m^T_{t+T} = \prod_{\tau=1}^{T} m_{t+\tau}$$

is a valid sdf for X^T_{t+T} wrt prices at t and $\mathcal{F}_t$. In other words,

$$\forall x_{t+T} \in X_{t+T}T : p^T_t(x_{t+T}) = \mathrm{E}_t[m^T_{t+T} x_{t+T}].$$

So if $N < S^T$, then $X^T_{t+T} \subsetneq \mathbb{R}^{S^T}$ and the market is *not* complete.

Rebalancing every period. With rebalancing every period, we can have $N < S^T$, *but* the market *may* still be dynamically complete. For each $\tau = t, \ldots, t + T - 1$,

$$p_\tau^{T+t-\tau}(x_{t+T}^i) = \mathrm{E}_\tau[m_{\tau+1} p_{\tau+1}^{T+t-\tau-1}(x_{t+T}^i)] \quad \text{for } i = 1, 2, \ldots, N,$$

and investors are able to generate

$$X_{t+T}^D = \left\{ x_{t+T} : \underbrace{\boldsymbol{\omega}_\tau' p_{\tau+1}^{T+t-\tau-1}(\boldsymbol{x}_{t+T})}_{\text{available to invest at } \tau+1} \underbrace{\begin{cases} = \boldsymbol{\omega}_{\tau+1}' p_{\tau+1}^{T+t-\tau-1}(\boldsymbol{x}_{t+T}) & \tau = t, \ldots, t+T-2, \\ = x_{t+T} & \tau = t+T-1 \end{cases}}_{\text{chosen portfolio at } \tau+1}, \right.$$

$$\left. \boldsymbol{\omega}_\tau \in \mathcal{F}_\tau, \quad \tau = t, \ldots, t+T-1 \right\},$$

where $p_{t+T}^0(\boldsymbol{x}_{t+T}) \equiv \boldsymbol{x}_{t+T}$.

It is easy to see that $X_{t+T}^D \supset X_{t+T}^T$. Now $X_{t+T}^D = \mathbb{R}^{S^T}$ and the market is dynamically complete iff

$$X_{\tau+1}(s_t, \ldots, s_\tau) = \{\boldsymbol{\omega}_\tau' p_{\tau+1}(\boldsymbol{x}_{t+T}) : \boldsymbol{\omega}_t \in \mathcal{F}_\tau\} = \mathbb{R}^{S_{\tau+1}}$$

for all $(s_t, \ldots, s_\tau)$ paths for all $\tau = t, \ldots, t + T - 1$. That is, the market is dynamically complete iff market is complete from τ to $\tau + 1$ for all $(s_t, \ldots, s_\tau)$ paths for all $\tau = t, \ldots, t + T - 1$.

Example of dynamically completing the market. Let $T = 3$, $S_\tau = 2$ for all τ, $N = 2$, and $S^T = 8$. At each point in time, there are two possible states at the next point in time, u or d, each equally likely: so prob$(u) = 0.5$ and prob$(d) = 0.5$. Define m^s to be the value taken by the 1-period sdf if state s occurs at the next point in time and assume

$$\begin{array}{ll} u & m^u = 0.4 \\ d & m^d = 1.2 \end{array}$$

Let π^s be the state price today for state s at the next point in time: so an agent pays π^s at time t for 1 in state s at time $(t + 1)$. So

$$\pi^u \begin{array}{l} 1 \\ 0 \end{array} \qquad\qquad \pi^d \begin{array}{l} 0 \\ 1 \end{array}$$

Now

$$\pi^u = \mathrm{E}[m 1_{\{s=u\}}] = \frac{1}{2} m^u \times 1 + \frac{1}{2} m^d \times 0 = 0.2$$
$$\pi^d = \mathrm{E}[m 1_{\{s=d\}}] = \frac{1}{2} m^u \times 0 + \frac{1}{2} m^d \times 1 = 0.6$$

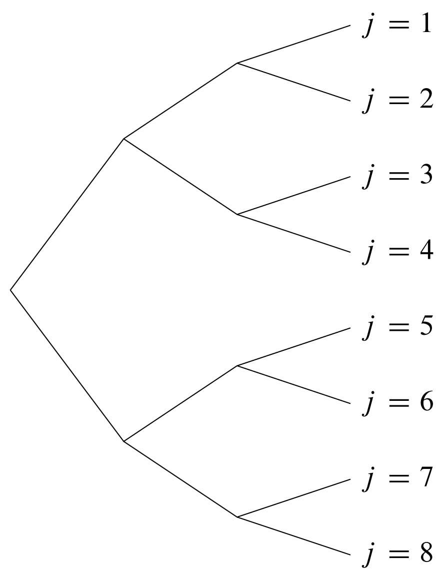
Let $\text{prob}^*(s)$ be the risk-neutral probability of state s. Then

$$\text{prob}^*(s) = \frac{\text{prob}(s)m^s}{\text{prob}(u)m^u + \text{prob}(d)m^d}$$

$$\text{prob}^*(u) = \frac{0.5 \times 0.4}{0.5 \times 0.4 + 0.5 \times 1.2} = 0.25$$

$$\text{prob}^*(u) = \frac{0.5 \times 1.2}{0.5 \times 0.4 + 0.5 \times 1.2} = 0.75$$

Let the payoff from an asset at $t + T$ be represented by a $S^T \times 1$ vector whose jth element is the payoff on the asset in the jth state.



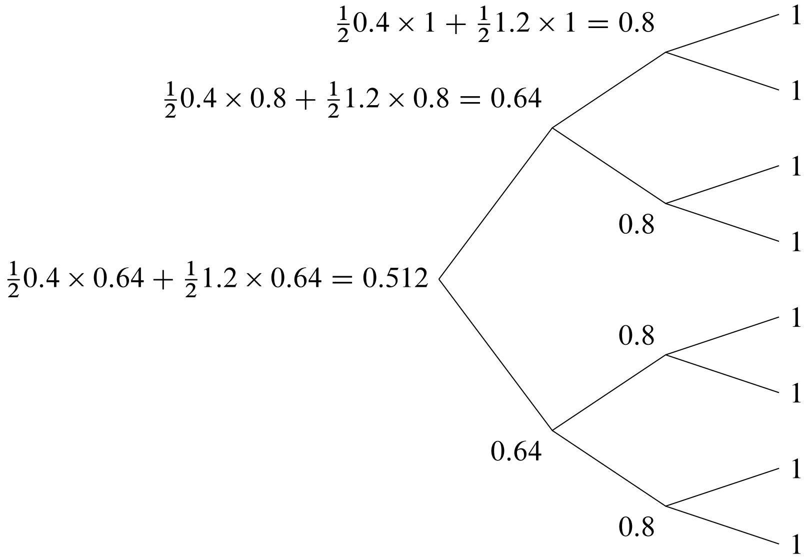
Let

$$q^j_{t+T} \equiv [0 \quad 0 \quad \dots \quad \underbrace{1}_{j\text{th element}} \quad \dots \quad 0]$$

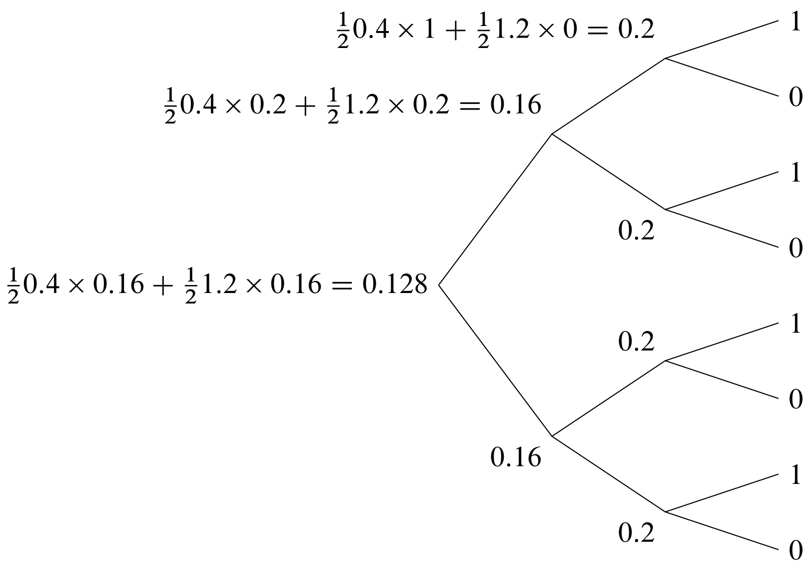
be the payoff on an asset that pays 1 in the jth state and 0 otherwise. The market is dynamically complete and $x^D_{t+T} = \mathbb{R}^8$ iff it is possible to create assets with payoff vectors q^j_{t+T} for all $j = 1, \cdots, S^T$.

Consider 3 assets with the following payoff vectors:

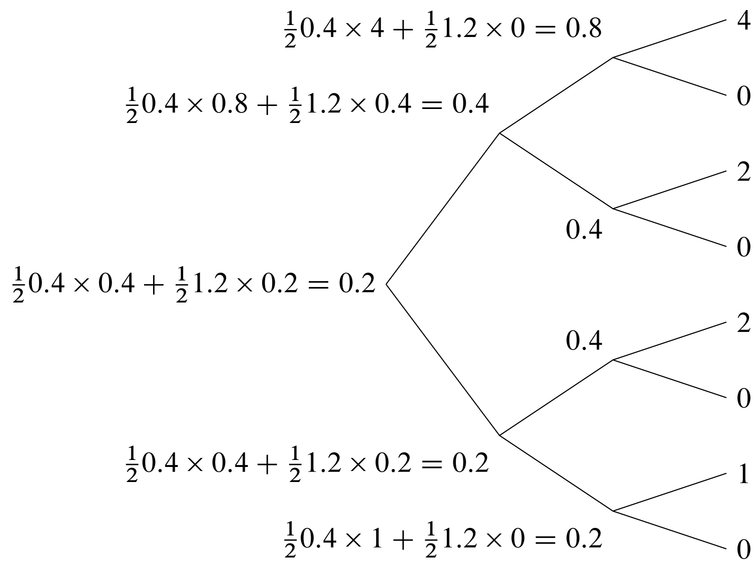
$x^1_{t+T} = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$



$x^2_{t+T} = [1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0]$



$x^3_{t+T} = [4 \quad 0 \quad 2 \quad 0 \quad 2 \quad 0 \quad 1 \quad 0]$



Economy A. Agents have access to assets 1 and 2 with payoffs x^1_{t+T} and x^2_{t+T}. It is not possible to create q^j_{t+T} for all $j = 1, 2, \cdots, 8$ and so the market is not dynamically complete and $x^D_{t+T} \subsetneq \mathbb{R}^8$.

Economy B. Agents have access to assets 1 and 3 with payoffs x^1_{t+T} and x^3_{t+T}. It is possible to create q^j_{t+T} for all $j = 1, 2, \cdots, 8$ and so the market is dynamically complete and $x^D_{t+T} = \mathbb{R}^8$.

CHAPTER 7

Utility Theory

7.1. Single period utility theory

We wish to use a concept of utility that is able to deal with uncertainty. So we introduce the von Neumann–Morgenstern utility function. The investor makes choices consistent with maximizing the expected value of the utility function. Utility is defined over either consumption or wealth, but can also be defined over a bundle of goods. von Neumann-Morgenstern utility functions can be developed axiomatically.

Suppose an individual cares about consumption c and define a lottery L as a set of consumption outcomes $(c_1, \ldots, c_m)$ and an associated set of probabilities $(\pi_1, \ldots, \pi_m)$. Let $\mathcal{L}$ be the space of lotteries. Elements of the set of consumption outcomes can be elements of $\mathcal{L}$.

Axiom 1 (Completeness): For every pair of lotteries, $L_1 \gtrsim L_2$ or $L_1 \lesssim L_2$.

Axiom 2 (Reflexivity): For every lottery, $L \gtrsim L$.

Axiom 3 (Transitivity): If $L_1 \gtrsim L_2$ and $L_2 \gtrsim L_3$, then $L_1 \gtrsim L_3$.

Axiom 4 (Continuity): For any $x, y, z \in \mathcal{L}$,

$$\{\pi \in [0, 1] : ((x, y), (\pi, (1 - \pi))) \gtrsim (z, 1)\}$$

and

$$\{\pi \in [0, 1] : ((x, y), (\pi, (1 - \pi))) \lesssim (z, 1)\}$$

are closed sets.

Axiom 5 (Independence): If $(x, 1) \sim (y, 1)$, then $((x, z), (\pi, 1 - \pi)) \sim ((y, z), (\pi, 1 - \pi))$.

Axiom 6 (Dominance): Let $L_1 = ((x_1, x_2), (\pi_1, 1 - \pi_1))$ and $L_2 = ((x_1, x_2), (\pi_2, 1 - \pi_2))$. If $x_1 \succ x_2$, then $L_1 \succ L_2 \Leftrightarrow \pi_1 > \pi_2$.

Axiom 7 (Not Necessary): If $x_1 \gtrsim x_2 \gtrsim x_3$, then there exists $\pi \in [0, 1]$ such that $(x_2, 1) \sim ((x_1, x_3), (\pi, 1 - \pi))$. The π is unique unless $x_1 \sim x_3$.

Theorem 27. *Under Axioms 1–6, there exists a utility function $u(\cdot)$ such that the choice of lottery by a decisionmaker corresponds to the lottery with the highest* $\mathrm{E}[u(\cdot)]$.

PROOF. Let L_h be the best lottery in $\mathcal{L}$ and L_w be the worst lottery in $\mathcal{L}$. Define $u(L_h) = 1$ and $u(L_w) = 0$. To find the utility of an intermediate lottery $L_i \in \mathcal{L}$, set $u(L_i) = p_i$, where p_i is defined by $((L_h, L_w), (p_i, 1 - p_i)) \sim (L_i, 1)$.

We need to check two things:

- Existence of such a p_i: The two sets

$$\{p \in [0, 1] : ((L_h, L_w), (p, 1 - p)) \gtrsim L_i\}$$

and

$$\{p \in [0,1] : ((L_h, L_w), (p, 1-p)) \precsim L_i\}$$

are closed (Axiom 4) and nonempty (since L_h is best and L_w is worst). Every point in [0, 1] is in one or the other of the two sets (Axiom 1). Since the unit interval is connected, there must be some p in both, but this will be just the desired p_i.

- Uniqueness of such a p_i: Suppose p_i and p_i' both satisfy the above definition. Then one must be larger than the other. By Axiom 6, the lottery that gives a bigger probability of L_h must be preferred to the one that gives a smaller probability. Hence p_i is unique and $u(\cdot)$ is well defined.

We next check that $u(\cdot)$ has the expected utility property. This can be shown as follows. By Axiom 5,

$$((L_x, L_y), (p, 1-p)) \sim ((((L_h, L_w), (p_x, 1-p_x)), ((L_h, L_w), (p_y, 1-p_y))), (p, 1-p)),$$

$$\sim ((L_h, L_w), (p_x p + p_y(1-p), \underbrace{(1-p_x)p + (1-p_y)(1-p)}_{1-p_x p - p_y(1-p)}))$$

$$\sim ((L_h, L_w), (pu(L_x) + (1-p)u(L_y), 1 - pu(L_x) - (1-p)u(L_y))),$$

so

$$u((L_x, L_y), (p, 1-p)) = pu(L_x) + (1-p)u(L_y),$$

as required.

Finally we verify that $u(\cdot)$ is a utility function. Suppose that $L_x \succ L_y$. Then

$$u(L_x) = p_x \ni L_x \sim ((L_h, L_w), (p_x, 1-p_x)),$$
$$u(L_y) = p_y \ni L_y \sim ((L_h, L_w), (p_y, 1-p_y)),$$

so by Axiom 6, $u(L_x) > u(L_y)$. □

A von Neumann–Morgenstern (vNM) utility function is identified up to a positive linear transformation, i.e. $u(\cdot)$ and $a + bu(\cdot)$ with $b > 0$ are equivalent. If the agent prefers more to less, $u(\cdot)$ is increasing.

7.2. Risk aversion

A decision-maker is said to be risk averse at a particular consumption level if she is *not* prepared to accept any actuarially fair, immediately resolved consumption gamble. For a given vNM utility function $u(c\cdot)$, the utility function is risk averse at c if $u(c) > \mathrm{E}[u(c+\varepsilon)]$ for all ε satisfying $\mathrm{E}[\varepsilon] = 0$ and $\mathrm{var}[\varepsilon] > 0$.

Theorem 28. *A decisionmaker is globally risk averse iff her vNM utility function is strictly concave at all consumption levels.*

Absolute risk aversion. For a utility function $u(\cdot)$, the absolute risk aversion measure is defined as

$$ARA(c) = -\frac{u''(c)}{u'(c)}.$$

It measures the dollar amount that the decision-maker would pay to avoid an actually fair gamble (per unit of dollar gamble variance), i.e.

$$\mathrm{E}[u(c+\varepsilon)] \approx u(c - \underbrace{\tfrac{1}{2}ARA(c)\,\mathrm{var}[\varepsilon]}_{q}),$$

where $\mathrm{E}[\varepsilon] = 0$. To show this, use Taylor series expansions of both sides:

$$\begin{aligned}\mathrm{E}[u(c) + \varepsilon u'(c) + \tfrac{1}{2}\varepsilon^2 u''(c)] &\approx u(c) - qu'(c)\\ \Rightarrow \tfrac{1}{2}\,\mathrm{var}[\varepsilon]u''(c) &\approx -qu'(c),\\ q &= \tfrac{1}{2}ARA(c)\,\mathrm{var}[\varepsilon].\end{aligned}$$

Relative risk aversion. For a utility function $u(\cdot)$, the relative risk aversion measure is defined as

$$RRA(c) = -\frac{u''(c)}{u'(c)}c.$$

It measures the amount (as a fraction of consumption) that the decision-maker would pay to avoid an actuarially fair gamble (per unit of variance of the gamble expressed as a fraction of consumption), i.e.

$$\mathrm{E}[u(c+e)] \approx u(c - \underbrace{c\tfrac{1}{2}RRA(c)\,\mathrm{var}[\varepsilon/c]}_{q})),$$

where $\mathrm{E}[\varepsilon] = 0$. To show this, notice that

$$\tfrac{1}{2}cRRA(c)\,\mathrm{var}[\varepsilon/c] = \tfrac{1}{2}ARA(c)\,\mathrm{var}[\varepsilon].$$

Comparison of $ARA(c)$ and $RRA(c)$. Consider:

- *CARA*: constant $ARA(c)$, so $RRA(c)$ is increasing in c.
- *CRRA*: constant $RRA(c)$, so $ARA(c)$ is decreasing in c.

Fix ε, vary c, and consider the dollar payment the decision-maker would pay:

$$\begin{aligned} q_{CARA(c)} &= \tfrac{1}{2}CARA\,\mathrm{var}[\varepsilon] \quad \text{a constant,}\\ q_{CRRA(c)} &= \tfrac{1}{2}\frac{CRRA}{c}\,\mathrm{var}[\varepsilon] \quad \text{which is decreasing in } c.\end{aligned}$$

Fix ε/c, vary c, and consider the payment as a fraction of c that the decision-maker would pay:

$$\begin{aligned} q_{CARA}(c)/c &= \tfrac{1}{2}cCARA\,\mathrm{var}[\varepsilon/c] \quad \text{which is increasing in } c,\\ q_{CRRA}(c)/c &= \tfrac{1}{2}CRRA\,\mathrm{var}[\varepsilon/c] \quad \text{a constant.}\end{aligned}$$

7.3. HARA utility function

The functional form of the hyperbolic absolute risk aversion (HARA) utility function is

$$u(w) = a\left(\frac{1-\varphi}{\varphi}\right)\left(\frac{w}{1-\varphi} - \hat{w}\right)^{\varphi} + b, \qquad \varphi \neq 0, \quad \frac{w}{1-\varphi} - \hat{w} > 0, \quad a > 0,$$

with

$$\begin{aligned}
u'(w) &= a\left(\frac{w}{1-\varphi} - \hat{w}\right)^{\varphi-1} > 0,\\
u''(w) &= -a\left(\frac{w}{1-\varphi} - \hat{w}\right)^{\varphi-2} < 0,\\
ARA(w) &= -\frac{u''(w)}{u'(w)} = \left(\frac{w}{1-\varphi} - \hat{w}\right)^{-1},\\
RRA(w) &= -\frac{u''(w)w}{u'(w)} = w\left(\frac{w}{1-\varphi} - \hat{w}\right)^{-1}.
\end{aligned}$$

HARA specializes to a number of important utility specifications.

Power utility. If $\hat{w} = 0$, we obtain power utility. Defining $\gamma = 1-\varphi$ and setting $a = (1-\varphi)^{-1+\varphi}$, $1-\varphi > 0$, then

$$u(w) = \frac{w^{1-\gamma}}{1-\gamma}, \qquad \gamma > 0, \quad w > 0, \quad RRA = \gamma.$$

Log utility. If $\hat{w} = 0$, $a = (1-\varphi)^{-1+\varphi}$ and $b = -1/\varphi$, letting $\varphi \to 0$, we obtain log utility as the limit since

$$u(w) = \frac{w^{\varphi} - 1}{\varphi},$$

so using L'Hôpital's rule,

$$\lim_{\varphi\to 0} u(w) = \lim_{\varphi\to 0} \frac{\frac{d}{d\varphi}(e^{\varphi \ln w} - 1)}{\frac{d}{d\varphi}\varphi} = \lim_{\varphi\to 0} \frac{w^{\varphi}\ln w}{1} = \ln w.$$

Note that since $u'(w) = \infty$, a power or log utility agent does not want to hold a wealth portfolio with a positive probability of a return of -1. Consequently, when a power or log utility agent has access to risky assets whose returns are multivariate normal and a riskless asset, the agent chooses to hold a portfolio 100% invested in the riskless asset.

Quadratic utility. If $\varphi = 2$ and $\hat{w} < 0$, we obtain quadratic utility. Setting $a = 1$ and $b = 0$ and defining $w^* = -\hat{w}$, we get

$$u(w) = -\tfrac{1}{2}(w^* - w)^2, \quad w < w^*.$$

Exponential utility. If

$$\hat{w} = \left(\frac{\varphi}{1-\varphi}\right)\frac{1}{\lambda}, \quad a = \left(\frac{\varphi-1}{\varphi}\right)^{\varphi-1}\lambda^{\varphi}, \quad b = 0,$$

then letting $\varphi \to \infty$, we obtain exponential utility since

$$\begin{aligned} u(w) &= \left(\frac{\varphi-1}{\varphi}\right)^{\varphi-1}\lambda^{\varphi}\frac{(1-\varphi)}{\varphi}\left(\frac{w}{1-\varphi} - \left(\frac{\varphi}{1-\varphi}\right)\frac{1}{\lambda}\right)^{\varphi} \\ &= -\left(\frac{\varphi-1}{\varphi}\right)^{\varphi}\lambda^{\varphi}\left(\frac{1}{\varphi-1}\left(\frac{\varphi}{\lambda} - w\right)\right)^{\varphi} \\ &= -\left(1 - \frac{\lambda w}{\varphi}\right)^{\varphi}, \end{aligned}$$

and so

$$\lim_{\varphi\to\infty} u(w) = \lim_{\varphi\to\infty} -\left(1 - \frac{\lambda w}{\varphi}\right)^{\varphi} = -e^{-\lambda w}, \quad ARA(w) = \lambda.$$

7.4. Utility theory with multiple periods

Use a time separable (or additive) utility function

$$\max \mathrm{E}\left[\sum_{\tau=0}^{\infty} \delta^{\tau} u(c_{t+\tau}) \,\middle|\, \mathcal{F}_t\right],$$

where $u(\cdot)$ is a vNM utility function and δ is a patience parameter.

Intertemporal elasticity of substitution. The intertemporal elasticity of substitution is the percentage change in consumption growth in response to a percentage change in one plus the interest rate.

With power utility, the intertemporal elasticity of substitution for consumption is equal to the inverse of RRA. To illustrate, ignore uncertainty and consider a 2-period problem,

$$\max_{c_t, c_{t+1}} \frac{c_t^{1-\gamma}}{1-\gamma} + \delta\frac{c_{t+1}^{1-\gamma}}{1-\gamma} \quad \text{s.t. } c_{t+1} = (W_t - c_t)(1+R).$$

The first-order condition is

$$\begin{aligned} c_t^{-\gamma} - \delta(1+R)c_{t+1}^{-\gamma} &= 0 \\ \Rightarrow \left(\frac{c_{t+1}}{c_t}\right)^{\gamma} &= \delta(1+R) \\ \Rightarrow \frac{c_{t+1}}{c_t} &= \delta^{1/\gamma}(1+R)^{1/\gamma} \\ \Rightarrow \ln\left(\frac{c_{t+1}}{c_t}\right) &= \frac{1}{\gamma}\ln\delta + \frac{1}{\gamma}\ln(1+R). \end{aligned}$$

Now the elasticity is given by

$$\frac{d \ln \frac{c_{t+1}}{c_t}}{d \ln(1+R)} = \frac{1}{\gamma},$$

as required. Why?

$$\frac{d\ln\frac{c_{t+1}}{c_t}}{d\ln(1+R)} = \frac{\dfrac{d\ln\frac{c_{t+1}}{c_t}}{d\frac{c_{t+1}}{c_t}}}{\dfrac{d\ln(1+R)}{d(1+R)}}\frac{d\frac{c_{t+1}}{c_t}}{d(1+R)} = \frac{d\frac{c_{t+1}}{c_t}\Big/\frac{c_{t+1}}{c_t}}{d(1+R)/(1+R)}.$$

7.5. Using dynamic programming to solve multiperiod problems

Finite-lived investor. Define

$$V_t(W_t, s_t) = \max_{\{c_\tau\}_{\tau=t}^{T-1}, \{\boldsymbol{\alpha}_\tau\}_{\tau=t}^{T-1}} \mathrm{E}\left[\sum_{\tau=t}^{T} \delta^{\tau-t} u(c_\tau) \mid s_t, W_t\right] \tag{27}$$

subject to

$$W_{\tau+1} = (W_\tau - c_\tau)(\boldsymbol{\alpha}'_\tau(\boldsymbol{R}_{\tau+1} - \mathbf{i}_N R^f_{\tau+1}) + R^f_{\tau+1}), \quad c_\tau, \boldsymbol{\alpha}_\tau \in \mathcal{F}_t(s_\tau, W_\tau) \tag{28}$$

for $\tau = t, \ldots, T-1$, where s_τ is the vector of state variables that affect the investor's happiness at time τ for $\tau = t, \ldots, T-1$.

We would like to convert the multiperiod problem into a single period problem. We can solve recursively working back from T. At time $(T-1)$,

$$V_{T-1}(W_{T-1}, s_{T-1}) = \max_{c_{T-1}, \boldsymbol{\alpha}_{T-1}} \mathrm{E}[u(c_{T-1}) + \delta u(W_T) \mid s_{T-1}, W_{T-1}]$$

subject to

$$W_T = (W_{T-1} - c_{T-1})(\boldsymbol{\alpha}'_{T-1}(\boldsymbol{R}_T - \mathbf{i}_N R^f_T) + R^f_T), \quad c_{T-1}, \boldsymbol{\alpha}_{T-1} \in \mathcal{F}(s_{T-1}).$$

At $T-2$,

$$V_{T-2}(W_{T-2}, s_{T-2}) = \max_{\substack{\{c_\tau\}_{\tau=T-2}^{T-1},\\ \{\boldsymbol{\alpha}\}_{\tau=T-2}^{T-1}}} \left\{ \mathrm{E}\left[u(c_{T-2}) + \delta u(c_{T-1}) + \delta^2 u(W_T) \,\middle|\, \begin{matrix} s_{T-2} \\ W_{T-2} \end{matrix} \right] \right\}$$

subject to

$$W_{\tau+1} = (W_\tau - c_\tau)(\boldsymbol{\alpha}'_\tau(\boldsymbol{R}_{\tau+1} - \mathbf{i}_N R^f_{\tau+1}) + R^f_{\tau+1}), \quad c_\tau, \boldsymbol{\alpha}_\tau \in \mathcal{F}(s_\tau)$$

for $\tau = T-2, T-1$. This time-$(T-2)$ value function can be written as:

$$V_{T-2}(W_{T-2}, s_{T-2}) =$$

$$\max_{\substack{c_{T-2},\\ \boldsymbol{\alpha}_{T-2}}} \left\{ \mathrm{E}\left[u(c_{T-2}) + \delta \underbrace{\max_{c_{T-1}, \boldsymbol{\alpha}_{T-1}} \left\{ \begin{matrix} \mathrm{E}\left[u(c_{T-1}) + \delta u(W_T) \,\middle|\, \begin{matrix} s_{T-1} \\ W_{T-1} \end{matrix} \right] \\ \text{s.t. } W_T = (W_{T-1} - c_{T-1}) \times \\ (\boldsymbol{\alpha}'_{T-1}(\boldsymbol{R}_T - \mathbf{i}_N R^f_T) + R^f_T), \\ c_{T-1}, \boldsymbol{\alpha}_{T-1} \in \mathcal{F}(s_{T-1}) \end{matrix} \right\}}_{V_{T-1}(W_{T-1}, s_{T-1})} \,\middle|\, \begin{matrix} s_{T-2} \\ W_{T-2} \end{matrix} \right] \right\}$$

subject to

$$W_{T-1} = (W_{T-2} - c_{T-2})(\boldsymbol{\alpha}'_{T-2}(\boldsymbol{R}_{T-1} - \mathbf{i}_N R^f_{T-1}) + R^f_{t-1}), \quad c_{T-2}, \boldsymbol{\alpha}_{T-2} \in \mathcal{F}(s_{T-2}).$$

Thus, the time-$(T-2)$ value function becomes

$$V_{T-2}(W_{T-2}, s_{T-2}) = \max_{c_{T-2}, \boldsymbol{\alpha}_{T-2}} \left\{ \mathrm{E}\left[u(c_{T-2} + \delta V_{T-1}(W_{T-1}, s_{T-1}) \,\middle|\, \begin{matrix} s_{T-2} \\ W_{T-2} \end{matrix} \right] \right\}$$

subject to

$$W_{T-1} = (W_{T-2} - c_{T-2})(\boldsymbol{\alpha}'_{T-2}(\boldsymbol{R}_{T-1} - \mathbf{i}_N R^f_{T-1}) + R^f_{T-1}), \quad c_{T-2}, \boldsymbol{\alpha}_{T-2} \in \mathcal{F}(s_{T-2}).$$

This holds for any $\tau = t, \ldots, T-1$, so

$$V_\tau(W_\tau, s_\tau) = \max_{c_\tau, \boldsymbol{\alpha}_\tau} \underbrace{\{\mathrm{E}[u(c_\tau) + \delta V_{\tau+1}(W_{\tau+1}, s_{\tau+1}) \mid s_\tau]\}}_{L(c_\tau, \boldsymbol{\alpha}_\tau; W_\tau, s_\tau)} \tag{29}$$

subject to

$$(28) W_{\tau+1} = (W_\tau - c_\tau) \underbrace{(\boldsymbol{\alpha}'_\tau(\boldsymbol{R}_{\tau+1} - \mathbf{i}_N R^f_{\tau+1}) + R^f_{\tau+1})}_{R^W_{\tau+1}}, \quad c_\tau, \boldsymbol{\alpha}_\tau \in \mathcal{F}(s_\tau), \quad V_T(\cdot) = u(\cdot).$$

PROOF. Fix $\{c^{**}_\tau\}_{\tau=t}^{T-\nu-1}, \{\boldsymbol{\alpha}^{**}_\tau\}_{\tau=t}^{T-\nu-1}$ as any arbitrary choice of c_τ and $\boldsymbol{\alpha}_\tau$ for $\tau = t, \ldots, T-\nu-1$.

Let $\{c^*_\tau\}_{\tau=T-\nu}^{T-1}, \{\boldsymbol{\alpha}^*_\tau\}_{\tau=T-\nu}^{T-1}$ be the solution to

$$V_{T-\nu}(W_{T-\nu}, s_{T-\nu}) = \max_{\{c_\tau\}_{\tau=T-\nu}^{T-1}} \mathrm{E}\left[\sum_{\tau=T-\nu}^{T} \delta^{\tau-(T-\nu)} u(c_\tau) \,\middle|\, \begin{matrix} s_{T-\nu} \\ W_{T-\nu} \end{matrix} \right] \tag{30}$$

subject to (28) for $\tau = T-\nu, \ldots, T-1$. Let $\{c^{**}_\tau\}_{\tau=t}^{T-\nu-1}, \{\boldsymbol{\alpha}^{**}_\tau\}_{\tau=t}^{T-\nu-1}$ be any other arbitrary choice. By definition

$$\mathrm{E}\left[\sum_{\tau=T-\nu}^{T} \delta^{\tau-(T-\nu)} u(c^*_\tau) \,\middle|\, \begin{matrix} s_{T-\nu} \\ W_{T-\nu} \end{matrix} \right] \geq \mathrm{E}\left[\sum_{\tau=T-\nu}^{T} \delta^{\tau-(T-\nu)} u(c^{**}_\tau) \,\middle|\, \begin{matrix} s_{T-\nu} \\ W_{T-\nu} \end{matrix} \right]$$

which means using the law of iterated expectations,

$$\begin{aligned} &\mathrm{E}\left[\sum_{\tau=t}^{T-\nu-1} \delta^{\tau-t} u(c^{**}_\tau) + \delta^{T-\nu-t} \left(\sum_{\tau=T-\nu}^{T} \delta^{\tau-(T-\nu)} u(c^*_\tau) \right) \,\middle|\, \begin{matrix} s_t \\ W_t \end{matrix} \right] \\ &\qquad \geq \mathrm{E}\left[\sum_{\tau=t}^{T-\nu-1} \delta^{\tau-t} u(c^{**}_\tau) + \delta^{T-\nu-t} \left(\sum_{\tau=T-\nu}^{T} \delta^{\tau-(T-\nu)} u(c^{**}_\tau) \right) \,\middle|\, \begin{matrix} s_t \\ W_t \end{matrix} \right]. \end{aligned}$$

Since this holds for any $\{c^{**}_\tau\}_{\tau=t}^{T-\nu-1}$, it holds for $\{c^*_\tau\}_{\tau=t}^{T-\nu-1}$, the optimal solution to (27). Hence the solution to (30) is also the solution to (27) for $\tau = T-\nu, \ldots, T-1$, so dynamic programming works for

$$u_t = u(c_t, c_{t-1}, \ldots, c_{t-s}).$$

which is habit persistence. But does it work for $u_t = u(c_t, c_{t+1}, \ldots, c_{t+s})$? □

It can be shown that if $u(\cdot)$ is increasing and concave, then $V_\tau(W_\tau, s_\tau)$ is increasing and concave in W_τ.

Envelope condition. We can show that

$$u'(c^*_\tau(W_\tau, s_\tau)) = V_{\tau,W}(W_\tau, s_\tau),$$

where $c^*_\tau(\cdot,\cdot)$ and $\boldsymbol{\alpha}^*_\tau(\cdot,\cdot)$ solve (29).

PROOF. The first-order condition for (29) wrt to $c_t au$ is given by:

$$\frac{\partial L(c^*_\tau(W_\tau, s_\tau), \boldsymbol{\alpha}^*_\tau(W_\tau, s_\tau); W_\tau, s_\tau)}{\partial c_\tau} = 0$$

$$u'(c^*_\tau(W_\tau, s_\tau)) - \delta\,\mathrm{E}[V_{\tau+1,W}(W_{\tau+1}, s_{\tau+1} R^W_{\tau+1} \mid s_\tau] = 0. \tag{31}$$

The first-order condition for (29) wrt to $\boldsymbol{\alpha}_t au$ is given by:

$$\frac{\partial L(c^*_\tau(W_\tau, s_\tau), \boldsymbol{\alpha}^*_\tau(W_\tau, s_\tau); W_\tau, s_\tau)}{\partial \boldsymbol{\alpha}_\tau} = \mathbf{0}$$

$$\mathrm{E}[V_{\tau+1,W}(W_{\tau+1}, s_{\tau+1})(\boldsymbol{R}_{\tau+1} - R^f_{\tau+1}\mathbf{i}_N) \mid s_\tau] = \mathbf{0}.$$

Now

$$\begin{aligned} V_{\tau,W}(W_\tau, s_\tau) &= \frac{dL(c^*_\tau(W_\tau, s_\tau), \boldsymbol{\alpha}^*_\tau(W_\tau, s_\tau); W_\tau, s_\tau)}{dW_\tau} \\ &= \frac{\partial L(\cdot)}{\partial W_\tau} + \underbrace{\frac{\partial L(\cdot)}{\partial c_\tau)}}_{=0} \frac{\partial c^*}{\partial W_\tau} + \underbrace{\frac{\partial L(\cdot)}{\partial \boldsymbol{\alpha}_\tau}}_{=0} \frac{\partial \boldsymbol{\alpha}^*}{\partial W_\tau} \\ &= \delta\,\mathrm{E}[V_{\tau+1,W}(W_{\tau+1}, s_{\tau+1}) R^W_{\tau+1} \mid s_\tau]. \end{aligned}$$

So using (31)

$$u'(c^*_\tau(W_\tau, s_\tau)) = V_{\tau,W}(W_\tau, s_\tau).$$

□

Infinite-lived investor. The problem at time t is

$$V_t(W_t, s_t) = \max \mathrm{E}\left[\sum_{\tau=0}^{\infty} \delta^\tau u(c_{t+\tau}) \mid s_t\right]$$

subject to (28) for $\tau = t, t+1, \ldots$. The problem at time $t+1$ is

$$V_{t+1}(W_{t+1}, s_{t+1}) = \max \mathrm{E}\left[\sum_{\tau=0}^{\infty} \delta^\tau u(c_{t+1+\tau}) \mid s_{t+1}\right]$$

subject to (28) for $\tau = t+1, t+2, \ldots$.

The problem at t looks the same as the problem at $t+1$, so if the solution to the right-hand side is finite, the Bellman equation becomes

$$V(W_\tau, s_\tau) = \max_{c_\tau, \boldsymbol{\alpha}_\tau} \{\mathrm{E}[u(c_\tau) + \delta V(W_{\tau+1}, s_{\tau+1}) \mid s_\tau]\}$$

subject to (28) for $\tau = t, t+1, \ldots$.

7.6. Campbell–Viceira approximate solution for the multi-period agent's problem

Consider a multi-period investor who solves the following problem:

$$V_t(W_t, x_t) = \max_{\{C_\tau\}_{\tau=t}^T \{\alpha_\tau\}_{\tau=t}^T}, \left\{ \mathrm{E}_t \left[\sum_{\tau=t}^T \delta^{\tau-t} \frac{C_\tau^{1-\gamma}}{1-\gamma} \right] \right\}, \quad C_\tau, \alpha_\tau \in \mathcal{F}_\tau \forall \tau.$$

s.t.

$$W_{\tau+1} = (W_\tau - C_\tau) R_{W,\tau+1} \tag{32}$$

and

$$R_{W,\tau+1} = \alpha_t (R_{1,\tau+1} - R_f) + R_f. \tag{33}$$

Let $r_{1,\tau+1} = \ln(R_{1,\tau+1})$, $r_{W,\tau+1} = \ln(R_{W,\tau+1})$, $r_f = \ln(R_f)$, $c_{\tau+1} = \ln(C_{\tau+1})$ and $w_{\tau+1} = \ln(W_{\tau+1})$. The return generating process satisfies:

$$\begin{aligned} \mathrm{E}_\tau[r_{1,\tau+1}] - r_f &= x_\tau, \\ x_{\tau+1} &= \mu + \varphi(x_\tau - \mu) + \eta_{\tau+1}, \end{aligned}$$

where $0 < \varphi < 1$ and μ are constants and $\eta_{\tau+1}$ is normal i.i.d. white noise with zero mean and standard deviation of σ_η. The conditional mean of $(\mathrm{E}_\tau[r_{1,\tau+1}] - r_f)$ is given by the state variable x_τ which evolves through time as an AR(1) process. Moreover, $r_{1,\tau+1}$ is conditionally homoskedastic. Letting $u_{\tau+1} = r_{1,\tau+1} - \mathrm{E}_\tau[r_{1,\tau+1}]$, assume that

$$\begin{aligned} \mathrm{var}_\tau[u_{\tau+1}] &= \sigma_u^2, \\ \mathrm{cov}_\tau[u_{\tau+1}, \eta_{\tau+1}] &= \sigma_{u\eta}. \end{aligned}$$

The Euler equation for $R_{i,t+1}$ that comes out of the first order condition for this investor's problem for $i = 1$, W and f says

$$1 = \mathrm{E}_t \left[\delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{i,t+1} \right]. \tag{34}$$

Let $V_{x,y,t} \equiv \mathrm{cov}_t[x_{t+1}, y_{t+1}]$. Assume $r_{1,\tau+1}$ and $c_{\tau+1} \equiv \ln C_{\tau+1}$ are multivariate normal conditional on time-τ information. Then (34) implies that for $i = W$,

$$\begin{aligned} 0 &= \ln\delta - \gamma\, \mathrm{E}_t[\Delta c_{t+1}] + \mathrm{E}_t[r_{1,t+1}] \\ &\quad + \tfrac{1}{2}(\gamma^2 \sigma_t^2[\Delta c_{t+1}] + \sigma_t^2[r_{1,t+1}] - 2\gamma\sigma_t[\Delta c_{t+1}, r_{1,t+1}]), \\ \mathrm{E}_t[r_{1,t+1}] &= -\ln\delta + \gamma\, \mathrm{E}_t[\Delta c_{t+1}] \\ &\quad - \tfrac{1}{2}\gamma^2\sigma_t^2[\Delta c_{t+1}] - \tfrac{1}{2}\sigma_t^2[r_{1,t+1}] + \gamma\sigma_t[\Delta c_{t+1}, r_{1,t+1}], \\ r_f &= -\ln\delta + \delta\, \mathrm{E}_t\, \Sigma c_{t+1} - \tfrac{1}{2}\sigma^2\sigma_t^2[\Delta c_{t+1}], \end{aligned}$$

and so

$$\mathrm{E}_t[r_{1,t+1}] - r_f + \tfrac{1}{2} V_{1,1,t}^2 = \gamma V_{1,c,t}. \tag{35}$$

But $c_{\tau+1}$ and $r_{1,\tau+1}$ are not multivariate normal conditional on time-τ information even if $r_{1,\tau+1}$ is conditionally normal. Even if $r_{1,t+1}$ and c_{t+1} are not multivariate normal conditional on the time-t information, it is possible to obtain (35) using log-linearizations.

Let $\Delta w_{\tau+1} = w_{\tau+1} - w_\tau$. We can use a log-linearization of the budget constraint (32) to obtain an approximate expression for $\Delta w_{\tau+1}$ that is linear in $r_{W,\tau+1}$ and $(c_\tau - w_\tau)$. We log-linearize the budget constraint by a first order Taylor expansion of $c_t - w_t$ around $\bar{c} - \bar{w}$. Letting $\rho = 1 - \exp(\bar{c} - \bar{w})$,

$$\begin{aligned}
W_{t+1} &= R_{W,t+1}(W_t - C_t), \\
W_{t+1}/W_t &= R_{W,t+1}\left(1 - \frac{C_t}{W_t}\right) \\
\Delta w_{t+1} &= r_{W,t+1} + \ln(1 - \exp(c_t - w_t)) \\
&= r_{W,t+1} + \ln(1 - \exp(\bar{c} - \bar{w})) + \frac{-\exp(\bar{c} - \bar{w})}{1 - \exp(\bar{c} - \bar{w})}(c_t - w_t - (\bar{c} - \bar{w})) \\
&= r_{W,t+1} + \left(1 - \frac{1}{\rho}\right)(c_t - w_t) + \left(\ln(\rho) - \left(1 - \frac{1}{\rho}\right)\ln(1 - \rho)\right). \qquad (36)
\end{aligned}$$

We can log-linearize the portfolio return equation (33) by a first order Taylor expansion of r˙W, t+1 with respect to $r_{1,t+1}$ around zero and r_f around zero:

$$\begin{aligned}
R_{W,t+1} &= \alpha_t(R_{1,t+1} - R_f) + R_f, \\
r_{W,t+1} &= \ln(\alpha_t \exp(r_{1,t+1}) - \alpha_t \exp(r_f) + \exp(r_f)) \\
&= \ln(\alpha_t \exp(0) - \alpha_t \exp(0) + \exp(0)) \qquad (37) \\
&\quad + \frac{\alpha_t}{\alpha_t \exp(0) - \alpha_t \exp(0) + \exp(0)}(r_{1,t+1} - 0) \\
&\quad + \frac{1 - \alpha_t}{\alpha_t \exp(0) - \alpha_t \exp(0) + \exp(0)}(r_f - 0) \\
&= \alpha_t(r_{1,t+1} - r_f) + r_f. \qquad (38)
\end{aligned}$$

We can use the approximate expressions derived in (36), (38) and (35) to show that the following approximate relation holds:

$$\alpha_t = \frac{\mathrm{E}_t[r_{1,t+1}] - r_f + \frac{1}{2}\sigma_u^2}{\gamma\sigma_u^2} - \frac{\mathrm{cov}_t[r_{1,t+1}, c_{t+1} - w_{t+1}]}{\sigma_u^2}. \qquad (39)$$

Combine (36) and (38) to get

$$\Delta w_{t+1} \approx \alpha_t(r_{1,t+1} - r_f) + r_f + \left(1 - \frac{1}{\rho}\right)(c_t - w_t) + k. \qquad (40)$$

Using (40) and

$$\Delta c_{t+1} = (c_{t+1} - w_{t+1}) - (c_t - w_t) + \Delta w_{t+1}, \qquad (41)$$

we can show that $V_{1c,t}$ can be written as a function of $V_{1,c-w,t}$, α_t and $V_{1,1,t}$:

$$\begin{aligned}
V_{1,c,t} &= \mathrm{cov}_t[r_{1,t+1}, \Delta c_{t+1}] \\
&= \mathrm{cov}_t[r_{1,t+1}, c_{t+1} - w_{t+1}] - \mathrm{cov}_t[r_{1,t+1}, c_t - w_t] \\
&\qquad + \mathrm{cov}_t[r_{1,t+1}, \alpha_t r_{1,t+1} + (1-\alpha_t) r_f + (1 - \tfrac{1}{\rho})(c_t - w_t) + k] \\
&= V_{1,c-w,t} + \alpha_t V_{1,1,t}.
\end{aligned}$$

Using (35),

$$\mathrm{E}_t[r_{1,t+1}] - r_f + \tfrac{1}{2} V_{1,1,t} = \gamma (V_{1,c-w,t} + \alpha_t V_{1,1,t}),$$

$$\alpha_t = \frac{\mathrm{E}_t[r_{1,t+1}] - r_f + \frac{1}{2} V_{1,1,t}}{V_{1,1,t}} \frac{1}{\gamma} - \frac{V_{1,c-w,t}}{V_{1,1,t}}$$

as required.

Suppose that the investor's life ended at time-$(t+1)$: i.e., $t = T-1$. How would the formula for α_t in (39) change? If the investor's life ended at time-$(t+1)$, then $C_{t+1} = W_{t+1}$ and so $c_{t+1} - w_{t+1} = 0$ and $V_{1,c-w,t} = 0$. Consequently, the the single-period agent's risky-asset allocation, which is known as the myopic demand, is given by:

$$\alpha_t = \frac{\mathrm{E}_t[r_{1,t+1}] - r_f + \frac{1}{2} V_{1,1,t}}{V_{1,1,t}} \frac{1}{\gamma}$$

The difference between the the risky-asset allocations by a multi-period agent and a single-period agent, which is given by $-\frac{V_{1,c-w,t}}{V_{1,1,t}}$, is known as the hedging demand and converges to 0 as t converges to $T-1$. Suppose $\sigma_{u\eta} = 0$. Since x_t is the only state variable for the agent's problem and recalling that the power utility problem is homogeneous in wealth, we know that $c_{t+1} - w_{t+1}$ only depends on x_{t+1}. Thus, if $\sigma_{u\eta} = 0$, then $V_{1,c-w,t} = 0$ and the hedging demand is zero. So the risky-asset return from t to $t+1$ needs to be $\mathcal{F}_t$-conditionally correlated with the state variable at $t+1$, x_{t+1}, for the hedging demand to be non-zero.

It is possible to obtain an explicit expression for α_t as a function of x_t. For simplicity, suppose that the agent is infinitely lived: i.e., $T = \infty$. It is possible to show that the solution for c_t satisfies

$$c_t - w_t = b_0 + b_1 x_t + b_2 x_t^2, \tag{42}$$

where b_0, b_1 and b_2 are constants. We can use (39) and the result

$$x_{\tau+1}^2 - \mathrm{E}_\tau[x_{\tau+1}^2] = (\eta_{\tau+1}^2 - \sigma_\eta^2) + (2\mu(1-\varphi) + 2\varphi x_\tau)\eta_{\tau+1}$$

to show that the solution for α_t is linear in x_t:

$$\alpha_t = a_0 + a_1 x_t.$$

We can obtain expressions for a_0 and a_1 in terms of γ, σ_u^2, $\sigma_{u\eta}$, μ, φ, b_1 and b_2. Taking (42) as given and noting that

$$x_{t+1}^2 - \mathrm{E}_t[x_{t+1}^2] = (\eta_{t+1}^2 - \sigma_\eta^2) + (2\mu(1-\varphi) + 2\varphi x_t)\eta_{t+1},$$

it follows that

$$\begin{aligned} V_{1,c-w,t} &= \operatorname{cov}_t[r_{1,t+1}, c_{t+1} - w_{t+1}] \\ &= \operatorname{cov}[u_{t+1}, b_1(x_{t+1} - \mathrm{E}_t\, x_{t+1}) \\ &\qquad + b_2(\eta_{t+1}^2 - \sigma_\eta^2 + (2\mu(1-\varphi) + 2\varphi x_t)\eta_{t+1}] \\ &= \operatorname{cov}[u_{t+1}, b_1\eta_{t+1} + b_2(2\mu(1-\varphi)) + 2\varphi x_t)\eta_{t+1}] \\ &= (b_1 + b_2(2\mu(1-\varphi) + 2\varphi x_t))\sigma_{u\eta}, \end{aligned}$$

and so

$$\begin{aligned} \alpha_t &= \frac{x_\tau + \frac{1}{2}\sigma_u^2}{\sigma_u^2}\frac{1}{\gamma} - \frac{1}{\sigma_u^2}(b_1 + b_2(2\mu(1-\varphi) + 2\varphi x_t)\sigma_{u\eta} \\ &= \underbrace{\frac{1}{2\gamma} - \frac{1}{\sigma_u^2} - \frac{1}{\sigma_u^2}b_2 2\mu(1-\varphi)\sigma_{u\eta}}_{a_0} \\ &\qquad + \underbrace{\left(\frac{1}{\gamma\sigma_u^2} - \frac{1}{\sigma_u^2}b_2 2\varphi\sigma_{u\eta}\right)}_{a_1} x_t \end{aligned}$$

as required.

It is worth noting that the analysis can be extended to:

- multiple risky assets.
- multiple state variables.
- Epstein-Zin preferences.

CHAPTER 8

Equilibrium Pricing: Representative Agent

8.1. Setting

There is a set of N assets available at t given by return vector $\boldsymbol{r}_{t+1}$. The set of payoffs available to the representative agent is given by

$$X_{t+1} = \{\boldsymbol{\omega}_t' \boldsymbol{r}_{t+1} : \boldsymbol{\omega}_t \in \mathcal{F}_t\},$$

where $\mathcal{F}_t = \mathcal{F}(s_t, W_t)$. The set of returns available to the representative agent is given by

$$R_{t+1} = \{\boldsymbol{\omega}_t' \boldsymbol{r}_{t+1} : \boldsymbol{\omega}_t \in \mathcal{F}_t, \boldsymbol{\omega}_t' \mathbf{i}_N = 1\}.$$

Again, let $\mathrm{E}_t[.] \equiv E[. \mid \mathcal{F}_t]$

Definition (Existence of a representative agent)**.** There exists a well-behaved utility function such that the solution to the problem of maximizing the function subject to the (per capita) wealth evolution equation for the economy delivers (per capita) aggregate consumption and market clearing portfolio allocations.

8.2. C–CAPM

Suppose there exists a multiperiod representative agent. We consider sufficient conditions for this to hold in chapter 10. The first-order condition for the representative agent's Bellman equation problem implies

$$u'(c_t^*(W_t, s_t)) = \delta\, \mathrm{E}_t[V_{t+1,W}(W_{t+1}, s_{t+1}) r_{t+1}^i] \quad \text{for all } r_{t+1}^i \in \boldsymbol{r}_{t+1}.$$

Using the envelope condition,

$$u'(c_t^*(W_t, s_t)) = \delta\, \mathrm{E}_t[u'(c_{t+1}^*(W_{t+1}, s_{t+1})) r_{t+1}^i] \quad \text{for all } r_{t+1}^i \in \boldsymbol{r}_{t+1}.$$

Rearranging, we obtain

$$p_t(x_{t+1}) = \mathrm{E}_t\left[\frac{\delta u'(c_{t+1}^*)}{u'(c_t^*)} x_{t+1}\right] \quad \text{for all } x_{t+1} \in X_{t+1}. \tag{43}$$

So $\delta u'(c_{t+1}^*)/u'(c_t^*)$ is a $\mathcal{F}_t$–conditional sdf for X_{t+1}, and so $\delta u'(c_{t+1}^*)/u'(c_t^*) = \mathrm{mrs}_{t+1}$ is a $\mathcal{F}_t$–conditional 1-factor βpm for R_{t+1}, i.e.

$$\mathrm{E}_t[r_{t+1}^i] = \alpha_t + \lambda_t \beta_t^{i,\mathrm{mrs}}$$

for all $r_{t+1}^i \in R_{t+1}$, where $\alpha_t, \lambda_t \in \mathcal{F}_t$ and

$$\beta_t^{i,\mathrm{mrs}} = \frac{\mathrm{cov}_t[r_{t+1}^i, \mathrm{mrs}_{t+1}]}{\mathrm{var}_t[\mathrm{mrs}_{t+1}]}.$$

Conditioning down (43),

$$\mathrm{E}[p_t(x_{t+1})] = \mathrm{E}\left[\frac{\delta u'(c^*_{t+1})}{u'(c^*_t)} x_{t+1}\right] \quad \text{for all } x_{t+1} \in X_{t+1}.$$

So mrs_{t+1} is also an unconditional 1-factor βpm for X_{t+1}.

8.3. CAPM: Single-period representative agent

The single period problem is

$$\max_{c_t, \boldsymbol{\alpha}_t} \{u(c_t) + \mathrm{E}_t[\delta u(c_{t+1})]\}$$

subject to

$$c_{t+1} = (W_t - c_t)\boldsymbol{\alpha}'_t \boldsymbol{r}_{t+1}, \quad \boldsymbol{\alpha}'_t \mathbf{i} = 1, \quad c_t, \boldsymbol{\alpha}_t \in \mathcal{F}_t.$$

The first order condition for the single period problem is

$$u'(c^*_t) = \mathrm{E}_t[\delta u'(c^*_{t+1}) r^i_{t+1}],$$

where $c^*_{t+1} = (W_t - c^*_t) r^W_{t+1}$ and r^W_{t+1} is the return on the agent's optimal portfolio.

Quadratic utility: Sufficient condition for CAPM. Quadratic utility is given by

$$u(c) = -\tfrac{1}{2}(A - c)^2, \quad u'(c) = A - c.$$

Substituting into first-order condition,

$$\begin{aligned}
A - c^*_t &= \mathrm{E}_t[\delta(A - c^*_{t+1}) r^i_{t+1}] \quad \text{for all } r^i_{t+1} \in R_{t+1} \\
\Leftrightarrow A - c^*_t &= \mathrm{E}_t[\delta(A - (W_t - c^*_t) r^W_{t+1}) r^i_{t+1}] \quad \text{for all } r^i_{t+1} \in R_{t+1} \\
\Rightarrow \text{there exists } a_t, b_t \in \mathcal{F}_t &\text{ such that } 1 = \mathrm{E}_t[(a_t + b_t r^W_{t+1}) r^i_{t+1}] \quad \text{for all } r^i_{t+1} \in R_{t+1}, \\
\Leftrightarrow r^W_{t+1} &\text{ is a } \mathcal{F}_t\text{–conditional } \beta\text{pm for all } r^i_{t+1} \in R_{t+1}.
\end{aligned}$$

i.e.,

$$\mathrm{E}_t[r^i_{t+1}] = \alpha_t + \lambda_t \beta^{i,W}_t \quad \text{for all } r^i_{t+1} \in R_{t+1}, \quad \alpha_t, \lambda_t \in \mathcal{F}_t,$$

and

$$\beta^{i,W}_t = \frac{\mathrm{cov}_t[r^i_{t+1}, r^W_{t+1}]}{\mathrm{var}_t[r^W_{t+1}]}.$$

Notice that $\lambda_t = \mathrm{E}_t[r^W_{t+1}] - \alpha_t$ because $r^W_{t+1} \in R_{t+1}$. Also note that $a_t, b_t \in \mathcal{F}_t \not\Rightarrow a_t, b_t \in \mathcal{F}^0_t$, so r^W_{t+1} need not constitute an unconditional βpm.

Conditionally multivariate normal returns: Sufficient condition for CAPM. Suppose $\boldsymbol{r}_{t+1}|\mathcal{F}_t$ has a multivariate normal distribution. First-order condition for single-period problem implies

$$1 = \mathrm{E}_t \left[\underbrace{\delta \frac{u'((W_t - c_t^*) r_{t+1}^W)}{u'(c_t^*)}}_{\mathrm{mrs}_{t+1}} r_{t+1}^i \right] \quad \text{for all } r_{t+1}^i \in R_{t+1}. \tag{44}$$

Since $\boldsymbol{r}_{t+1}|\mathcal{F}_t$ is normally distributed, both r_{t+1}^W and r_{t+1}^i are normally distributed for all $r_{t+1}^i \in R_{t+1}$. Stein's lemma says that if x and y are normally distributed, then

$$\mathrm{cov}[g(x), y] = \mathrm{E}[g'(x)] \, \mathrm{cov}[x, y].$$

Rewrite (44) as

$$1 = \mathrm{cov}_t \left[\delta \frac{u'((W_t - c_t^*) r_{t+1}^W)}{u'(c_t^*)}, r_{t+1}^i \right] + \mathrm{E}_t \left[\delta \frac{u'((W_t - c_t^*) r_{t+1}^W)}{u'(c_t^*)} \right] \mathrm{E}_t[r_{t+1}^i].$$

$$\Rightarrow \mathrm{E}_t[r_{t+1}^i] = \frac{1}{\mathrm{E}_t[\mathrm{mrs}_{t+1}]} - \frac{1}{\mathrm{E}_t[\mathrm{mrs}_{t+1}]} \mathrm{cov}_t \left[\delta \frac{u'((W_t - c_t^*) r_{t+1}^W)}{u'(c_t^*)}, r_{t+1}^i \right].$$

Using Stein's lemma,

$$\mathrm{E}_t[r_{t+1}^i] = \underbrace{\frac{1}{\mathrm{E}_t[\mathrm{mrs}_{t+1}]}}_{\alpha_t} \underbrace{- \frac{1 \times \mathrm{var}_t[r_{t+1}^W]}{\mathrm{E}_t[\mathrm{mrs}_{t+1}]} \mathrm{E}_t \left[\delta \frac{u''(c_{t+1}^*)(W_t - c_t^*)}{u'(c_t^*)} \right]}_{\lambda_t} \underbrace{\frac{\mathrm{cov}_t[r_{t+1}^W, r_{t+1}^i]}{\mathrm{var}_t[r_{t+1}^W]}}_{\beta_t^{i,W}}.$$

So r_{t+1}^W is a $\mathcal{F}_t$-conditional βpm for R_{t+1}.

8.4. CAPM: Multiperiod representative agent

The multiperiod Bellman equation problem is

$$\max_{c_t, \boldsymbol{\alpha}_t} \{u(c_t, s_t) + \delta \, \mathrm{E}[V_{t+1}(W_{t+1}, s_{t+1}) \,|\, s_t, W_t]\}$$

subject to

$$W_{t+1} = (W_t - c_t) \boldsymbol{\alpha}_t' \boldsymbol{r}_{t+1}, \quad \boldsymbol{\alpha}_t' \mathbf{i} = 1, \quad c_t, \boldsymbol{\alpha}_t \in \mathcal{F}_t.$$

If

$$V_{t+1}(W_{t+1}, s_{t+1}) \equiv V_{t+1}(W_{t+1}), \tag{45}$$

then

- The Bellman equation has the same form as the single period problem.
- In both problems, the agent cares about consumption at t and only wealth at $t + 1$.
- We can show that if $u(\cdot)$ is increasing and concave, then $V_{t+1}(W_{t+1}, s_{t+1})$ is increasing and concave in its first argument.

- If sufficient conditions for (3) to hold for all t are satisfied, then we can treat the multiperiod problem as a series of single period problems in which the agent only cares about consumption today and wealth tomorrow. If, in addition, sufficient conditions to obtain the CAPM with a single period representative agent are imposed, we obtain the CAPM with a new multiperiod representative agent.

Proposition 29. *A sufficient condition for (45) to hold for all t is that $u(\cdot)$ is* not *state dependent and return distributions from t to $t+1$ are not forecastable using the $\mathcal{F}_t$ information; i.e.,*

$$u(c_t, s_t) \equiv u(c_t) \quad \textit{and} \quad p(\boldsymbol{r}_{t+1}|s_t) = p(\boldsymbol{r}_{t+1}). \tag{A}$$

PROOF. It is sufficient to show that $V_{t+1}(W_{t+1}) \Rightarrow V_t(W_t)$.

$$\begin{aligned} V_t(W_t, s_t) &= \max_{c_t, \boldsymbol{\alpha}_t} \{u(c_t) + \delta\, \mathrm{E}[V_{t+1}(W_{t+1}) \mid s_t, W_t]\} \\ &= \max_{c_t, \boldsymbol{\alpha}_t} \{u(c_t) + \delta\, \mathrm{E}[V_{t+1}(W_{t+1})|W_t]\} = V_t(W_t) \end{aligned}$$

subject to

$$W_{t+1} = (W_t - c_t)\boldsymbol{\alpha}_t' \boldsymbol{r}_{t+1}, \quad \boldsymbol{\alpha}_t' \mathbf{i} = 1, \quad c_t, \boldsymbol{\alpha}_t \in \mathcal{F}_t$$

with solution $c_t^* = c_t^*(W_t)$ and $\boldsymbol{\alpha}_t^* = \alpha_t^*(W_t)$. □

Sufficient conditions for CAPM.

- (A) plus $\boldsymbol{r}_{t+1}$ multivariate normal ⇒ CAPM holds.
- (A) plus $u(\cdot)$ quadratic ⇒ CAPM holds. Here we need to show that $u(\cdot)$ quadratic ⇒ $V_{t+1}(\cdot)$ quadratic (see Cochrane, chapter 9).

8.5. Special case: Multiperiod log utility

It can be shown (see Problem Set 6) that a T-period agent with log utility $u(c) = \ln(c)$ and patience parameter δ has a value function of the form

$$V_t(W_t, s_t) = A_t(s_t) + B_t \ln(W_t)$$

with

$$B_t = \frac{1 - \delta^{T-t+1}}{1 - \delta}$$

which is not sufficient for (45), and optimal consumption that satisfies $c_t^* = W_t / B_t$. However, it can also be shown that

$$\frac{\delta u'(c_{t+1}^*)}{u'(c_t^*)} = \frac{\delta c_t^*}{c_{t+1}^*} = \frac{\delta W_t B_{t+1}}{W_{t+1}(1 - 1/B_t) r_{t+1}^W B_t)} = \frac{1}{r_{t+1}^W}$$

It follows that a modified CAPM with $\frac{1}{r_{t+1}^W}$ as the factor in the βpm rather than r_{t+1}^W holds both $\mathcal{F}_t$-conditionally and unconditionally for R_{t+1} when the representative agent has log utility, irrespective of whether the representative agent has a single or multiperiod horizon or whether (A) holds.

8.6. ICAPM: Multiperiod representative agent

A sufficient condition for the intertemporal CAPM (ICAPM) to hold is that $r_{t+1}, s_{t+1} | s_t, W_t$ multivariate normal.

Lemma (Multivariate Stein's lemma)**.** *Suppose $y, x_1, \ldots, x_2$ is multivariate normal. Then with appropriate restrictions on the function g,*

$$\operatorname{cov}[g(x_1, \ldots, x_s), y] = \sum_{k=1}^{s} \mathrm{E}[g_k(x_1, \ldots, x_s)] \operatorname{cov}[x_k, y].$$

The first-order condition for the multiperiod Bellman equation problem is

$$u'(c_t^*(W_t, s_t)) = \mathrm{E}_t[\delta u'(c_{t+1}^*(W_{t+1}, s_{t+1})) r_{t+1}^i] \quad \text{for all } r_{t+1}^i \in R_{t+1},$$

$$1 = \mathrm{E}_t \left[\underbrace{\frac{\delta u'(c_{t+1}^*(W_{t+1}, s_{t+1}))}{u'(c_t^*)}}_{\mathrm{mrs}_{t+1}} r_{t+1}^i \right] \quad \text{for all } r_{t+1}^i \in R_{t+1},$$

where $W_{t+1} = (W_t - c_t^*) r_{t+1}^W$.

Define

$$c_{t+1,W}^* = \frac{\partial c_{t+1}^*}{\partial W_{t+1}}, \quad c_{t+1,k}^* = \frac{\partial c_{t+1}^*}{\partial s_{k,t+1}},$$

where $s_{k,t+1}$ is the kth element of $\boldsymbol{s}_{t+1}$. From above we can rewrite the first-order condition as

$$\forall r_{t+1}^i \in R_{t+1} : \mathrm{E}_t[r_{t+1}^i] = \frac{1}{\mathrm{E}_t[\mathrm{mrs}_{t+1}]} - \frac{1}{\mathrm{E}_t[\mathrm{mrs}_{t+1}]} \operatorname{cov}_t \left[\delta \frac{u'(c_{t+1}^*(W_{t+1}, s_{t+1}))}{u'(c_t^*)}, r_{t+1}^i \right].$$

Using Stein's lemma,

$$\begin{aligned}
\mathrm{E}_t[r_{t+1}^i] &= \frac{1}{\mathrm{E}_t[\mathrm{mrs}_{t+1}]} - \frac{\delta}{\mathrm{E}_t[\mathrm{mrs}_{t+1}]} \Bigg\{ \mathrm{E}_t \left[\frac{u''(c_{t+1}^*)}{u'(c_t^*)} c_{t+1,W}^* (W_t - c_t^*) \right] \operatorname{cov}_t[r_{t+1}^W, r_{t+1}^i] \\
&\qquad + \sum_{k=1}^{s} \mathrm{E}_t \left[\frac{u''(c_{t+1}^*)}{u'(c_t^*)} c_{t+1,k}^* \right] \operatorname{cov}_t[s_{k,t+1}, r_{t+1}^i] \Bigg\} \\
&= \frac{1}{\mathrm{E}_t[\mathrm{mrs}_{t+1}]} + \hat{\boldsymbol{\lambda}}' \operatorname{cov}_t[\underbrace{\begin{bmatrix} r_{t+1}^W \\ \boldsymbol{s}_{t+1} \end{bmatrix}}_{\boldsymbol{s}_{t+1}^W}, r_{t+1}^i] \quad \text{for all } r_{t+1}^i \in R_{t+1} \\
&= \underbrace{\frac{1}{\mathrm{E}_t[\mathrm{mrs}_{t+1}]}}_{\alpha_t} + \underbrace{\hat{\boldsymbol{\lambda}}_t{}' \operatorname{cov}_t[\boldsymbol{s}_{t+1}^W, \boldsymbol{s}_{t+1}^{W\prime}]}_{\boldsymbol{\lambda}_t'} \times \\
&\qquad \underbrace{(\operatorname{cov}_t[\boldsymbol{s}_{t+1}^W, \boldsymbol{s}_{t+1}^{W\prime}])^{-1} \operatorname{cov}_t[\boldsymbol{s}_{t+1}^W, r_{t+1}^i]}_{\beta_t^{i,s^W}} \quad \text{for all } r_{t+1}^i \in R_{t+1}.
\end{aligned}$$

So $\begin{bmatrix} r^W_{t+1} \\ \boldsymbol{s}_{t+1} \end{bmatrix} = \boldsymbol{s^W_{t+1}}$ constitutes a conditional βpm for R_{t+1}.

CHAPTER 9

Arbitrage Pricing Theory

9.1. Preliminaries

Economy. Consider an economy defined as follows. There is a single period with prices at t and payoffs at $t+1$, and the information set available at t is $\mathcal{F}$. There exists a risk free asset x^f, with associated return r^f. The set of payoffs is x^i for $i = 1, 2, 3, \ldots, \infty$ with associated prices $p(x^i)$ and returns r^i. Define the payoff space

$$X^N = \{\omega^f x^f + \omega^1 x^1 + \ldots + \omega^N x^N : \omega^i \in \mathcal{F}, \quad i = f, 1, \ldots, N\}.$$

Payoffs satisfy a K factor structure, so for any i,

$$x^i = \mathrm{E}[x^i] + \tilde{\boldsymbol{\beta}}_i' \boldsymbol{f} + \tilde{e}^i, \tag{46}$$

where $\mathrm{E}[\tilde{e}^i] = 0$, $\operatorname{cov}[\tilde{e}^i, \boldsymbol{f}] = 0$, $\mathrm{E}[\boldsymbol{f}] = \mathbf{0}$, $\mathrm{E}[\tilde{e}^i \tilde{e}^j] = 0$ for $i \neq j$, $\mathrm{E}[(\tilde{e}^i)^2] < \bar{\sigma}^2_{\tilde{e}}$, and

$$\mathrm{E}\left[\left(\frac{\tilde{e}^i}{p(x^i)}\right)^2\right] < \bar{\sigma}^2_{\epsilon}.$$

The K factor structure is a restriction on the covariance matrix of returns.

Define M^N to be the set of sdf for X^N; i.e.,

$$M^N = \{m : \mathrm{E}[mx] = p(x) \text{ for all } x \in X^N\}.$$

We can see that $M^N \supset M^{N+1} \supset M^{N+2} \ldots$.

Fixing N. For fixed N,

$$\begin{aligned}
r^i &= \mathrm{E}[r^i] + \boldsymbol{\beta}_i' \boldsymbol{f} + e^i, \quad i = 1, 2, \ldots, N,\\
p(r^i) &= \mathrm{E}[r^i] p(1) + \boldsymbol{\beta}_i' \mathrm{E}[m\boldsymbol{f}] + \mathrm{E}[me^i], \qquad \text{for all } m \in M^N,\\
\mathrm{E}[r^i] &= r^f + \boldsymbol{\beta}_i'(-r^f \mathrm{E}[m\boldsymbol{f}]) - r^f \mathrm{E}[me^i]\\
&= r^f + \boldsymbol{\beta}_i' \boldsymbol{\lambda} + u^i,
\end{aligned}$$

where $e^i = \tilde{e}^i / p(x^i)$ and $\boldsymbol{\beta}_i' = \tilde{\boldsymbol{\beta}}_i' / p(x^i)$.

Factors. f^k satisfy one of the following for each $k = 1, 2, \ldots, K$:

- $f^k \notin X^N$ for all N, e.g. $f^k = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N r^i$.
- $$f^k \begin{cases} \notin X^N & N < N^k \\ \in X^N & N \geq N^k \end{cases}, \quad \text{e.g. } f^k = \frac{1}{500} \sum_{i=1}^{500} r^i.$$

Define M^f to be the set of sdf for $\boldsymbol{f}$; i.e.,

$$M^f = \{m : \mathrm{E}[m\boldsymbol{f}] = p(\boldsymbol{f})\}.$$

Letting $N^f = \max_k\{N_k : k = 1, 2, \ldots, K\}$, we can see that $M^f \supset M^{N^f} \supset M^{N^f+1} \supset M^{N^f+2} \ldots$.

9.2. Pricing error bounds

Fixing N. For a given N, x^i with $\sigma^2[\tilde{e}^i] > 0$, a K factor structure and $r^f > \underline{u} > 0$, we can find $p(\cdot)$ and associated M^N such that there exists an $m \in M^N$ such that $|u^i| > \underline{u}$.

Case 1: $\boldsymbol{f} \notin X^N$

Assign "prices" to $\boldsymbol{f}$, $p(\boldsymbol{f})$, which has an associated M^f defined as

$$M^f = \{m : \mathrm{E}[m\boldsymbol{f}] = p(\boldsymbol{f})\}.$$

and $f^* = \boldsymbol{\beta}^*\boldsymbol{f} \in M^f$ where f^* is the unique sdf associated with $p(\cdot)$ that lies in the space generated by $\boldsymbol{f}$. Can define the $p(\cdot)$ for X^N associated with a given $\tilde{m}^f \in M^f$ as follows:

$$\forall x \in X^N : p(x) = \mathrm{E}[\tilde{m}^f x].$$

Consider $m^f = f^* + \alpha^i\tilde{e}^i$. Now $m^f \in M^f$ since $\mathrm{E}[f^*\tilde{e}^i] = 0$ and so

$$\mathrm{E}[m^f\boldsymbol{f}] = \mathrm{E}[(f^* + \alpha^i\tilde{e}^i)\boldsymbol{f}] = \mathrm{E}[f^*\boldsymbol{f}] = p(\boldsymbol{f}).$$

Further,

$$\mathrm{E}[m^f\tilde{e}^i] = \underbrace{\mathrm{E}[f^*\tilde{e}^i]}_{=0} + \mathrm{E}[\alpha^i\tilde{e}^i\tilde{e}^i] = \alpha^i\,\mathrm{E}[(\tilde{e}^i)^2],$$

and so for $|\alpha^i|$ sufficiently large, it can easily be shown, using $e^i = \tilde{e}^i/p(x^i)$, that

$$\mathrm{E}[m^f e^i] = \mathrm{E}[m^f\frac{\tilde{e}^i}{p(x^i)}] = \frac{\alpha^i\sigma^2[\tilde{e}^i]}{p(x^i)} > \underline{u}\frac{1}{r^f}$$

as required. See figure 10 on page 79. Choose $p(\cdot)$ so that $p(x) = \mathrm{E}[m^f x]$ for all $x \in X^N$.

Case 2: $\boldsymbol{f} \in X^N$

Same as above, but the "prices" assigned to $\boldsymbol{f}$ become their actual prices under the chosen price functional $p(\cdot)$.

Limiting arguments fixing m. Fix the sdf for the economy m and consider $\hat{r} \in X^N$ such that

$$\hat{r} = \mathrm{E}[\hat{r}] + \hat{\boldsymbol{\beta}}\boldsymbol{f} + \hat{e}, \quad \mathrm{E}[\hat{r}] = r^f + \hat{\boldsymbol{\beta}}\boldsymbol{\lambda} + \hat{u}, \quad p(\hat{r}) = 1.$$

We can show that $|\hat{u}| \to 0$ as $\sigma[\hat{e}] \to 0$ since

$$\mathrm{E}[m\hat{e}] = \sigma[m]\sigma[\hat{e}]\rho_{me} < \sigma[m]\sigma[\hat{e}] \to 0$$

as $\sigma[\hat{e}] \to 0$.

Fix the sdf m for the economy and consider

$$\hat{r}_N = \frac{1}{N}\sum_{i=1}^{N} r^i$$

where

$$\hat{r}_N = \mathrm{E}[\hat{r}_N] + \hat{\boldsymbol{\beta}}_N \boldsymbol{f} + \hat{e}_N, \quad \mathrm{E}[\hat{r}_N] = r^f + \hat{\boldsymbol{\beta}}_N \boldsymbol{\lambda} + \hat{u}_N.$$

We can show that $|\hat{u}_N| \to 0$ as $N \to \infty$ since

$$\sigma^2[\hat{e}_N] = \frac{1}{N^2}\sum_{i=1}^{N}\sigma^2[e^i] < \frac{1}{N^2} N\bar{\sigma}_e^2 \to 0$$

as $N \to \infty$.

Limiting arguments without fixing m. Let $p(\cdot)$ be the pricing functional for the economy (i.e., defined on X^N for all N). For each N, no arbitrage says it is associated with a set of sdfs M^N. If we consider

$$\hat{r}_N = \frac{1}{N}\sum_{i=1}^{N} r^i$$

without fixing the sdf m associated with the $p(\cdot)$ for the economy, we cannot show that $|\hat{u}_N| \to 0$ as $N \to \infty$ for all $p(\cdot)$ defined on X^N for all N. We do this by finding a $p(\cdot)$ defined on X^N for all N for which $|\hat{u}_N| \nrightarrow 0$ as $N \to \infty$

Case 1: Assign "prices" to $\boldsymbol{f}$, $p(\cdot)$, which has an associated M^f and $f^* = \boldsymbol{\beta}^* \boldsymbol{f} \in M^f$ where f^* is the unique sdf associated with $p(\cdot)$ that lies in the space generated by $\boldsymbol{f}$. Consider

$$m^N = f^* + \sum_{i=1}^{N}\alpha^i \tilde{e}^i,$$

which has an associated pricing functional $p^N(\cdot)$. Now

$$\begin{aligned}
p^N(\tilde{\boldsymbol{\beta}}_i' \boldsymbol{f}) &= p(\tilde{\boldsymbol{\beta}}_i' \boldsymbol{f}) \quad \text{for all } N,\\
p^N(\tilde{e}^i) &= \mathrm{E}[m^N \tilde{e}^i]\\
&= \mathrm{E}[(f^* + \sum_{j=1}^{N}\alpha^j \tilde{e}^j)\tilde{e}^i]\\
&= \alpha^i \sigma^2[\tilde{e}^i] = p(\tilde{e}^i) \quad \text{for all } N.
\end{aligned}$$

So $\boldsymbol{f}$ and any $\tilde{e}^i$ is priced the same by m^N for all $N \geq i$ as required for the sequence of m^Ns to be associated with a $p(\cdot)$ defined on X^N for all N. But notice that since

$$\mathrm{E}[m^N e^i] = \mathrm{E}[m^N \frac{\tilde{e}^i}{p(x^i)}] = \frac{\alpha^i \sigma^2[\tilde{e}^i]}{p(x^i)},$$

any u^i sequence between r^f and 0 can be obtained from this $p(\cdot)$ by appropriately choosing the α^i sequence. Since

$$\hat{u}_N = \frac{1}{N}\sum_{i=1}^{N} u^i,$$

we can obtain any $\hat{u}_N$ sequence between r^f and 0 by appropriately choosing an α^i sequence (so that $|\alpha^i|\sigma^2[\tilde{e}^i]$ is bounded below). Notice that $|\alpha^i|\sigma^2[\tilde{e}^i]$ bounded below implies that $\sigma^2[m^N]$ is increasing without bound in N. But the implication is that when $\sigma^2[m^N]$ is allowed to increase without bound in N, any desired $\hat{u}_N$ sequence between r^f and 0 can be obtained by appropriately choosing the $p^N(\cdot)$ sequence.

Case 2: Same as above, but the "prices" assigned to $\boldsymbol{f}$ become their actual prices under the chosen price functional $p(\cdot)$.

9.3. Asymptotic no arbitrage

We need to restrict the set of stochastic discount factors further to place restrictions on $\hat{u}$s. This can be done with asymptotic no arbitrage.

Definition (Asymptotic no arbitrage)**.** Asymptotic no arbitrage is attained if there does not exist a sequence of zero cost portfolios r_N^z such that $\mathrm{E}\, r_N^2 \to \infty$ and $\sigma^2[r_N^2] \to 0$ as $N \to \infty$.

Asymptotic no arbitrage is stronger than no arbitrage or the law of one price.

For a given N, run the following cross-sectional regression,

$$\mathrm{E}[r^i] = \lambda_N^0 + \boldsymbol{\lambda}'_N \boldsymbol{\beta}^i + \eta_N^i.$$

We know that $\sum_{i=1}^N \eta_N^i = 0$ and $\sum_{i=1}^N \eta_N^i \boldsymbol{\beta}^i = \mathbf{0}$. Note that η_N^i can be thought of a pricing error in the same way that u^i can be thought of as a pricing error. Consider a portfolio with weights $h_N \eta_N^i$ for r^i,

$$\begin{aligned}\mathrm{E}[r^p] = \sum_{i=1}^N h_N \eta_N^i \,\mathrm{E}\, r^i &= h_N \left(\sum_{i=1}^N \left(\eta_N^i \lambda_N^0 + \eta_N^i \boldsymbol{\lambda}'_N \boldsymbol{\beta}^i + (\eta_N^i)^2 \right) \right) \\ &= h_N \sum_{i=1}^N (\eta_N^i)^2,\end{aligned}$$

$$r^p - \mathrm{E}[r^p] = \sum_{i=1}^N h_N (\eta_N^i \boldsymbol{\beta}^{i\prime} \boldsymbol{f} + \eta_N^i e^i) = h_N \sum_{i=1}^N \eta_N^i e^i,$$

$$\sigma^2[r^p] = h_N^2 \sum_{i=1}^N (\eta_N^i)^2 \sigma^2[e^i] < h_N^2 \sum_{i=1}^N (\eta_N^i)^2 \bar{\sigma}_e^2.$$

Choosing $h_N = (\sum_{i=1}^N (\eta_N^i)^2)^{-\gamma}$, we obtain

$$\mathrm{E}[r^p] = \left(\sum_{i=1}^N (\eta_N^i)^2 \right)^{1-\gamma}, \quad \sigma^2[r^p] < \bar{\sigma}_e^2 \left(\sum_{i=1}^N (\eta_N^i)^2 \right)^{1-2\gamma},$$

so asymptotic no arbitrage implies

$$\lim_{N\to\infty} \sum_{i=1}^N (\eta_N^i)^2 = c < \infty.$$

Otherwise take $\frac{1}{2} < \gamma < 1$ and obtain $\mathrm{E}[r^p] \to \infty$ and $\sigma^2[r^p] \to 0$. Note that asymptotic no arbitrage implies that, for any finite number of assets, the sum of the squared pricing errors can be arbitrarily large

For specific functional forms, an expected utility maximizing investor may *not* find it utility improving to take advantage of asymptotic no arbitrage opportunities. For example, consider the following sequence of zero cost portfolios:

$$r_N^z = \begin{cases} -N & \text{with probability } \frac{1}{N^3} \\ N & \text{with probobabity } 1 - \frac{2}{N^3} \\ 3N & \text{with probobabity } \frac{1}{N^3}. \end{cases}$$

For these portfolios, $\mathrm{E}[r_N^z] = N \to \infty$ as $N \to \infty$, and $\sigma^2[r_N^z] = \frac{8}{N} \to 0$ as $N \to \infty$, which makes this an asymptotic arbitrage opportunity. However,

$$\mathrm{E}[-e^{-r_N^z}] = \underbrace{-e^N \frac{1}{N^3}}_{\to -\infty} - \underbrace{e^{-N}\left(1 - \frac{1}{2N^3}\right)}_{\to 0} - \underbrace{e^{-3N}\frac{1}{N^3}}_{\to 0} \to -\infty.$$

We need to bound utility from below for asymptotic arbitrage opportunities to be attractive. A utility is unbounded from below if $u(c) \to -\infty$ as $c \to -\infty$.

9.4. Law of one price plus a restriction on the variance of $m \in M^N$

Imposing the law of one price plus a restriction on the variance of $m \in M^N$ implies a restriction on the maximum Sharpe ratio attainable using X^N and the factors, since

$$\forall \hat{r} \in R_{t+1} : \frac{\mathrm{E}[\hat{r}] - r^f}{\sigma[\hat{r}]} \leq \frac{\sigma[m]}{\mathrm{E}[m]}$$

whenever m is an sdf for R_{t+1}.

Consider for a given N

$$\hat{r} = \mathrm{E}[\hat{r}] + \hat{\beta} \boldsymbol{f} + \hat{e},$$

where

$$\mathrm{E}\,\hat{r} = r^f + \hat{\boldsymbol{\beta}}\boldsymbol{\lambda} + \hat{u}, \quad \hat{r} \in X^N.$$

We can show that for a given N: $|\hat{u}| \to 0$ as $\sigma^2[\hat{e}] \to 0$ for all $p^N(\cdot)$ defined on X^N by an associated m^N that satisfies $\sigma^2[m^N] < A$.

Consider

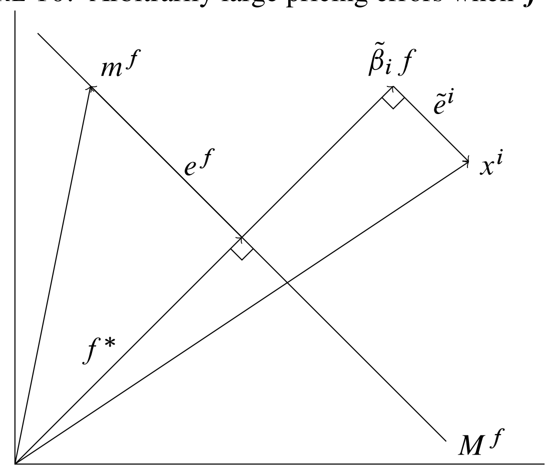
$$\hat{r}_N = \frac{1}{N}\sum_{i=1}^{N} r^i$$

where

$$\hat{r}_N = \mathrm{E}[\hat{r}_N] + \hat{\boldsymbol{\beta}}_N \boldsymbol{f} + \hat{e}_N, \quad \mathrm{E}[\hat{r}_N] = r^f + \hat{\boldsymbol{\beta}}_N \boldsymbol{\lambda} + \hat{u}_N$$

as before. We can show that: $|\hat{u}_N| \to 0$ as $N \to \infty$ for any $p(\cdot)$ defined on X^N by an associated m^N that satisfies $\sigma^2[m^N] < A$ for all N. Before, $\sigma^2[m^N]$ increasing in N without bound allowed $|\hat{u}_N|$ to increase as $\sigma^2[\hat{e}_N] \to 0$. This suggests that an upper bound on $\sigma^2[m^N]$, i.e. on the maximum attainable Sharpe ratio, may be enough to get $|\hat{u}_N| \to 0$ as $\sigma^2[\hat{e}_N] \to 0$ (which in fact it is).

FIGURE 10. Arbitrarily large pricing errors when $\boldsymbol{f} \notin X^N$



CHAPTER 10

Heterogeneity and Equilibrium: Homogenous Beliefs

10.1. Existence of a representative agent

Complete markets imply that a representative agent exists. This will be illustrated in a one-period setting, but can be generalized to a multiperiod setting.

Competitive equilibrium. The competitive equilibrium problem consists of:

- 2 consumers, A and B.
- One good.
- Two dates, $t = 0$ and 1.
- Consumption takes place only at $t = 1$.
- Endowments at date 0 are w^A and w^B.
- States of nature are indexed by $s = 1, 2, \ldots, S$.
- $p(s)$ is the probability of state s.
- $\pi(s)$ is the state-price for an Arrow–Debreu security, i.e. a security that pays 1 if state s occurs and 0 otherwise.
- Complete markets, so $\pi(s)$ is unique and there exists an Arrow–Debreu security for every state.
- Utility functions are strictly increasing and strictly concave von Neumann–Morgenstern utility function, denoted $u_A(\cdot)$ and $u_B(\cdot)$.

Consumer A's problem is

$$\max_{c_A(s)} \sum_{s=1}^{S} p(s) u_A(c_A(s)) \quad \text{s.t.} \sum_{s=1}^{S} \pi(s) c_A(s) = w_A.$$

The first-order condition is

$$p(s)u'_A(c_A(s)) - \lambda_A \pi(s) = 0 \quad \text{for } s = 1, 2, \ldots, S, \tag{1}$$
$$\sum_{s=1}^{S} \pi(s) c_A(s) - w_A = 0.$$

Let $c_A^*(s)$ and λ_A^* be the solution. Consumer B's problem is analogous and again let $c_B^*(s)$ and λ_B^* be the solution.

Recall the definition of a representative agent from chapter 8. To establish the existence of a representative agent, we need to show that there exists a vNM utility function such that aggregate wealth $w_A + w_B = w$ as the wealth constraint gives aggregate consumption $c_A^*(s) + c_B^*(s) = c^*(s)$ as the solution.

Central planner's problem. The central planner's problem is

$$\max_{c_A(s),c_B(s)} \sum_{s=1}^{S} p(s)\{\alpha u_A(c_A(s)) + \beta u_B(c_B(s))\} \quad (\alpha, \beta > 0)$$

subject to

$$\sum_{s=1}^{S}(c_A(s) + c_B(s))\pi(s) = w_A + w_B.$$

The first-order conditions are

$$\begin{aligned}
&\alpha p(s)u'_A(c_A(s)) - \lambda\pi(s) = 0,\\
&\beta p(s)u'_B(c_B(s)) - \lambda\pi(s) = 0,\\
&\sum_{s=1}^{S}(c_A(s) + c_B(s))\pi(s) = w_A + w_B.
\end{aligned}$$

Let us consider $\alpha = 1/\lambda_A^*$ and $\beta = 1/\lambda_B^*$. Notice that λ_A^* and λ_B^* are positive since they are shadow prices. Take $\lambda = 1, c_A(s) = c_A^*(s), c_B(s) = c_B^*(s)$ as a candidate solution. The first-order condition for consumer A's problem (1) $\Leftrightarrow$

$$\frac{1}{\lambda_A^*} p(s)u'_A(c_A(s)) - 1\pi(s) = 0,$$

and similarly for consumer B. Consumers A and B's budget constraints being satisfied by $c_A^*(s)$ and $c_B^*(s)$ implies that the central planner's budget constraint is satisfied. Thus, the solution to the central planner's problem with α set equal to $1/\lambda_A^*$ and β set equal to $1/\lambda_B^*$ corresponds to the solutions to the problems of consumers A and B.

Obtaining the representative agent. Consider the function

$$u(c) = \max_{c_A}\left\{\frac{1}{\lambda_A^*}u_A(c_A) + \frac{1}{\lambda_B^*}u_B(c_B)\right\} \quad \text{s.t. } c_A + c_B = c.$$

It can be shown that $u(\cdot)$ inherits increasing and concave from $u_A(\cdot)$ and $u_B(\cdot)$. Let us rewrite the central planner's problem with $\alpha = 1/\lambda_A^*$ and $\beta = 1/\lambda_B^*$; we are interested in this problem since $c_A^*(s)$

and $c_B^*(s)$ are the solution.

$$\begin{aligned}
&\max_{c_A(s),c_B(s)} \sum_{s=1}^{S} p(s)\left\{\frac{1}{\lambda_A^*}u_A(c_A(s)) + \frac{1}{\lambda_B^*}u_B(c_B(s))\right\} \\
&\qquad \text{s.t.} \sum_{s=1}^{S}(c_A(s)+c_B(s))\pi(s) = w_A + w_B \\
&= \max_{c(s)} \left\{\sum_{s=1}^{S} p(s)\left\{\max_{c_A(s)} \left\{\frac{1}{\lambda_A^*}u_A(c_A(s)) + \frac{1}{\lambda_B^*}u_B(c_B(s))\right\}_{\text{s.t. } c_A(s)+c_B(s)=c(s)}\right\}\right\} \\
&\qquad \text{s.t.} \sum_{s=1}^{S} c(s)\pi(s) = w_A + w_B = w \\
&= \max_{c(s)} \left\{\sum_{s=1}^{S} p(s)u(c(s))\right\} \quad \text{s.t.} \sum_{s=1}^{S} c(s)\pi(s) = w.
\end{aligned}$$

So $u(\cdot)$ is a vNM utility function for the representative agent. Notice that λ_A^* depends on w_A and λ_B^* depends on w_B, so the λs change with the distribution of endowments.

Pareto optimality in complete markets. Consider the following problem,

$$\max_{c_A(s),c_B(s)} \sum_{s=1}^{S} p(s)u_A(c_A(s))$$

subject to

$$\begin{aligned}
\sum_{s=1}^{S} \pi(s)(c_A(s)+c_B(s)) &= w, \\
\sum_{s=1}^{S} p(s)u_B(c_B(s)) &= \sum_{s=1}^{S} p(s)u_B(c_B^*(s)).
\end{aligned}$$

The first-order conditions are

$$\begin{aligned}
p(s)u_A'(c_A(s)) - \lambda_1\pi(s) &= 0, \\
\lambda_1\pi(s) + \lambda_2 p(s)u_B'(c_B(s)) &= 0
\end{aligned}$$

plus the 2 constraints. Consumption allocations that satisfy this problem are Pareto optimal. We can see that $c_A(s) = c_A^*(s)$, $c_B(s) = c_B^*(s)$, $\lambda_1 = \lambda_A*$ and $\lambda_2 = -\frac{\lambda_A^*}{\lambda_B^*}$ satisfy the first-order condition, and so the competitive consumption allocations in a complete market are Pareto optimal.

Implications. If consumers have uninsurable idiosyncratic labor income, then markets may not be complete and there may not be a representative agent.

10.2. K-fund separation

Definition (K-fund separation)**.** Suppose an economy has N assets. An economy exhibits K-fund separation when every agent in the economy holds a combination of the same K portfolios, where K is less than N.

It is possible to achieve K-fund separation via two channels: restrictions on the utility functions of all agents in the economy, which is know as utility-based separation; and, restrictions on the joint return distribution for all assets in the economy. Typically, K-fund separation implies the existence of a representative agent.

10.3. Single period setting: Utility-based separation ⇒ representative agent exists

The following sets of restrictions on utility functions establish K-fund separation and the existence of a representative agent, in single-period setting.

Quadratic utility. Quadratic utility delivers 2-fund separation because any investor's portfolio lies on the minimum variance frontier. Any portfolio on the minimum variance frontier can be represented as a portfolio of 2 assets where those 2 assets are on the frontier. This holds with or without a riskless asset.

Power utility. All consumers having the same RRA γ and different endowments w_0 delivers one-fund separation, since all investors hold the same portfolio irrespective of endowments.

HARA utility. Assume that all consumers have the same $\varphi = 1-\gamma$ coefficients, but have different subsistence levels $\hat{w}$. If the risk-free rate exists, we get 2-fund separation.

Investor i's problem is

$$\max_{\{\alpha_j^i\}_{j=1}^N} \mathrm{E}\left[\frac{(w_1^i - \hat{w}^i)^{1-\gamma}}{1-\gamma}\right] \quad \text{s.t. } w_1^i = w_0^i \sum_{j=1}^N \alpha_j^i r^j, \quad \sum_{j=1}^N \alpha_j = 1,$$

where asset 1 is the riskless asset. Suppose the investor places $\frac{\hat{w}^i}{r^f}$ in asset 1. This leaves $w_0^i - \frac{\hat{w}^i}{r^f}$ to invest in the N assets, so

$$w_1^i = \left(\frac{\hat{w}^i}{r^f}\right) r^f + \left(w_0^i - \frac{\hat{w}^i}{r^f}\right) \sum_{j=1}^N \hat{\alpha}_j^i r^j, \quad \sum_{j=1}^N \hat{\alpha}_j^i = 1.$$

We need to show 2-fund separation. We can think of r^f and the portfolio given by $\hat{\alpha}^i$ as the 2 funds and need to show that $\hat{\alpha}_j^i = \hat{\alpha}_j$, which does not depend on $\hat{w}^i$ or w_0^i.

We can rewrite investor i's problem as

$$\max_{\{\hat{\alpha}_j^i\}_{i=1}^N} \mathrm{E}\left[\frac{\left(\hat{w}^i + \left(w_0^i - \frac{\hat{w}^i}{r^f}\right)\sum_{j=1}^N \hat{\alpha}_j^i r^j - \hat{w}^i\right)^{1-\gamma}}{1-\gamma}\right] \quad \text{s.t.} \sum_{j=1}^N \hat{\alpha}_j^i = 1$$

$$= \max_{\{\hat{\alpha}_j^i\}_{i=1}^N} \left(w_0^i - \frac{\hat{w}^i}{r^f}\right)^{1-\gamma} \mathrm{E}\left[\frac{\left(\sum_{j=1}^N \hat{\alpha}_j^i r^j\right)^{1-\gamma}}{1-\gamma}\right] \quad \text{s.t.} \sum_{j=1}^N \hat{\alpha}_j^i = 1,$$

where the term inside the expectation does not depend on w_0^i or $\hat{w}^i$. So $\{\hat{\alpha}_j^i\}_{i=1}^N$ does not depend on w_0^i or $\hat{w}^i$.

10.4. Multiperiod setting: Utility based separation ⇒ representative agent exists

Utility-based fund separation is harder to establish in a multiperiod setting. The value function depends on state variables as well as wealth and this dependence typically varies with age. For example, power utility implies

$$V_t(s_t, w_t) = \max_{c_t, \boldsymbol{\alpha}_t} \left\{ \mathrm{E}\left[\frac{A_{t+1}(s_{t+1})}{1-\gamma} w_{t+1}^{1-\gamma} \mid s_t, w_t\right] + \frac{c_t^{1-\gamma}}{1-\gamma} \right\}.$$

The first-order condition is

$$(w_t - c_t)^{-\gamma}\, \mathrm{E}[\underbrace{A_{t+1}(s_{t+1})}_{\text{depends on age}} (r_{t+1}^w)^{-\gamma}(\boldsymbol{r} - \mathbf{i}_N r^f) \mid s_t, w_t] = 0.$$

One fund separation is likely to break down and the age distribution become important. This is not a problem if the return distribution does not depend on the state.

10.5. Single period setting: Distribution based separation ⇒ representative agent exists

The only restriction on utility is von Neumann–Morgenstern utility. In a single-period setting, the joint return distribution being a K fund separating distribution is both necessary and sufficient for K fund separation. Before defining a K-fund separating distribution, the following results will prove useful for the the rest of this section.

Mean-variance spanning. The following theorem can be proven.

Theorem 30. *Consider a $N \times 1$ vector of risky returns $\boldsymbol{R}$, a $K \times 1$ vector of portfolios $\boldsymbol{R}_p$ formed from $\boldsymbol{R}$, and a linear regression of $\boldsymbol{R}$ on $\boldsymbol{R}_p$ and a constant:*

$$\boldsymbol{R} = \boldsymbol{a} + B\boldsymbol{R}_p + \boldsymbol{e}$$

where $\boldsymbol{a}$ and B are respectively an $N \times 1$ vector and an $N \times K$ matrix of regression coefficients, $\mathrm{E}[e] = 0$*, and cov$[\boldsymbol{R}_p, \boldsymbol{e}'] = 0$. $\boldsymbol{R}_p$ is said to span $\boldsymbol{R}$ in a mean-variance sense if the minimum-variance frontier for $\boldsymbol{R}_p$ exactly coincides with the minimum-variance frontier for $\boldsymbol{R}$. $\boldsymbol{R}_p$ spans $\boldsymbol{R}$ in a mean-variance sense iff $\boldsymbol{a} = 0$ and $B\boldsymbol{i}_K = \boldsymbol{i}_N$.*

The proof is the answer to question 3 of Problem Set 9.

Second-order stochastic dominance. Define second-order stochastic dominance as follows:

Definition. (Second-order stochastic dominance) A random variable z_2 strictly second-order stochastically dominates another random variable z_1 if the distribution of z_1 equals the distribution of $(z_2+\eta+\epsilon)$ where η and ϵ are random variables such that:

- $\eta \leq 0$;
- $\mathrm{E}[\epsilon \mid z_2 + \eta] = 0$; and,
- at least one of the following is true: $\mathrm{prob}(\eta < 0) > 0$ or $\mathrm{var}[\epsilon] > 0$.

The following theorem can be proven:

Theorem 31. *Suppose a random variable z_2 strictly second-order stochastically dominates another random variable z_1. If the von Neumann Morgenstern utility function of an investor is monotone increasing and concave, then she will prefer z_2 to z_1 if comparing the two in isolation.*

The proof is the answer to question 1 of Problem Set 9.

K **fund separating distribution.** A vector of asset returns $\underset{N\times 1}{\boldsymbol{R}}$ constitutes a K-fund separating distribution iff

$$\exists \tilde{Y}, \underset{(K-1)\times 1}{\tilde{\boldsymbol{Z}}}, \underset{N\times(K-1)}{B}, \underset{N\times 1}{\tilde{\boldsymbol{\epsilon}}}, \{\boldsymbol{\alpha}_j\}_{j=1}^{K}$$

such that

i. $\boldsymbol{R} = Y\mathbf{i}_N + B\boldsymbol{Z} + \boldsymbol{\epsilon}$ and $\mathrm{E}[\boldsymbol{\epsilon}] = \mathbf{0}$.
ii. $\forall i, \forall \xi : \mathrm{E}[\epsilon_i \mid Y + \xi\boldsymbol{Z}] = 0$ where ϵ_i is the ith element of $\boldsymbol{\epsilon}$.
iii. $\boldsymbol{\alpha}_j'\boldsymbol{\epsilon} = 0$ for $j = 1, 2, \ldots, K$.
iv. $\boldsymbol{\alpha}_j'\mathbf{i}_N = 1$ for $j = 1, 2, \ldots, K$.
v. Defining $\alpha = \begin{bmatrix} \boldsymbol{\alpha}_1' \\ \vdots \\ \boldsymbol{\alpha}_K' \end{bmatrix}$, the two matrices

$$\begin{bmatrix} \mathbf{i}_N & \vdots & B \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{i}_K & \vdots & \alpha B \end{bmatrix}$$

generate submatrices from any set of columns with the same rank.

Condition ii. implies that $\forall i, \forall \xi : \mathrm{cov}[\epsilon_i, Y + \xi\boldsymbol{Z}] = 0$. Condition v. means that $\begin{bmatrix} \mathbf{i}_N & \vdots & B \end{bmatrix}$ of rank K implies that $\begin{bmatrix} \mathbf{i}_K & \vdots & \alpha B \end{bmatrix}$ is of full rank. $\{\boldsymbol{\alpha}_j\}_{j=1}^{K}$ gives the K funds.

Understanding condition v.

- Condition v. is the weakest condition that implies that any loading vector $\underset{(K-1)\times 1}{\boldsymbol{b}}$ that can be achieved using the N assets can also be achieved using the K portfolios.

- Can use the N assets to create an arbitrary asset:

$$R_q = \underset{1\times N}{\boldsymbol{\alpha}_q'} \underset{N\times 1}{\boldsymbol{R}} = Y + \underbrace{(\boldsymbol{\alpha}_q' B)}_{\underset{1\times(K-1)}{\boldsymbol{b}_q'}} \boldsymbol{Z} + \underbrace{\boldsymbol{\alpha}_q' \boldsymbol{\epsilon}}_{\epsilon_q}.$$

- Define

$$\underset{K\times 1}{\boldsymbol{R}_p} \equiv \underset{K\times N}{\alpha} \underset{N\times 1}{\boldsymbol{R}} \quad \text{and} \quad \underset{K\times(K-1)}{B_p} \equiv \underset{K\times N}{\alpha} \underset{N\times(K-1)}{B}.$$

For the K funds:

$$\alpha \boldsymbol{R} = Y \underbrace{\alpha \mathbf{i}_N}_{\mathbf{i}_K} + \underbrace{\alpha B}_{B_p} \boldsymbol{Z} + \underbrace{\alpha \boldsymbol{\epsilon}}_{\underset{K\times 1}{\mathbf{0}}},$$

$$||$$

$$\boldsymbol{R}_p = Y \mathbf{i}_K + B_p \boldsymbol{Z} + 0.$$

Can create an arbitrary asset using the K funds: $\{\boldsymbol{\alpha}_j\}_{j=1}^K$

$$R_j = \underset{1\times K}{\hat{\boldsymbol{\alpha}}_j'} \underset{K\times 1}{\boldsymbol{R}_p} = Y + \underbrace{\hat{\boldsymbol{\alpha}}_j' B_p}_{\boldsymbol{b}_j} \boldsymbol{Z} + 0.$$

- Condition v. is the weakest condition that ensures that any $\boldsymbol{b}_q$ that can be obtained using the N assets can be obtained using the K portfolios (i.e., that $\boldsymbol{b}_j$ can be made equal to any feasible $\boldsymbol{b}_q$).
- For example, if $\left[\mathbf{i}_N \,\vdots\, B\right]$ is of rank K, then any $\boldsymbol{b}_q \in \mathbb{R}^{K-1}$ is possible using the N assets. We need $\left[\mathbf{i}_K \,\vdots\, \alpha B\right]$ to be of rank K to be able to obtain any $\boldsymbol{b}_j \in \mathbb{R}^{K-1}$ using the K portfolios.
- For example, suppose

$$\begin{bmatrix} \mathbf{i}_N & B \end{bmatrix} = \begin{bmatrix} 1 & b_{11} & \cdots & b_{1(K-1)} \\ 1 & b_{21} & \cdots & b_{2(K-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & b_{N1} & \cdots & b_{N(K-1)} \end{bmatrix} = \begin{bmatrix} 1 & b_1 & b_{12} & \cdots & b_{1(K-1)} \\ 1 & b_1 & b_{22} & \cdots & b_{2(K-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & b_1 & b_{N2} & \cdots & b_{N(K-1)} \end{bmatrix},$$

which means that any $\boldsymbol{b}_q$ formed from the N assets has a first element equal to b_1 and the remaining $K-2$ elements that can take any values. Condition v. ensures that the same is true for portfolios that can be formed from the K funds. Take the first 2 columns of $\left[\mathbf{i}_N \,\vdots\, B\right]$ and

$\left[\mathbf{i}_K \vdots \alpha B\right]$. For condition v. to be satisfied, both must have a rank of 1:

$$\underset{N\times 2}{\begin{bmatrix} 1 & b_1 \\ 1 & b_1 \\ \vdots & \vdots \\ 1 & b_1 \end{bmatrix}} \qquad \underset{K\times 2}{\begin{bmatrix} 1 & & b_1 \\ 1 & \alpha & b_1 \\ \vdots & & \vdots \\ 1 & & b_1 \end{bmatrix}} = \underset{K\times 2}{\begin{bmatrix} 1 & b_1 \\ 1 & b_1 \\ \vdots & \vdots \\ 1 & b_1 \end{bmatrix}}$$

$$\text{rank of } 1 \quad = \quad \text{rank of } 1$$

Take the first column and any other column except the second of $\left[\mathbf{i}_N \vdots B\right]$ and $\left[\mathbf{i}_K \vdots \alpha B\right]$:

$$\underset{N\times 2}{\begin{bmatrix} 1 & b_{1k} \\ 1 & b_{2k} \\ \vdots & \vdots \\ 1 & b_{Nk} \end{bmatrix}} \qquad \underset{K\times 2}{\begin{bmatrix} 1 & & b_{1k} \\ 1 & \alpha & b_{2k} \\ \vdots & & \vdots \\ 1 & & b_{Nk} \end{bmatrix}}$$

for $k = 2, 3, \dots, K-1$. Condition v. says the rank of these 2 matrices must be the same.

Proof of sufficiency. Since condition v. ensures that any loadings $\boldsymbol{b}_q$ that can be attained using the N assets can also be attained using the K portfolios, it follows that for any available asset, there exists a portfolio of the K portfolios with the same loading vector that second order stochastically dominates the asset and thus, by theorem 31 is preferred to the asset by any investor whose preferences can be represented by a monotonically increasing and concave von Neumann–Morgenstern utility function.

K fund separation implies a $(K-1)$ factor βpm. Condition i. implies that

$$\mathrm{E}[\boldsymbol{R}] = \mathrm{E}[Y]\mathbf{i}_N + B\,\mathrm{E}[\boldsymbol{Z}].$$

K fund separation implies a $(K-1)$ factor βpm using $(K-1)$ portfolios formed from the K funds.

Suppose $Y \neq R_f$ and $\begin{bmatrix}\mathbf{i}_N & B\end{bmatrix}$ is of rank K. We can form K portfolios, D, of the K funds such that

$$\underset{(K-1)\times K}{\begin{bmatrix} \hat{\boldsymbol{\alpha}}_1^{D\prime} \\ \vdots \\ \hat{\boldsymbol{\alpha}}_{K-1}^{D\prime} \end{bmatrix}} \underbrace{\alpha B}_{K\times(K-1)} = \begin{bmatrix} \alpha_1^{D\prime} \\ \vdots \\ \alpha_{K-1}^{D\prime} \end{bmatrix} B = \begin{bmatrix} b_{11}^D & 0 & 0 & \cdots & 0 \\ 0 & b_{22}^D & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & b_{K-1,K-1}^D \end{bmatrix},$$

a $(K-1) \times (K-1)$ diagonal matrix with (j, j) element b_{jj}^D and

$$\underset{1\times K}{\hat{\boldsymbol{\alpha}}_K^{D\prime}} \underbrace{\alpha B}_{K\times(K-1)} = \underset{1\times N}{\boldsymbol{\alpha}_K^{D\prime}} \underset{N\times(K-1)}{B} = \underset{1\times(K-1)}{\mathbf{0}'}.$$

Let

$$\boldsymbol{R}^D = \begin{bmatrix} \boldsymbol{\alpha}_1^{D\prime} \\ \vdots \\ \boldsymbol{\alpha}_{K-1}^{D\prime} \end{bmatrix} \boldsymbol{R}.$$

Know

$$R_K^D = \alpha_K^{D\prime} \boldsymbol{R} = Y.$$

Now

$$R_j^D = Y + b_{jj}^D z_j \Rightarrow R_j^D - Y = b_{jj}^D z; \quad j = 1, 2, \ldots, K-1. \tag{47}$$

Let R_i be an arbitrary return constructed from R and

$$\boldsymbol{b}^{D\prime} = \begin{bmatrix} b_{11}^D & \cdots & b_{K-1,K-1}^D \end{bmatrix}.$$

Conditions i. and ii. imply that a regression of R_i on a constant, Y and $\boldsymbol{Z}$ yields coefficients of 0, 1 and $\boldsymbol{b}_i'$, where $\boldsymbol{b}_i'$ is the ith row of B. So using (47), a regression of R_i on a constant, Y and $\boldsymbol{R}^D - Y\mathbf{i}_{K-1}$ yields coefficients of 0, 1 and $\boldsymbol{b}_i./\boldsymbol{b}^{D\prime}$, where ./ represents element by element division: i.e.,

$$R_i = \underbrace{\alpha_i}_{=0} + \underbrace{\beta_{iY}}_{=1} Y + \underbrace{\boldsymbol{\beta}_{i,D}'}_{=\boldsymbol{b}_i'./\boldsymbol{b}^{D\prime}} (\boldsymbol{R}^D - Y\mathbf{i}_{K-1}) + \underbrace{u_i}_{=\epsilon_i}.$$

So a regression of R_i on a constant, Y and $\boldsymbol{R}^D$ yields coefficients of 0, $1 - (\boldsymbol{b}_i'./\boldsymbol{b}^{D\prime}\mathbf{i}_{K-1})$ and $(\boldsymbol{b}_i'./\boldsymbol{b}^{D\prime})$: i.e.,

$$R_i = \underbrace{\hat{\alpha}_i}_{=0} + \underbrace{\hat{\beta}_{iY}}_{=1-\hat{\boldsymbol{\beta}}_{i,D}'\mathbf{i}_{K-1}} Y + \underbrace{\hat{\boldsymbol{\beta}}_{i,D}'}_{=\boldsymbol{b}_i'./\boldsymbol{b}^{D\prime}} \boldsymbol{R}^D + \underbrace{\hat{u}_i}_{=\epsilon_i}.$$

From theorem 30, we know that this means that the minimum variance frontier for Y and $\boldsymbol{R}^D$ is the same as for $\boldsymbol{R}$. We also know that the MVF for a given set of assets has a point for each asset for which the weight in that asset is zero. So there is a portfolio on the MVF for Y and $\boldsymbol{R}^D$ which has a weight in Y of 0. That portfolio is also on the MVF for $\boldsymbol{R}$. Thus a linear combination of $\boldsymbol{R}^D$ lies on the MVF for $\boldsymbol{R}$ and so $\boldsymbol{R}^D$ constitutes a $(K-1)$-factor βpm for $\boldsymbol{R}$ using earlier results.

Alternately, suppose $Y = R^f$. Then the same result can be obtained.

Other results.

- K-fund separating distribution is stronger than APT assumptions on the joint return distribution.
- The 2 fund separation separating distribution is the weakest distributional assumption that delivers the CAPM. The 2 funds lie on the MVF if the first 2 moments exist.
- $\boldsymbol{R}$ multivariate normal implies that $\boldsymbol{R}$ satisfies 2 fund separating distribution.

10.6. Multiperiod setting: Distribution based separation ⇒ representative agent exists

Let $\boldsymbol{r}$ be a $N \times 1$ vector of returns and $\boldsymbol{k}$ be an $S \times 1$ vector of S state variables. Normality of returns $\boldsymbol{r}$ and the S state variables that agents care about $\boldsymbol{k}$ is sufficient to obtain $(S+2)$-fund separation.

Multifactor math (with no riskless asset). Any solution to the following problem is a multifactor minimum variance MMV portfolio.

$$\min_{\boldsymbol{x}} \quad \sigma^2[r_e] = \boldsymbol{x}' \underset{N\times N}{\Sigma_{rr}} \boldsymbol{x} \tag{48}$$

subject to

$$\begin{aligned} \underset{S\times N}{B}\underset{N\times 1}{\boldsymbol{x}} &= \underset{S\times 1}{\boldsymbol{b}_e} \\ \boldsymbol{\mu}_r'\boldsymbol{x} &= \mathrm{E}[r_e] \\ \mathbf{i}_N\boldsymbol{x} &= 1 \end{aligned}$$

where

- Σ_{rr} is the $N \times N$ covariance matrix of returns.
- $B = (\Sigma_{kk})^{-1}\Sigma_{kr}$ is an $S \times N$ matrix of regression coefficients from regressing $\boldsymbol{r}$ on $\boldsymbol{k}$, the vector of the S state variables $k_s, s = 1, 2, \ldots, S$ (b_{si} is the $\{s, i\}$ element and $\boldsymbol{b}_i$ is the ith row).
- Σ_{kk} is the $S \times S$ covariance matrix of the state variables.
- Σ_{kr} is the $S \times N$ covariance matrix of the state variables and the returns.
- $\boldsymbol{\mu}_r$ is the $N \times 1$ mean return vector.
- $E[r_e]$ and $\boldsymbol{b}_e$ are the MMV portfolio's target expected return and regression coefficient vector respectively.
- $\boldsymbol{x}$ is the $N \times 1$ vector of portfolio weights for the MMV portfolio.

The following results do not require normality or risk aversion. They only assume:

- infinite divisibility.
- Σ_{rr}, $\boldsymbol{\mu}_r$ and B are given.
- any $\begin{bmatrix} \boldsymbol{b}_e \\ \mathrm{E}[r_e] \end{bmatrix}$ combination is feasible.

It can be shown that the following properties hold for MMV portfolios:

i. $r_e = \boldsymbol{x}'\boldsymbol{r}$ is a MMV portfolio (i.e. $\boldsymbol{x}$ is a solution to (48)) ⇔

$$\mathrm{E}[r_i] = \gamma_0 + \boldsymbol{\gamma}'\boldsymbol{b}_i + \gamma_e \operatorname{cov}[r_i, r_e], \quad i = 1, 2, \ldots, N, \tag{49}$$

where

$$\boldsymbol{b}_i = \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{Si} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_S \end{bmatrix}.$$

ii. Take any $S + 2$ MMV portfolios $e_1, e_2, \ldots, e_{S+2}$ which satisfy

$$\underset{(S+2)\times(S+2)}{\begin{bmatrix} \mathrm{E}[r_{e_1}] & \mathrm{E}[r_{e_2}] & \cdots & \mathrm{E}[r_{e_{S+2}}] \\ \boldsymbol{b}_{e_1} & \boldsymbol{b}_{e_2} & \cdots & \boldsymbol{b}_{e_{S+2}} \\ 1 & 1 & \cdots & 1 \end{bmatrix}} \quad \text{nonsingular,}$$

i.e. any $S+2$ MMV portfolio from which a portfolio with any given $\begin{bmatrix} \boldsymbol{b}_e \\ \mathrm{E}[r_e] \end{bmatrix}$ can be formed, then any portfolio of these $S+2$ is MMV and any MMV portfolio can be formed from these $S+2$ portfolios. These $S+2$ MMV portfolios form a basis set for the set of MMV portfolios.

iii. Define a multifactor efficient ME portfolio to be a MMV portfolio that also solves

$$\max_{\boldsymbol{x}} \quad \mathrm{E}[r_e] = \boldsymbol{\mu}_r' \boldsymbol{x}$$

subject to

$$\begin{aligned} \underset{S\times N}{B} \underset{N\times 1}{\boldsymbol{x}} &= \underset{S\times 1}{\boldsymbol{b}_e} \\ \sigma^2[r_e] &= \boldsymbol{x}' \underset{N\times N}{\Sigma_{rr}} \boldsymbol{x} \\ \mathbf{i}_N \boldsymbol{x} &= 1 \end{aligned}$$

It follows that

$$r_e \text{ is a ME portfolio } \not\Leftarrow\Rightarrow r_e \text{ is a MMV portfolio.}$$

iv. All positively weighted portfolios of ME portfolios are ME.

v. The MVF for any $S+2$ MMV portfolios that form a basis set equals the MVF for the N assets, i.e. any $S+2$ MMV portfolios that form a basis set span the N assets in a mean–variance sense. Thus, using theorem 30:

$$r_i = \underbrace{\alpha_i^*}_{=0} + \underbrace{\beta_{i,e_1}^*}_{=(1-\sum_{j=2}^{S+2}\beta_{i,e_j})} r_{e_1} + \sum_{j=2}^{S+2} \beta_{i,e_j} r_{e_j} + \epsilon_i,$$

where

$$\begin{aligned} E[\epsilon_i] &= 0, \\ E[\epsilon_i r_{e_j}] &= 0, \quad j = 1, 2, \ldots, S+2. \end{aligned}$$

vi. For any $(S+2)$ MMV portfolios that form a basis set, the following relation holds using (49):

$$r_i - r_{e_1} = \alpha_i + \sum_{j=2}^{S+2} (r_{e_j} - r_{e_1}) \beta_{i-e_1, e_j - e_1} + \epsilon_i, \quad i = 1, 2, \ldots, N, \tag{50}$$

where

$$\mathrm{E}[\epsilon_i] = 0,$$

$$\mathrm{E}[\epsilon_i (r_{e_j} - r_{e_1})] = 0, \quad j = 2, \ldots, S+2,$$

and

$$\beta_{i-e_1, e_j - e_1} = \beta_{i,e_j}^*.$$

So

$$\mathrm{E}[r_i - r_{e_1}] = \sum_{j=2}^{S+2} \beta_{i-e_1, e_j - e_1}(\mathrm{E}[r_{e_j} - r_{e_1}]). \tag{51}$$

vii. For any $(S+2)$ MMV portfolios $e_1, \ldots, e_{S+2} \ni$

$$\mathrm{cov}[r_{e_1}, r_{e_j}] = 0, \quad j = 2, \ldots, S+2, \tag{52}$$

and which form a basis set, (50) and (51) hold and

$$r_i = \hat{\alpha}_{e_1} + \sum_{j=2}^{S+2} r_{e_j} \beta_{i,e_j} + \nu_v, \quad i = 1, 2, \ldots, N,$$

where

$$\mathrm{E}[\nu_i] = 0$$

$$\mathrm{E}[\nu_i r_{e_j}] = 0, \quad j = 2, \ldots, S+2,$$

$$\beta_{i,e_j} = \beta^*_{i,e_j}, \quad j = 2, \ldots, S+2,$$

and

$$\hat{\alpha}_{e_1} = \mathrm{E}[r_{e_1}]\left(1 - \sum_{j=2}^{S+2} \beta_{i,e_j}\right),$$

since (52)

$$\Rightarrow \mathrm{E}[\beta^*_{i,e}(r_{e_1} - \mathrm{E}\, r_{e_1}) r_{e_j}] = 0, \quad j = 2, \ldots, S+2.$$

viii. The following portfolios are MMV:

- All minimum variance MV portfolios are MMV (so the global minimum variance GMV portfolio and the minimum variance portfolio at the point on the MVF tangential to a line from the origin in $\{\mathrm{E}[R], \sigma[R]\}$ space are both MMV).
- The S "mimicking" portfolios for the S state variables whose portfolio weights satisfy the following expression are MMV:

  $$X = \underbrace{\Sigma_{rr}{}^{-1} B'}_{N \times S} \;./\; \underbrace{(\mathbf{i}'_N \Sigma_{rr}{}^{-1} B')}_{1 \times S}$$

  where
  ./ is element-by-element division,

  $$X = [\boldsymbol{x}_1 \boldsymbol{x}_2 \ldots \boldsymbol{x}_S],$$

  and
  $\boldsymbol{x}_s$ is the $N \times 1$ vector of portfolio weights for the "mimicking" portfolio for the sth state variable, $s = 1, 2, \ldots, S$.

So $N \times 1$ vector of portfolio weights for the "mimicking" portfolio for the sth state variable is given by:

$$\boldsymbol{x}_s = \underbrace{\Sigma_{rr}{}^{-1}\boldsymbol{b}_s}_{N\times 1} ./(\mathbf{i}_N' \Sigma_{rr}{}^{-1}\boldsymbol{b}_s)$$

where $\boldsymbol{b}_s$ is the sth row of B. These are the "mimicking" portfolios described in Fama's article. The $N \times 1$ vector of portfolio weights for "mimicking" portfolio for the sth state variable, $\boldsymbol{x}_s$ is the solution to the following problem with b_{se} chosen appropriately, which is an MMV portfolio problem:

$$\min_{\boldsymbol{x}} \quad \sigma^2[r_e] = \boldsymbol{x}' \underset{N\times N}{\Sigma_{rr}} \boldsymbol{x} \tag{53}$$

subject to

$$\underbrace{\boldsymbol{b}_s}_{1\times N} \boldsymbol{x} = b_{se}$$

$$\mathbf{i}_N \boldsymbol{x} = 1$$

- The S "mimicking" portfolios for the S state variables whose portfolio weights satisfy the following expression are MMV:

$$\tilde{X} = \underbrace{\Sigma_{rr}{}^{-1}\Sigma_{rk}}_{N\times S} ./ \underbrace{(\mathbf{i}_N' \Sigma_{rr}{}^{-1}\Sigma_{rk})}_{1\times S}$$

where

$$\tilde{X} = [\tilde{\boldsymbol{x}_1}\tilde{\boldsymbol{x}_2}\ldots\tilde{\boldsymbol{x}_S}],$$

and

$\tilde{\boldsymbol{x}_s}$ is the $N \times 1$ vector of portfolio weights for the "mimicking" portfolio for the sth state variable, $s = 1, 2, \ldots, S$.

So $N \times 1$ vector of portfolio weights for the sth the "mimicking" portfolio for the sth state variable is given by:

$$\tilde{\boldsymbol{x}_s} = \underbrace{\Sigma_{rr}{}^{-1}\boldsymbol{\Sigma}_{rs}}_{N\times 1} ./(\mathbf{i}_N' \Sigma_{rr}{}^{-1}\boldsymbol{\Sigma}_{rs})$$

where $\boldsymbol{\Sigma}_{rs}$ is the sth column of Σ_{rk}. The $N \times 1$ vector of portfolio weights for "mimicking" portfolio for the sth state variable, $\tilde{\boldsymbol{x}_s}$ is the solution to the following problem, with the σ_{se} chosen appropriately:

$$\min_{\boldsymbol{x}} \quad \sigma^2[r_e] = \boldsymbol{x}' \underset{N\times N}{\Sigma_{rr}} \boldsymbol{x} \tag{54}$$

subject to

$$\underbrace{\boldsymbol{\Sigma}_{rs}}_{1\times N} \boldsymbol{x} = \sigma_{se}$$

$$\mathbf{i}_N \boldsymbol{x} = 1$$

where $\boldsymbol{\Sigma}_{rs}$ is the sth row of Σ_{rk}.

- The minimum variance portfolio for any given $\boldsymbol{b}_e$ is MMV.

Multifactor math (with a riskless asset). We can formulate an analogous minimum variance problem to (48). All the properties of MMV portfolios in the no riskless asset case carry over though since the GMV portfolio for the risky assets is not on the MVF for the risky assets and the riskless asset, the GMV portfolio is not MMV.

Merton's ICAPM (no riskless asset).

Assumptions.

i. The number of securities N available at t is much greater than the number of state variables.

ii.

$$\boldsymbol{r}_{t+1} = \begin{bmatrix} r_{1,t+1} \\ \vdots \\ r_{N,t+1} \end{bmatrix} \quad \text{and} \quad \boldsymbol{k}_{t+1} = \begin{bmatrix} k_{1,t+1} \\ \vdots \\ k_{S,t+1} \end{bmatrix}$$

are multivariate normally distributed conditional on time-t information.

iii. Complete agreement.

iv. Perfect capital markets.

v. Unrestricted short-selling.

vi. Investors are risk averse.

vii. Investors care about $\boldsymbol{k}_{t+1}$ as well as W_{t+1}.

Equilibrium. Assumptions ii. and vii. ⇒ an individuals utility depends on

$$\{\mathrm{E}[r_p], \sigma[r_p], b_{p1}, \ldots, b_{pS}\}.$$

Assumptions ii., vii. and vi. ⇒ all individuals hold ME portfolios. Assumptions iv., v. and i. ⇒ the MMV and ME set satisfy all the properties described in the multifactor math section. With assumption iii. ⇒ all individuals see the same $\boldsymbol{\mu}_r$ vector, Σ_{rr} matrix and B matrix. Each individual's portfolio as a positive weight in the market portfolio (since the sum of all individuals' portfolios equals the market portfolio and all individuals have positive wealth) and each individual's portfolio is ME ⇒ (using multifactor math result iv.) that the market portfolio is ME.

So M, the S mimicking portfolios and a MMV portfolio uncorrelated with them form a basis set for the MMV portfolios, which ⇒ (multifactor math results v., vi. and vii.)

$$\mathrm{E}[r_i - r_z] = \beta_{i,M}\,\mathrm{E}[r_M - r_z] + \sum_{s=1}^{S} \beta_{i,s}\,\mathrm{E}[r_s - r_z], \quad i = 1, 2, \ldots, N, \tag{55}$$

where $\beta_{i,M}$ are coefficients from

$$r_i = \alpha_i + \beta_{i,M} r_M + \sum_{s=1}^{S} \beta_{i,s} r_s + \nu_i,$$

$$\mathrm{E}[\nu_i r_s] = 0 \quad s = 1, \ldots, S, \quad , \mathrm{E}[\nu_i] = 0, \mathrm{E}[\nu_i r_M] = 0,$$

or

$$r_i - r_z = \hat{\alpha}_i + \beta_{i,M}(r_M - r_z) + \sum_{s=1}^{S} \beta_{i,s}(r_s - r_z) + \hat{\nu}_i,$$
$$\mathrm{E}[\hat{\nu}_i(r_s - r_z)] = 0 \quad s = 1, \ldots, S, \quad \mathrm{E}[\nu_i r_M] = 0, \quad \mathrm{E}[\nu_i] = 0,$$

with (55) implying $\hat{\alpha}_i = 0$.

Although the market portfolio is ME, it almost surely is not mean-variance efficient, i.e. not E.

The same arguments go through if there is a riskless asset, except replace r_z with r_f, but it is not possible to sign the expected excess return on the market since $b_{M,s}$s need not be zero.

CHAPTER 11

Term Structure Models

11.1. Notation

- $P_{n,t}$ is the time-t price of an n-period discount bound paying 1 at time $t+n$.
- $p_{n,t} = \ln P_{n,t}$.
- $Y_{n,t} = (\frac{1}{P_{n,t}})^{1/n} - 1$ is the discretely compounded per-period yield at t on an n-period discount bond.
- $y_{n,t} = -\frac{1}{n} p_{n,t}$ is the continuously compounded per-period yield at t on an n-period discount bond.
- $R_{n,t,t+\tau} = (\frac{P_{n-\tau,t+\tau}}{P_{n,t}})^{1/n} - 1$ is the discretely compounded per-period holding period return from t to $t+\tau$ on an n-period discount bond.
- $r_{n,t,t+\tau} = \frac{1}{\tau}(p_{n-\tau,t+\tau} - p_{n,t})$ is the continuously compounded per-period holding period return from t to $t+\tau$ on an n-period discount bond.
- $R_{n,t+1} = \frac{P_{n-1,t+1}}{P_{n,t}} - 1$ is the discretely compounded holding period return from t from $t+1$ on an n-period discount bond.
- $r_{n,t+1} = (p_{n-1,t+1} - p_{n,t}) = ny_{n,t} - (n-1)y_{n-1,t+1}$ is the continuously compounded holding period return from t from $t+1$ on an n-period discount bond.
- $F_{n,n+\tau,t}$ is the τ-period forward rate available at t starting at $t+n$ expressed as a discretely compounded per-period rate.
- $f_{n,n+\tau,t}$ is the τ-period forward rate available at t starting at $t+n$ expressed as a continuously compounded per-period rate.
- $F_{n,t}$ is the 1-period forward rate available at t starting at $t+n$ expressed as a discretely compounded per-period rate.
- $f_{n,t}$ is the 1-period forward rate available at t starting at $t+n$ expressed as a continuously compounded per-period rate.

11.2. Basics

An identity.

$$\begin{aligned} ny_{n,t} &= \tau r_{n,t,t+\tau} + (n-\tau)y_{n-\tau,t+\tau} \\ &= \sum_{j=0}^{\tau-1} r_{n-j,t+j+1} + (n-\tau)y_{n-\tau,t+\tau} \\ &= \sum_{j=0}^{n-1} r_{n-j,t+j+1}. \end{aligned}$$

A no arbitrage relation. Consider the following two strategies entered into at time 0:

- Buy $(n+\tau)$-period discount bonds with a face value of $e^{(n+\tau)y_{n+\tau,t}}$, which costs -1: pays off $(n+\tau)y_{n+\tau,t}$ at time $(n+\tau)$, which is certain at time 0.
- Buy n-period discount bonds with a face value of $e^{ny_{n,t}}$, which costs -1, and roll the proceeds into the $f_{n,n+\tau,t}$ forward rate: pays off n $y_{n,t} + \tau f_{n,n+\tau,t}$ at time $(n+\tau)$, which is certain at time 0.

Both strategies produce a certain cashflow at time $(n+\tau)$ using time 0 information, so

$$(n+\tau)y_{n+\tau,t} = ny_{n,t} + \tau f_{n,n+\tau,t}.$$

11.3. Expectations hypothesis

Log form of the expectations hypothesis.

Definition. (Log form of the expectations hypothesis) The log form of the expectations hypothesis says:

$$\mathrm{E}_t[r_{n,t+1}] = y_{1,t} + \kappa_n, \tag{56}$$

$$\mathrm{E}_t[y_{1,t+n}] = f_{n,t} + \gamma_n, \tag{57}$$

$$y_{n,t} = \frac{1}{n}\sum_{j=0}^{n-1}\mathrm{E}_t[y_{1,t+j}] + \xi_n. \tag{58}$$

Proposition 32. *(56) holds for all n and for all t* $\Leftrightarrow$ *(57) holds for all n and for all t* $\Leftrightarrow$ *(58) for all n and for all t.*

PROOF. See question 1 of Problem set 10. □

Pure expectations hypothesis (not log form).

Definition. (One-period form of the pure expectations hypothesis) The one-period form of the pure expectations hypothesis says:

$$1 + Y_{1,t} = \mathrm{E}_t[1 + R_{n,t+1}] \tag{59}$$

$$\Rightarrow 1 + Y_{1,t} = (1 + Y_{n,t})^n\,\mathrm{E}_t[(1 + Y_{n-1,t+1})^{-(n-1)}]. \tag{60}$$

Definition. (n-period form of the pure expectations hypothesis) The n-period form of the pure expectations hypothesis says:

$$(1 + Y_{n,t})^n = \mathrm{E}_t[(1 + Y_{1,t})(1 + Y_{1,t+1})\cdots(1 + Y_{1,t+n-1})] \quad \text{for all } n. \tag{61}$$

CLM claim that this implies

$$1 + F_{n-1,t} = \mathrm{E}_t[1 + Y_{1,t+n-1}] \tag{62}$$

and

$$(1 + Y_{n,t})^n = (1 + Y_{1,t})\,\mathrm{E}_t[(1 + Y_{n-1,t+1})^{n-1}]. \tag{63}$$

Notice that (60) and (63) *cannot* hold simultaneously when future discount bond yields are uncertain today.

11.4. Using sdfs to price discount bonds

Recall that $M_{\tau+1}$ is a sdf for $X_{\tau+1}$ using $\mathcal{F}_\tau$ if

$$\forall X^i_{\tau+1} \in X_{\tau+1} : \mathrm{E}[M_{\tau+1}X^i_{\tau+1} \mid \mathcal{F}_t] = p_\tau(X^i_{\tau+1}).$$

No arbitrage implies that there exists an sdf that is strictly positive. Equivalently, if there exists an sdf $M_{\tau+1} > 0$, then $\mathrm{E}[M_{\tau+1}X^i_{\tau+1} \mid \mathcal{F}_\tau]$ generates prices that satisfy no arbitrage for all $X^i_{\tau+1} \in X_{\tau+1}$.

An n-period discount bond at t pays off 1 at $t+n$, so its price at $t+n-1$ is

$$\mathrm{E}_{t+n-1}[M_{t+n}1] = p_{t+n-1}(1) = P_{1,t+n-1}.$$

Moreover, its price at $t+n-2$ is

$$\begin{aligned}
\mathrm{E}_{t+n-2}[M_{t+n-1}P_{1,t+n-1}] &= P_{2,t+n-2} \\
\Rightarrow \mathrm{E}_{t+n-2}[M_{t+n-1}\,\mathrm{E}_{t+n-1}[M_{t+n}]] &= P_{2,t+n-2} \\
\Rightarrow \mathrm{E}_{t+n-2}[M_{t+n-1}M_{t+n}] &= P_{2,t+n-2}.
\end{aligned}$$

Recursively, its price at t is given by

$$\mathrm{E}_t[\prod_{i=1}^{n} M_{t+i}] = P_{n,t}.$$

So by specifying a process for M_τ with $M_\tau > 0$ for all τ, we can generate arbitrage-free discount bond prices.

11.5. Affine-yield models: Assumptions

Assume throughout that $m_{\tau+1} = \ln M_{\tau+1}$ is normal conditioning on a set of state variables available at τ, x_τ. Make assumptions that imply that discount bond prices and yields in logs at τ are linear in the set of state variables x_τ, and $p_{n,\tau+1}$ is normal conditional on x_τ for all n and all τ. It follows that

$$P_{n,\tau} = \mathrm{E}_t[M_{t+1}P_{n-1,t+1}] \tag{64}$$

and so

$$p_{n,t} = \mathrm{E}_t[m_{t+1} + p_{n-1,t-1}] + \tfrac{1}{2}\sigma^2_t[m_{t+1} + p_{n-1,t-1}]. \tag{65}$$

11.6. A homoskedastic single-factor affine-yield model

The following is an example of an affine-yield model. It is a homoskedastic single-factor model. The model is closely related to the continuous time model of Vasicek (1977).

Environment. The generating process for m_{t+1} is

$$-m_{t+1} = x_t + \epsilon_{t+1}, \quad \epsilon_{t+1} \sim N(0, \sigma_\epsilon^2).$$

The generating process for x_t is AR(1),

$$x_{t+1} = (1-\phi)\mu + \phi x_t + \xi_{t+1}, \quad \xi_{t+1} \sim N(0, \sigma_\xi^2).$$

Innovations to m_{t+1} and x_{t+1} may be correlated,

$$\epsilon_{t+1} = \beta\xi_{t+1} + \eta_{t+1}, \quad \eta_{t+1} \sim N(0, \sigma_\eta^2), \quad \mathrm{cov}[\eta_{t+1}, \xi_{t+1}] = 0.$$

Pricing. When $n = 1$,

$$p_{n-1,t+1} = p_{0,t+1} = \ln(1) = 0.$$

So

$$\begin{aligned} p_{1,t} &= \mathrm{E}_t[m_{t+1}] + \tfrac{1}{2}\sigma_t^2[m_{t+1}] \\ &= -x_t + \tfrac{1}{2}(\beta^2\sigma_\xi^2 + \sigma_\eta^2), \end{aligned} \tag{66}$$

and so

$$y_{1,t} = x_t - \tfrac{1}{2}(\beta^2\sigma_\xi^2 + \sigma_\eta^2). \tag{67}$$

Guess that

$$-p_{n,\tau} = A_n + B_n x_\tau \tag{68}$$

for all n and all t, and prove by induction that the guess is correct.

$n = 0$: $A_0 = B_0 = 0$.

$n = 1$: $A_1 = -\frac{1}{2}(\beta^2\sigma_\xi^2 + \sigma_\eta^2), \quad B_1 = 1$.

Need to show that $p_{n-1,\tau}$ satisfies (68) implies that $p_{n,\tau}$ satisfies (68). Suppose $p_{n-1,t+1}$ satisfies (68). Then

$$\begin{aligned} m_{t+1} + p_{n-1,t+1} &= -x_t - \beta\xi_{t+1} - \eta_{t+1} - A_{n-1} - B_{n-1}((1-\phi)\mu + \phi x_t + \xi_{t+1}) \\ &= -(B_{n-1}(1-\phi)\mu + A_{n-1}) - (1 + B_{n-1}\phi)x_t \\ &\quad - (\beta + B_{n-1})\xi_{t+1} - \eta_{n+1}, \end{aligned}$$

and so

$$\begin{aligned} \mathrm{E}_t[m_{t+1} + p_{n-1,t+1}] &= -(B_{n-1}(1-\phi)\mu + A_{n-1}) - (1 + B_{n-1}\phi)x_t, \\ \sigma_t^2[m_{t+1} + p_{n-1,t+1}] &= (\beta + B_{n-1})^2\sigma_\xi^2 + \sigma_\eta^2. \end{aligned}$$

Substituting into (65),

$$\begin{aligned} -p_{n,t} &= B_{n-1}(1-\phi)\mu + A_{n-1} + (1 + B_{n-1}\phi)x_t - \tfrac{1}{2}((\beta + B_{n-1})^2\sigma_\xi^2 + \sigma_\eta^2), \\ A_n + B_n x_t &= B_{n-1}(1-\phi)\mu + A_{n-1} - \tfrac{1}{2}((\beta + B_{n-1})^2\sigma_\xi^2 + \sigma_\eta^2) + (1 + B_{n-1}\phi)x_t, \end{aligned}$$

and so

$$A_n = A_{n-1} + B_{n-1}(1-\phi)\mu - \tfrac{1}{2}((\beta + B_{n-1})^2\sigma_\xi^2 + \sigma_\eta^2)), \tag{69}$$

$$B_n = 1 + B_{n-1}\phi. \tag{70}$$

as required for $p_{n,t}$ to satisfy (68).

Notice that (70) together with $B_0 = 0$ implies that

$$B_n = \frac{1-\phi^n}{1-\phi}.$$

for all n.

Implications. Notice that

$$\begin{aligned} y_{1,t} &= x_t - \tfrac{1}{2}\beta^2\sigma_\xi^2 + \sigma_\eta^2, \\ y_{n,t} &= \tfrac{1}{n}(A_n + B_n x_t), \quad B_n > 0. \end{aligned} \tag{71}$$

So x_t tells you the level of the yield curve, in particular the spot rate.

Notice

$$\begin{aligned} p_{n-1,t+1} &= -(n-1)y_{n-1,t+1} \\ &= -A_n - B_n x_{t+1}, \\ \text{and} \quad r_{n,t+1} &= p_{n-1,t+1} - p_{n,t} \\ &= -A_{n-1} - B_{n-1}x_{t+1} - p_{n,t} \\ &= -A_{n-1} - p_{n,t} - B_{n-1}((1-\phi)\mu + \phi x_t) - B_{n-1}\xi_{t+1} \end{aligned}$$

which is normally distributed using time-t information. So a positive shock to the level of the yield curve at $t+1$ (i.e. x_{t+1}) conditioning on time t information leads to a negative shock to $p_{n-1,t+1}$ and $r_{n,t+1}$ conditioning on time-t information so $\mathrm{cov}_t[r_{n,t+1}, x_{t+1}] < 0$.

Moreover, $-B_{n-1}$ is the coefficient from a conditional time-t regression of log 1-period return for an n-period discount bond from t to $t+1$ on the state variable at $t+1$ since $-B_{n-1} < 0$: so the regression of $r_{n,t+1}$ on x_{t+1} using time-t conditioning has a slope coefficient of $-B_{n-1}$. Note that the regression explains 100% of the variation in $r_{n,t+1}$.

Why is $-B_{n-1} < 0$ and $\mathrm{cov}_t[r_{n,t+1}, x_{t+1}] < 0$? The answer is as follows:

$$x_{t+1} \uparrow \quad \Rightarrow \quad \mathrm{E}_{t+1}[m_{t+\tau}] \downarrow \quad \Rightarrow \quad y_{n-1,t+1} \uparrow \quad \Rightarrow \quad p_{n-1,t+1} \downarrow \quad \Rightarrow \quad r_{n,t+1} \downarrow$$

for $\tau = 2, 3, \ldots$.

Log expectations hypothesis. The model delivers the log expectations hypothesis. From (64),

$$\begin{aligned} 0 &= \mathrm{E}_t[m_{t+1} + r_{n,t+1}] + \tfrac{1}{2}\sigma_t^2[m_{t+1} + r_{n,t+1}] \\ \Rightarrow \mathrm{E}_t[r_{n,t+1}] &= -\mathrm{E}_t[m_{t+1}] - \tfrac{1}{2}\sigma_t^2[m_{t+1}] - \tfrac{1}{2}\sigma_t^2[r_{n,t+1}] - \sigma_t[m_{t+1}, r_{n,t+1}] \end{aligned}$$

for any $n > 1$. Now for $n = 1$, $r_{1,t+1} = y_{1,t}$ and so for any $n > 1$,

$$y_{1,t} = -\mathrm{E}_t[m_{t+1}] - \tfrac{1}{2}\sigma_t^2[m_{t+1}],$$

and so

$$
\begin{aligned}
\mathrm{E}_t[r_{n,t+1}] - y_{1,t} &= -\tfrac{1}{2}\sigma_t^2[r_{n,t+1}] - \sigma_t[m_{t+1}, r_{n,t+1}] \\
&= -\tfrac{1}{2}B_{n-1}^2\sigma_\xi^2 - \sigma_t[-\beta\xi_{t+1}, -B_{n-1}\xi_{t+1}] \\
&= -\tfrac{1}{2}B_{n-1}^2\sigma_\xi^2 - B_{n-1}\beta\sigma_\xi^2, \qquad (72)
\end{aligned}
$$

or

$$\mathrm{E}_t[r_{n,t+1}] - y_{1,t} + \tfrac{1}{2}\sigma_t^2[r_{n,t+1}] = -B_{n-1}\beta\sigma_\xi^2,$$

which is the same for all t.

Intuition for the functional form of B_n. Since the log expectations hypothesis holds, we know that:

$$p_{n,t} = -\sum_{j=0}^{n-1}\mathrm{E}_{t+1}[y_{1,t+j}] - n\xi_n.$$

So how x_t affects

$$-\sum_{j=0}^{n-1}\mathrm{E}_t[y_{1,t+j}],$$

determines how x_t affects $p_{n,t}$. We also know that $y_{1,t+\tau}$ varies one-for-one with $x_{t+\tau}$ from (71). So how x_t affects

$$-\sum_{j=0}^{n-1}\mathrm{E}_t[x_{t+j}],$$

determines how x_t affects $p_{n,t}$. We can show that:

$$\mathrm{E}[x_{t+j}|x_t] = (1-\phi^j)\mu + \phi^j x_t$$

and

$$\sum_{j=0}^{n-1}\phi^j = \frac{1-\phi^n}{1-\phi}.$$

Thus, we can see explicitly why B_n has the functional form that it does. Moreover, we see that ϕ affects B_n because it affects $\mathrm{E}[x_{t+\tau}|x_t]$. Note that:

- $\frac{dB_n}{dn} > 0$ as would be expected since, as n increases, you discount back \$1 for more periods at appropriate future interest rates given x_t to get $p_{n,t}$, which causes $p_{n,t}$ to decline in n for given x_t.
- $\frac{dB_n}{d\phi} = \frac{d}{d\phi}(\sum_{j=0}^{n-1}\phi^j) > 0$ as would be expected since, as ϕ increases, $\mathrm{E}[x_{t+\tau}|x_t]$ increases for fixed τ, which means that, as ϕ increases, you discount back \$1 at higher future interest rates given x_t to get $p_{n,t}$ which causes $p_{n,t}$ to decline in ϕ for given x_t.

Risk premium. Since $-B_{n-1}$ is the coefficient from regressing $r_{n,t+1}$ on x_{t+1} using time-t conditioning, we can regard $\beta\sigma_\xi^2$ as the risk premium for bearing x_{t+1}-risk.

Notice that β determines the sign of the risk premium on the discount bonds:

$$\beta \begin{Bmatrix} < \\ > \end{Bmatrix} 0 \Leftrightarrow \mathrm{E}_t[r_{n,t+1}] - y_{1,t} + \tfrac{1}{2}\sigma_t^2[r_{n,t+1}] \begin{Bmatrix} > \\ < \end{Bmatrix} 0.$$

We know that $\mathrm{cov}_t[p_{n,t+1}, x_{t+1}] < 0$ and $\mathrm{cov}_t[r_{n,t+1}, x_{t+1}] < 0$. Now

$$\beta < 0 \Leftrightarrow \mathrm{cov}_t[m_{t+1}, x_{t+1}] > 0 \Leftrightarrow \mathrm{cov}[r_{n,t+1}, m_{t+1}] < 0,$$

which implies

$$\mathrm{E}_t[r_{n,t+1}] - y_{1,t} + \tfrac{1}{2}\sigma_t^2[r_{n,t+1}] > 0.$$

Note that

$$\mathrm{sign}\{\beta\} = \mathrm{sign}\{\rho_t[m_{t+1}, m_{t+\tau}]\}, \quad \text{for } \tau > 1.$$

So

$$\mathrm{sign}\{\mathrm{E}_t[r_{t+1}^n] - y_{1,t} + \tfrac{1}{2}\sigma_t^2[r_{n,t+1}]\} = -\,\mathrm{sign}\{\rho_t[m_{t+1}, m_{t+\tau}]\}.$$

Role of σ_η^2. Both $y_{n,t} - y_{1,t}$, the log yield spread of an n-period discount bond less a 1-period discount bond, and $\mathrm{E}_t[r_{n,t+1}] - y_{1,t}$, the expected 1-period log return on an n-period discount bond less the log yield of a 1-period discount bond, are unaffected by σ_η^2. Only A_n is affected by σ_η^2. Since η_{t+1} is conditionally uncorrelated with x_{t+1} (using time-t conditioning), η_{t+1} is not correlated with time-$(t+1)$ discount bond prices (using time-t conditioning). Thus η_{t+1} only affects the expected 1-period discount bond returns through its effect on $\mathrm{E}_t[M_{t+1}]$. This effect is the same for all discount bonds.

Forward rate curve. Know that

$$\begin{aligned} f_{n,t} &= p_{n,t} - p_{n+1,t} \\ &= -p_{1,t} + (\mathrm{E}_t[p_{n,t+1}] - p_{n+1,t} + p_{1,t}) - (\mathrm{E}_t[p_{n,t+1}] - p_{n,t}) \\ &= y_{1,t} + (\mathrm{E}_t[r_{n+1,t+1}] - y_{1,t}) - (\mathrm{E}_t[p_{n,t+1}] - p_{n,t}). \end{aligned}$$

So using (65), (72) and the fact that (68) implies

$$\begin{aligned} \mathrm{E}_t[p_{n,t+1}] - p_{n,t} &= B_n\,\mathrm{E}_t[\Delta x_{t+1}] \\ &= B_n(1-\phi)(\mu - x_t), \end{aligned}$$

it follows that

$$
\begin{aligned}
f_{n,t} &= x_t - \tfrac{1}{2}(\beta^2\sigma_\xi^2 + \sigma_\eta^2) - \tfrac{1}{2}B_n^2\sigma_\xi^2 - B_n\beta\sigma_\xi^2 + B_n(1-\phi)(\mu - x_t) \\
&= \mu + (B_n(1-\phi) - 1)(\mu - x_t) - \tfrac{1}{2}(\beta + B_n)^2\sigma_\xi^2 - \tfrac{1}{2}\sigma_\eta^2 \\
&= \mu - \tfrac{1}{2}\sigma_\eta^2 + \phi^n(x_t - \mu) - \tfrac{1}{2}\sigma_\xi^2\left[\beta + \frac{1-\phi^n}{1-\phi}\right]^2 \qquad (14) \\
&= \left[\mu - \tfrac{1}{2}\sigma_\eta^2 - \left(\beta + \frac{1}{1-\phi}\right)^2 \frac{\sigma_\xi^2}{2}\right] + \left[(x_t - \mu) + \frac{1 + \beta(1-\phi)}{(1-\phi)^2}\sigma_\xi^2\right]\phi^n \qquad (15) \\
&\quad - \left[\left(\frac{1}{1-\phi}\right)^2 \frac{\sigma_\xi^2}{2}\right]\phi^{2n}.
\end{aligned}
$$

Three possible shapes are possible:

- Rising throughout.
- Falling throughout (inverted).
- Rising at first and then falling (hump-shaped).

Other shapes are *not* possible, e.g. falling at first and then rising (inverted hump-shaped).

We can see that the effect of x_t on $f_{n,t}$ is given by $\phi^n x_t$, which is consistent with the log expectations hypothesis holding, i.e. $\mathrm{E}_t[y_{1,t+n}] = f_{n,t} + \gamma_n$, since

$$
\begin{aligned}
\mathrm{E}_t[y_{1,t+n}] &= \mathrm{E}_t[x_{t+n} - \tfrac{1}{2}(\beta^2\sigma_\xi^2 + \sigma_\eta^2)] \\
&= (1-\phi)\mu\sum_{j=1}^{n}\phi^{j-1} + \phi^n x_t - \tfrac{1}{2}(\beta^2\sigma_\xi^2 + \sigma_\eta^2).
\end{aligned}
$$

Equilibrium interpretation of M_{t+1} in the model. Consider a power utility representative agent. It follows that the representative agent's marginal rate of substitution is a valid sdf for the economy:

$$
M_{t+1} = \delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}.
$$

Take logs to obtain

$$
m_{t+1} = \ln(\delta) - \gamma\Delta c_{t+1}.
$$

Take

$$
\begin{aligned}
x_t &= \mathrm{E}_t[-m_{t+1}] = -\ln\delta + \gamma\,\mathrm{E}_t[\Delta c_{t+1}], \\
\epsilon_{t+1} &= -m_{t+1} - \mathrm{E}_t[-m_{t+1}] \\
&= \gamma(\Delta c_{t+1} - \mathrm{E}_t[\Delta c_{t+1}]).
\end{aligned}
$$

So if expected log consumption growth is an AR(1) process, the homoskedastic single-factor affine-yield model characterizes the term-structure dynamics for this economy.

We know that

$$
\mathrm{cov}_t[r_{n,t+1}, x_{t+1}] = \mathrm{cov}_t[r_{n,t+1}, \gamma\,\mathrm{E}_{t+1}[\Delta c_{t+2}]] < 0.
$$

Why is $\text{cov}_t[r_{n,t+1}, \gamma\, \text{E}_{t+1}[\Delta c_{t+2}]] < 0$? The answer is as follows:

$$\gamma\, \text{E}_{t+1}[\Delta c_{t+2}] \uparrow \quad \Rightarrow \text{E}_{t+1}[\ln(\delta) - \gamma \Delta c_{t+\tau}] \downarrow \quad \Rightarrow y_{n-1,t+1} \uparrow \quad \Rightarrow p_{n-1,t+1} \downarrow \quad \Rightarrow r_{n,t+1} \downarrow$$

for $\tau = 2, 3, \ldots$.

Now from above, we know that

$$\beta \begin{Bmatrix} < \\ > \end{Bmatrix} 0 \Leftrightarrow \text{E}_t[r_{n,t+1}] - y_{1,t} + \tfrac{1}{2}\sigma_t^2[r_{n,t+1}] \begin{Bmatrix} > \\ < \end{Bmatrix} 0.$$

which is consistent with economic intuition because

$$\beta < 0$$

$$\Leftrightarrow$$

$$\text{cov}_t[m_{t+1}, x_{t+1}] = \text{cov}_t[-\gamma \Delta c_{t+1}, \gamma\, \text{E}_{t+1}[\Delta c_{t+2}]] > 0$$

$$\text{or} \quad \text{cov}_t[\Delta c_{t+1}, \text{E}_{t+1}[\Delta c_{t+2}]] < 0$$

$$\Leftrightarrow \text{ since } \text{cov}_t[r_{n,t+1}, \gamma\, \text{E}_{t+1}[\Delta c_{t+2}]] < 0,$$

$$\text{cov}_t[r_{n,t+1}, m_{t+1}] = \text{cov}_t[r_{n,t+1}, -\gamma \Delta c_{t+1}] < 0 \quad \text{or} \quad \text{cov}_t[r_{n,t+1}, \Delta c_{t+1}] > 0.$$

11.7. A square-root single-factor affine-yield model

The model is the discrete time analog of the Cox–Ingersoll–Ross continuous-time term structure model.

Environment. The generating process for m_{t+1} is

$$-m_{t+1} = x_t + x_t^{1/2}\epsilon_{t+1} = x_t + x_t^{1/2}\beta\xi_{t+1}.$$

The generating process for x_{t+1} is

$$x_{t+1} = (1 - \phi)\mu + \phi x_t + x_t^{1/2}\xi_{t+1},$$

where ϵ_{t+1} and ξ_{t+1} are distributed as before.

Pricing. Solution looks similar to previous case,

$$-p_{n,t} = A_n + B_n x_t,$$

where A_n and B_n can be solved recursively.

Implications. Log expectations hypothesis no longer holds since $\text{E}_t[r_{n,t+1}] - y_{1,t}$ depends on x_t. The forward rate curve can only take the same 3 shapes as above.

CHAPTER 12

Stylized Facts and Empirical Regularities

12.1. Representative agent puzzles

Equity premium puzzle. Using a standard power utility function, aggregate consumption growth is too smooth and not sufficiently correlated with equity returns to explain the magnitude of the equity Sharpe ratio for reasonable values of the relative risk aversion coefficient.

Recall that for a representative agent economy whose representative agent has power utility:

$$\begin{aligned}
\mathrm{E}_t[R^e_{t+1} - R^f_{t+1}] &= -\frac{1}{\mathrm{E}_t[M_{t+1}]}\,\mathrm{cov}_t[M_{t+1}, R^e_{t+1}],\\
\frac{\mathrm{E}_t[R^e_{t+1} - R^f_{t+1}]}{\sigma_t[R^e_{t+1}]} &= -\frac{\sigma_t[M_{t+1}]}{\mathrm{E}_t[M_{t+1}]}\rho_t[M_{t+1}, R^e_{t+1}]\\
&= -\frac{\sigma_t\left[\delta\left(\dfrac{C_{t+1}}{C_t}\right)^{-\gamma}\right]}{\mathrm{E}_t\left[\delta\left(\dfrac{C_{t+1}}{C_t}\right)^{-\gamma}\right]}\rho_t\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}, R^e_{t+1}\right]
\end{aligned}$$

where
R^e_{t+1} is the return on the value-weighted equity portfolio from t to $t+1$,
R^f_{t+1} is the riskfree return from t to $t+1$,
C_τ is aggregate consumption at τ, and
γ is the RRA coefficient for the representative agent.

The puzzle is driven by the joint distribution of $R^e_{t+1} - R^f_{t+1}$ and C_{t+1} and does not depend on equity cash flows. It is related to the Hansen–Jagannathan bound though the bound is too weak due to the low correlation between C_{t+1} and R^e_{t+1}. To understand the importance of risk aversion, consider the log-normal model introduced earlier:

$$\begin{aligned}
&\mathrm{E}_t[M_{t+1}R^i_{t+1}] = 1,\\
&\mathrm{E}_t\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} R^i_{t+1}\right] = 1,\\
&\mathrm{E}_t[\exp\{\ln(\delta) - \gamma\Delta c_{t+1} + r^i_{t+1}\}] = 1,\\
&\exp\{\ln(\delta) - \gamma\,\mathrm{E}_t[\Delta c_{t+1}] + \mathrm{E}_t[r^i_{t+1}] + \tfrac{1}{2}\sigma^2_t[-\gamma\Delta c_{t+1} + r^i_{t+1}]\} = 1,\\
&\mathrm{E}_t[r^i_{t+1}] + \tfrac{1}{2}\sigma^2_t[r^i_{t+1}]\\
&\qquad = -\ln\delta + \gamma\,\mathrm{E}_t[\Delta c_{t+1}] - \tfrac{1}{2}\gamma^2\sigma^2_t[\Delta c_{t+1}] + \gamma\sigma_t[\Delta c_{t+1}, r^i_{t+1}],
\end{aligned}$$

which implies

$$
\begin{aligned}
\frac{\mathrm{E}_t[r^e_{t+1}] - r^f_{t+1} + \frac{1}{2}\sigma^2_t[r^e_{t+1}]}{\sigma_t[r^e_{t+1}]} &= \gamma\rho_t[r^e_{t+1}, \Delta c_{t+1}]\sigma_t[\Delta c_{t+1}], \\
\frac{0.0601 - 0.0183 + \frac{1}{2}\times 0.1774^2}{0.1774} &= \gamma \times 0.4979 \times 0.0328, \\
0.324 &= \gamma \times 0.0163, \\
\gamma &= 20
\end{aligned}
$$

where the data values are taken from Table 8.1 of CLM on page 308. Mehra and Prescott (1985) discuss the experimental evidence and conclude that reasonable values for γ lie between 0.5 and 10. So a value of 20 for γ is too high.

Risk-free rate puzzle. The risk-free rate is too low given mean and standard deviation of aggregate consumption growth. We know that

$$
R^f_{t+1} = \frac{1}{\mathrm{E}_t[M_{t+1}]} = \frac{1}{\mathrm{E}_t\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]}.
$$

Using the log-normal model,

$$
\mathrm{E}_t[r^f_{t+1}] = -\ln(\delta) + \gamma\,\mathrm{E}_t[\Delta c_{t+1}] - \frac{\gamma^2}{2}\sigma^2_t[\Delta c_{t+1}].
$$

Assuming homoskedastic Δc_{t+1}, we can take unconditional expectations to obtain

$$
\begin{aligned}
\mathrm{E}[r^f_{t+1}] &= -\ln(\delta) + \gamma\,\mathrm{E}[\Delta c_{t+1}] - \frac{\gamma^2}{2}\sigma^2_{\Delta c} \\
&= -\ln(\delta) + \gamma 0.0172 - \frac{\gamma^2}{2}0.0328^2 \\
&= -\ln(\delta) + \gamma 0.0172 - \gamma^2 0.00053792
\end{aligned}
$$

where again the the data values are taken from Table 8.1 of CLM on page 308.

Notice that $\mathrm{E}[r^f_{t+1}]$ has three components:

i. $\mathrm{E}[r^f_{t+1}]$ is high if δ is low.
ii. $\mathrm{E}[r^f_{t+1}]$ is high if $\mathrm{E}[\Delta c_{t+1}]$ is high, since the representative agent must be offered a high interest rate to overcome her desire to borrow to reduce $\mathrm{E}_t[\Delta c_{t+1}]$. This effect depends on the elasticity of substitution, which is the inverse of relative risk aversion in power utility model.
iii. $\mathrm{E}[r^f_{t+1}]$ is low if $\sigma^2_{\Delta c}$ is high since a low interest rate must be offered to overcome the representative agent's high precautionary saving motive.

Notice that γ has two effects:

i. As γ increases, elasticity of substitution $\frac{1}{\gamma}$ decreases, so a higher r^f_{t+1} must be offered for the investor to have a given wedge $\mathrm{E}[\Delta c_{t+1}]$ between consumption today and tomorrow.
ii. As γ increases, the precautionary saving motive becomes stronger and so a smaller r^f_{t+1} needs to be offered for a given $\sigma^2_{\Delta c}$.

In particular,

$$\frac{d\,\mathrm{E}[r^f_{t+1}]}{d\gamma} = \mathrm{E}[\Delta c_{t+1}] - \gamma\sigma^2_{\Delta c}$$
$$= 0.0172 - \gamma 0.00107584$$

For low values of γ, $\mathrm{E}[r^f_{t+1}]$ is increasing in γ, but for γ sufficiently high (higher than approximately 16), $\mathrm{E}[r^f_{t+1}]$ is decreasing in γ. Assuming δ is approximately 1,we can see that for $\gamma = 10$, the $\mathrm{E}[r^f_{t+1}]$ implied by the representative-agent model is 0.118, while a γ value lower than 1.11 is needed to deliver a riskfree lower than the data value of 0.0183. However, a γ value that low implies a much lower equity Sharpe ratio than is observed in the data. A γ value of 20 delivers an equity Sharpe ratio roughequal to that in the data but is implausibly high and implies a riskfree rate of 0.128 which is much higher than its data value.

Notice that the volatility of $\mathrm{E}_t[M_{t+1}]$ determines the volatility of R^f_{t+1}, which is an additional restriction the data imposes on a candidate sdf.

12.2. Equity volatility puzzle

Equity returns are too volatile given the joint distribution of equity dividends and aggregate consumption to be explained by a power utility representative agent with reasonable risk aversion. Notice that in contrast to the equity premium puzzle, the equity volatility puzzle depends on the the joint distribution of equity dividends and aggregate consumption. Because of this, we need to use an endowment economy to derive market clearing prices given an exogenously specified joint process for aggregate consumption and equity dividend.

Here is the endowment economy used by Mehra and Prescott (1985) Recall that if

$$\text{(TC)} \qquad \lim_{T\to\infty} \mathrm{E}_t\left[\left(\prod_{j=1}^{T} M_{t+j}\right) P^i_{t+T}\right] = 0$$

then

$$\mathrm{E}_t\left[\sum_{\tau=1}^{\infty}\left(\prod_{j=1}^{\tau} M_{t+j}\eta^i_{t+j}\right)\right] = \frac{P^i_t}{D^i_t},$$

where $\eta^i_{t+j} = D^i_{t+j}/D^i_{t+j-1}$.

Consider the following economy:

i. Infinitely lived representative agent with power utility: So

$$M_{t+j} = \delta\left(\frac{C_{t+j}}{C_{t+j-1}}\right)^{-\gamma} \quad \text{and} \quad \left(\prod_{j=1}^{\tau} M_{t+j}\right) = \delta^\tau\left(\frac{C_{t+\tau}}{C_t}\right)^{-\gamma}.$$

ii. State space: S states at each point in time with transition matrix $\underset{S\times S}{\Pi}$ that satisfies $\Pi\mathbf{i}_S = \mathbf{i}_S$.

iii. Endowment economy: Exogenously specify D_t and D_{t+1}, where

$$\frac{D_{\tau+1}}{D_\tau} = G_{\tau+1}$$

is a stationary process.

iv. Equilibrium: Market clearing in the goods markets implies $D_\tau = C_\tau$ for all τ.

v. Pricing: Any asset in this economy can be priced as follows:

$$\mathrm{E}_t\left[\sum_{\tau=1}^{\infty}\delta^\tau\left(\prod_{j=1}^{\tau}G_{t+j}^{-\gamma}\eta_{t+j}^i\right)\right] = \frac{P_t^i}{D_t^i} = P/D_t^i,$$

for which (TC) holds.

vi. Firm i: η_{t+1}^i is a stationary process.

Prices can be solved in closed form as follows.

i. $\boldsymbol{G}$ is the $S\times 1$ vector whose sth element is the G_τ realisation if the sth state occurs at τ. $\boldsymbol{\eta}^i$ is the $S\times 1$ vector whose sth element is the η_τ^i realisation if the sth state occurs at τ.

ii. In this economy, P/D_τ^i only depends on s_τ.

iii. We know that

$$\begin{aligned}
\mathrm{E}_t[\delta G_{t+1}^{-\gamma}R_{t+1}^i] &= 1,\\
\mathrm{E}_t[\delta G_{t+1}^{-\gamma}(P_{t+1}^i + D_{t+1}^i)] &= P_t^i,\\
\mathrm{E}_t\left[\delta G_{t+1}^{-\gamma}\left(\frac{P_{t+1}^i}{D_{t+1}^i}\frac{D_{t+1}^i}{D_t^i} + \frac{D_{t+1}^i}{D_t^i}\right)\right] &= \frac{P_t^i}{D_t^i},\\
\mathrm{E}_t[\delta G_{t+1}^{-\gamma}\eta_{t+1}^i(1 + P/D_{t+1}^i)] &= P/D_t^i.
\end{aligned}$$

iv. In vector form:

$$\Pi(\delta\boldsymbol{G}^{-\gamma}.*\boldsymbol{\eta}^i.*(\mathbf{i}_S + \boldsymbol{P}/\boldsymbol{D}^i)) = \boldsymbol{P}/\boldsymbol{D}^i,$$

where $\boldsymbol{P}/\boldsymbol{D}^i$ is the $S\times 1$ vector whose sth element is the P/D_τ^i value if the sth state occurs at τ. The $.*$ operator represents element-by-element multiplication.

v. A closed-form solution for $\boldsymbol{P}/\boldsymbol{D}^i$ is obtained as follows.

$$\Pi(\delta\boldsymbol{G}^{-\gamma}.*\boldsymbol{\eta}^i) + ((\delta\boldsymbol{G}^{-\gamma}.*\boldsymbol{\eta}^i)'.*\Pi)\boldsymbol{P}/\boldsymbol{D}^i = \boldsymbol{P}/\boldsymbol{D}^i,$$

$$\Pi(\delta\boldsymbol{G}^{-\gamma}.*\boldsymbol{\eta}^i) = (I_S - (\delta\boldsymbol{G}^{-\gamma}.*\boldsymbol{\eta}^i)'.*\Pi)\boldsymbol{P}/\boldsymbol{D}^i,$$

$$\boldsymbol{P}/\boldsymbol{D}^i = (I_s - (\delta\boldsymbol{G}^{-\gamma}.*\boldsymbol{\eta}^i)'.*\Pi)^{-1}\Pi(\delta\boldsymbol{G}^{-\gamma}.*\boldsymbol{\eta}^i).$$

12.3. Cross section of expected stock returns

CAPM can be rejected. Conditional CAPM does better than unconditional CAPM. Firm and stock characteristics can help explain the cross-section over and above CAPM beta:

- Market capitalization
- Book to market
- Liquidity measures
- Past returns

12.4. Equity return predictability

Publicly available information can be used to predict stock returns. A long list of information variables can be used:

- Lagged returns
- Aggregate dividend–price ratio
- Aggregate earnings–price ratio
- Aggregate book-to-market ratio
- The share of equity in new finance
- Term spread: spread between long-term and short-term government bond yields
- Default spread: spread between low-grade and high-grade bond yields
- Recent changes in short-term interest rates
- The level of consumption relative to wealth

Many of these variables are related to the stage of the business cycle and predict counter-cyclical variation in stock returns.

Predictive regressors typically explain a larger fraction of the variance of longer-horizon returns. Long horizon returns have variances that are less than linear in horizon: negative autocorrelations.

Since consumption growth is close to iid in the data, standard consumption models have difficulty delivering the variation in conditional Sharpe ratios seen in the data.

12.5. A VAR framework for thinking about return predictability

Suppose

$$\underset{N\times 1}{r_{t+1}} = \underset{N\times 1}{b} + \underset{N\times 1}{a}\,\underset{1\times 1}{d_t} + \underset{N\times 1}{e_{t+1}},$$
$$d_{t+1} = \rho d_t + v_{t+1}.$$

where

$$\operatorname{cov}[r_{t+1}] = \Sigma_r, \quad \operatorname{cov}[e_{t+1}] = \Sigma_e = \Sigma_r - aa'\sigma_d^2, \quad \sigma^2[d_t] = \sigma_d^2,$$

$$\sigma^2[v_{t+1}] = \sigma_v^2 = (1-\rho^2)\sigma_d^2, \quad \operatorname{cov}[e_{t+1}, v_{t+1}] = \underset{N\times 1}{\sigma_{ev}}, \text{ and } \underset{(N+1)\times(N+1)}{\Sigma_{ev}} = \begin{bmatrix} \Sigma_e & \sigma_{ev}' \\ \sigma_{ev} & \sigma_v^2 \end{bmatrix}.$$

Then

$$d_{t+2} = \rho d_{t+1} + v_{t+2} = \rho(\rho d_t + v_{t+1}) + v_{t+2},$$
$$d_{t+3} = \rho^3 d_t + \rho^2 v_{t+1} + \rho v_{t+2} + v_{t+3},$$
$$d_{t+\tau} = \rho^\tau d_t + \sum_{i=1}^{\tau-1} \rho^i v_{t+\tau-i} + v_{t+\tau} = \rho^\tau d_t + v_{t+\tau}^\tau$$

where

$$\sigma^2[v_{t+\tau}^\tau] = \sum_{i=0}^{\tau-1} \rho^{2i}\sigma_v^2 = \left(\frac{1-\rho^{2\tau}}{1-\rho^2}\right)\sigma_v^2.$$

Note that

$$\sigma^2[d_{t+\tau}] = \sigma_d^2 = \rho^{2\tau}\sigma_d^2 + \left(\frac{1-\rho^{2\tau}}{1-\rho^2}\right)(1-\rho^2)\sigma_d^2 = \sigma^2[\rho^\tau d_t + v_{t+\tau}^\tau]$$

as required.

Now

$$\begin{aligned} r_{t+2} &= b + a(\rho d_t + v_{t+1}) + e_{t+2}, \\ r_{t+3} &= b + a(\rho^2 d_t + \rho v_{t+1} + v_{t+2}) + e_{t+3}, \\ r_{t+i} &= b + a\rho^{i-1} d_t + a\sum_{j=0}^{i-2} \rho^j v_{t+i-1-j} + e_{t+i}, \end{aligned}$$

and so

$$\begin{aligned} \sum_{i=1}^{\tau} r_{t+i} &= \tau b + a\sum_{i=1}^{\tau}\rho^{i-1} d_t + a\sum_{i=1}^{\tau}\sum_{j=0}^{i-2}\rho^j v_{t+i-1-j} + \sum_{i=1}^{\tau} e_{t+i} \\ &= b\tau + a\left(\frac{1-\rho^\tau}{1-\rho}\right) d_t + a\sum_{i=1}^{\tau-1}\left(\sum_{j=0}^{i-1}\rho^j\right) v_{t+\tau-i} + \sum_{i=1}^{\tau} e_{t+i} \\ &= b\tau + a\underbrace{\left(\frac{1-\rho^\tau}{1-\rho}\right)}_{<\tau} d_t + \underbrace{\sum_{i=1}^{\tau-1}\left[\left(\frac{1-\rho^i}{1-\rho}\right) a v_{t+\tau-i} + e_{t+\tau-i}\right] + e_{t+\tau}}_{e_{t+\tau}^\tau} . \end{aligned}$$

12.6. Term structure facts

The log expectations hypothesis says that

$$\kappa_n + y_{1,t} = \mathrm{E}_t[r_{n,t+1}]. \tag{73}$$

By definition,

$$r_{n,t+1} = y_{n,t} - (n-1)(y_{n-1,t+1} - y_{n,t}), \tag{74}$$

so substitute (74) into (73),

$$\begin{aligned} \kappa_n + y_{1,t} &= \mathrm{E}_t[y_{n,t} - (n-1)(y_{n-1,t+1} - y_{n,t})] \\ \Rightarrow \frac{1}{n-1}(y_{n,t} - y_{1,t}) - \frac{1}{n-1}\kappa_n &= \mathrm{E}_t[y_{n-1,t+1} - y_{n,t}], \end{aligned}$$

which suggests the regression

$$y_{n-1,t+1} - y_{n,t} = \alpha_n^{\Delta y} + \beta_n^{\Delta y}\left(\frac{y_{n,t} - y_{1,t}}{n-1}\right) + \varepsilon_{n,t}^{\Delta y},$$

where the log expectations hypothesis says $\beta_n^{\Delta y} = 1$.

In the data, we find

i. $\beta_n^{\Delta y}$ is small and positive for small n.

ii. $\beta_n^{\Delta y}$ is negative for large n.

Note that

$$y_{n,t} = r_{n,t+1} + (n-1)(y_{n-1,t+1} - y_{n,t}) = y_{n,t},$$
$$\frac{1}{n-1}(r_{n,t+1} - y_{1,t}) + y_{n-1,t+1} - y_{n,t} = \frac{1}{n-1}(y_{n,t} - y_{1,t}).$$

So consider the regression:

$$\frac{1}{n-1}(r_{n,t+1} - y_{1,t}) = \alpha_n^r + \beta_n^r \left(\frac{y_{n,t} - y_{1,t}}{n-1} \right) + \varepsilon_{n,t}^r,$$

where by construction $\beta_n^r = 1 - \beta_n^{\Delta y}$. The log expectations hypothesis says $\beta_n^r = 0$ and the empirical findings for $\beta_n^{\Delta y}$ imply β_n^r is positive and increasing in n.

CHAPTER 13

New models to address stylized facts about asset prices

13.1. Habit persistence preferences

Three features. There are three important dimensions to habit models:

- Difference vs. ratio habit models.
  Ratio:
  $$E\left[\sum_{j=0}^{\infty} \delta^j \frac{(C_{t+j}/X_{t+j})^{1-\gamma}}{1-\gamma} \,\middle|\, \mathcal{F}_t\right],$$
  where X_{t+j} is the habit at time $t+j$.
  Difference:
  $$E\left[\sum_{j=0}^{\infty} \delta^j \frac{(C_{t+j}-X_{t+j})^{1-\gamma}}{1-\gamma} \,\middle|\, \mathcal{F}_t\right].$$
- Internal vs. external habit models.
  External habit models: habit is determined by current and past consumption of other agents. In equilibrium, with homogeneity, current and past consumption of other agents will equal the current and past consumption of the optimizing agent.
  Internal habit models: habit is determined by the current and past consumption of the optimizing agent.
- Specification of the habit function:
  $$X_{t+j} = f(C_{t+j}, C_{t+j-1}, \ldots).$$

Example. Utility specification: difference.

$$E\left[\sum_{j=0}^{\infty} \delta^j \frac{(C_{t+j}-aC_{t+j-1})^{1-\gamma}}{1-\gamma} \,\middle|\, \mathcal{F}_t\right]$$

The first-order condition for external habit is

$$E[\delta(C_{\tau+1}-aC_{\tau})^{-\gamma} R^i_{\tau+1} \,|\, \mathcal{F}_\tau] = (C_\tau - aC_{\tau-1})^{-\gamma},$$

and so

$$E\left[\delta\left(\frac{C_{\tau+1}-aC_\tau}{C_\tau - aC_{\tau-1}}\right)^{-\gamma} R^i_{\tau+1} \,\middle|\, \mathcal{F}_t\right] = 1.$$

The first-order condition for internal habit is

$$\mathrm{E}\left[\left(\delta(C_{\tau+1}-aC_{\tau})^{-\gamma}-a\delta^2(C_{\tau+2}-aC_{\tau+1})^{-\gamma}\right)R^i_{\tau+1}\,\middle|\,\mathcal{F}_\tau\right]$$
$$=\mathrm{E}\left[((C_\tau-aC_{\tau-1})^{-\gamma}-a\delta(C_{\tau+1}-aC_\tau)^{-\gamma})\,\middle|\,\mathcal{F}_\tau\right].$$

Notice that for external habit,

$$RRA=-\frac{Cu_{CC}}{u_C}=\frac{\gamma C(C-aC_{-1})^{-\gamma-1}}{(C-aC_{-1})^{-\gamma}}=\gamma\frac{C}{C-aC_{-1}}>\gamma,$$

so habit can increase risk aversion, but

$$R^f_{t+1}=\frac{1}{\mathrm{E}\left[\delta\left(\dfrac{C_{t+1}-aC_t}{C_t-aC_{t-1}}\right)^{-\gamma}\,\middle|\,\mathcal{F}_t\right]},$$

which is too volatile when γ is set large enough to deliver the equity premium puzzle.

Campbell–Cochrane habit model. This is an external difference model. Define

$$S_t=\frac{C_t-X_t}{C_t},$$

so the first-order condition is given by

$$\mathrm{E}\left[\delta\left(\frac{S_{t+1}}{S_t}\frac{C_{t+1}}{C_t}\right)^{-\gamma}R^i_{t+1}\,\middle|\,\mathcal{F}_t\right]=1.$$

Assume $c_t=\ln(C_t)$ follows

$$\Delta c_{t+1}=g+\nu_{t+1},$$

where $\nu_{t+1}\sim N(0,\sigma^2_\nu)$, and specify an AR(1) model for $s_{t+1}=\ln(S_{t+1})$:

$$s_{t+1}=\min(s_{max},(1-\phi)\bar{s}+\phi s_t+\lambda(s_t)\nu_{t+1}). \tag{75}$$

So X_t is a complex nonlinear function of current and past consumption. The advantage of this specification is that consumption never falls below habit. Campbell and Cochrane show that at $s_{t+1}=\bar{s}$,

$$x_{t+1}=\ln X_{t+1}=\left[h+\frac{g}{1-\phi}\right]+(1-\phi)\sum_{j=0}^{\infty}\phi^j c_{t-j},$$

where h is the steady-state value of x_t-c_t.

The risk free rate is given by

$$\begin{aligned}
r^f_{t+1}&=\ln R^f_{t+1}\\
&=-\ln[\mathrm{E}[\exp\{\ln\delta-\gamma(s_{t+1}-s_t+\Delta c_{t+1})\}\,|\,\mathcal{F}_t]]\\
&=-\ln\delta+-\ln[\mathrm{E}[\exp\{-\gamma((1-\phi)(\bar{s}-s_t)+g)-\gamma(1+\lambda(s_t))\nu_{t+1}\}\,|\,\mathcal{F}_t]]\\
&=-\ln\delta+\gamma g-\gamma(1-\phi)(s_t-\bar{s})-\tfrac{1}{2}\gamma^2(1+\lambda(s_t))^2\sigma^2_\nu.
\end{aligned}$$

The second and third terms capture the effect of the investor's intertemporal saving motive on r^f_{t+1}. Regarding the third term, if s_t is low, marginal utility in period t is higher, but marginal utility in period

$t+1$ is lower as s_t reverts to its mean. So the agent wants to borrow, which forces r^f_{t+1} up. The fourth term captures a precautionary saving motive that stems from the volatility of marginal utility at $t+1$. It motivates the agent to save, which forces r^f_{t+1} down.

Campbell–Cochrane set

$$\lambda(s_t) = \frac{1}{\bar{s}}\sqrt{1-2(s_t-\bar{s})}-1, \qquad s_t \leq s_{max},$$

where

$$\bar{s} = \sigma_v\sqrt{\frac{\gamma}{1-\phi}} \qquad \text{and} \qquad s_{max} = \bar{s} + \tfrac{1}{2}(1-\bar{s}^2),$$

which causes the effects of the third and fourth terms to exactly offset, leaving the risk free rate constant. When s_t is low, this functional form for $\lambda(.)$ causes the precautionary saving motive to increase by just enough to offset the investor's increased desire to borrow because marginal utility at t is higher than at $t+1$. Thus, the model has time varying risk aversion

$$RRA = -\frac{Cu_{CC}}{u_C} = \frac{\gamma}{S_t} \geq \gamma,$$

with a constant risk free rate

$$r^f_{t+1} = -\ln\delta + \gamma g - \frac{\gamma}{2}(1-\phi).$$

13.2. Idiosyncratic labor income

If agents in an economy receive idiosyncratic uninsurable labor income, a representative agent may not exist for the economy. So using a representative agent model to price assets can produce expected returns that deviate from those found in the economy. Recall that the equity premium puzzle says that aggregate consumption is too smooth and not sufficiently correlated with equity returns to generate Sharpe ratios for equity as high as seen in US for any reasonable risk aversion coefficient. Perhaps idiosyncratic uninsurable labor income can help explain the equity premium puzzle. This subsection presents a simple model to examine this question.

Model setup. Consider a 1-period world with two points in time, 0 and 1, when agents consume. There is a continuum of agents who are all identical at time 0. Each agent maximizes the CRRA utility function

$$\frac{c_0^{1-\gamma}}{1-\gamma} + \mathrm{E}\left[\frac{c_1^{1-\gamma}}{1-\gamma}\right], \tag{76}$$

where c_t is consumption at time t and γ is relative risk aversion. All agents are endowed with consumption at time 0 that is normalized to equal 1. There are 2 aggregate states at time 1, each equally likely: a good state denoted by G and a bad state denoted by B. In the good state, half the agents receive an endowment of $\mu_G(1+\varphi_G)$ and the other half receive an endowment of $\mu_G(1-\varphi_G)$, where $0<\varphi_G<1$. So in the good state, each agent has a 50% chance of receiving $\mu_G(1+\varphi_G)$ and a 50% chance of receiving $\mu_G(1-\varphi_G)$. In the bad state, half the agents receive an endowment of $\mu_B(1+\varphi_B)$ and the other half receive an endowment of $\mu_B(1-\varphi_B)$, where $0<\varphi_B<1$. So in the bad state, each agent has a 50% chance of receiving $\mu_B(1+\varphi_B)$ and a 50% chance of receiving $\mu_B(1-\varphi_B)$. Note that $\mu_G > \mu_B > 0$. There are two assets: equity and a riskless asset. A unit of equity pays μ_G in the good state and μ_B in the bad state, and offers an expected total return of $(1+\bar{R}^e)$. The riskless asset offers a certain total return of $(1+R^f)$. There are no assets available whose payoffs depend on the agent's realized endowment within a state.

Average per capita endowment. Define

$$\begin{aligned}\phi_{Gu} &\equiv 1+\varphi_G,\\ \phi_{Gd} &\equiv 1-\varphi_G,\\ \phi_{Bd} &\equiv 1-\varphi_B, \qquad\qquad \text{and,}\\ \phi_{Bu} &\equiv 1+\varphi_B.\end{aligned}$$

The average per capita endowment in the good state is given by:

$$\mathrm{E}[\frac{1}{2}\mu_G\varphi_{Gu}+\frac{1}{2}\mu_G\varphi_{Gd}|G]=\mu_G.$$

The average per capita endowment in the bad state is given by:

$$\mathrm{E}[\frac{1}{2}\mu_B\varphi_{Bu}+\frac{1}{2}\mu_B\varphi_{Bd}|B]=\mu_B.$$

Equity return. Let $(1+R^e_G)$ be the total return on equity in the good state, $(1+R^e_B)$ be the total return on equity in the bad state, and P_0 be the price of one unit of equity at time 0. We know that

$$1+R^e_G=\frac{\mu_G}{P_0} \qquad \text{and} \qquad 1+R^e_B=\frac{\mu_B}{P_0}.$$

Further, we know that

$$P_0=\frac{\frac{1}{2}(\mu_G+\mu_B)}{1+\bar{R}^e}.$$

It follows that the total return on the equity in the good and bad states can be written as

$$1+R^e_G=(1+\bar{R}^e)\frac{2\mu_G}{\mu_G+\mu_B}=(1+\bar{R}^e)\left(1+\frac{\mu_G-\mu_B}{\mu_G+\mu_B}\right)=(1+\bar{R}^e)(1+\xi)$$

and

$$1+R^e_B=(1+\bar{R}^e)\frac{2\mu_B}{\mu_G+\mu_B}=(1+\bar{R}^e)\left(1-\frac{\mu_G-\mu_B}{\mu_G+\mu_B}\right)=(1+\bar{R}^e)(1-\xi),$$

respectively, where $\xi\equiv\frac{\mu_G-\mu_B}{\mu_G+\mu_B}$.

Riskfree rate and the equity premium. Let

$$\bar{r}^e\equiv\frac{1+\bar{R}^e}{1+R^f}.$$

All agents are ex ante identical. An agent's first order conditions imply the following expression for the total return on the risk free asset:

$$1+R^f=4\left[(\mu_G\phi_{Gu})^{-\gamma}+(\mu_G\phi_{Gd})^{-\gamma}+(\mu_B\phi_{Bu})^{-\gamma}+(\mu_B\phi_{Bd})^{-\gamma}\right]^{-1}$$

An agent's first order condition with respect to the total return on the equity is given by

$$\begin{aligned}1 = \tfrac{1}{4}\big[&(\mu_G\phi_{Gu})^{-\gamma}(1+\bar{R}^e)(1+\xi) + (\mu_G\phi_{Gd})^{-\gamma}(1+\bar{R}^e)(1+\xi) \\ &+ (\mu_B\phi_{Bu})^{-\gamma}(1+\bar{R}^e)(1-\xi) + (\mu_B\phi_{Bd})^{-\gamma}(1+\bar{R}^e)(1-\xi)\big].\end{aligned}$$

This first order condition can be used to obtain the following expression for the expected total return on the equity:

$$\begin{aligned}1+\bar{R}^e &= 4\left[\mu_G^{-\gamma}(1+\xi)(\phi_{Gu}^{-\gamma}+\phi_{Gd}^{-\gamma}) + \mu_B^{-\gamma}(1-\xi)(\phi_{Bu}^{-\gamma}+\phi_{Bd}^{-\gamma})\right]^{-1} \\ &= 2(\mu_G+\mu_B)\left[\mu_G^{1-\gamma}(\phi_{Gu}^{-\gamma}+\phi_{Gd}^{-\gamma}) + \mu_B^{1-\gamma}(\phi_{Bu}^{-\gamma}+\phi_{Bd}^{-\gamma})\right]^{-1}.\end{aligned}$$

It follows that

$$\begin{aligned}\bar{r}^e &= \frac{\mu_G^{-\gamma}\overbrace{((1+\varphi_G)^{-\gamma}+(1-\varphi_G)^{-\gamma})}^{\Omega_G} + \mu_B^{-\gamma}\overbrace{((1+\varphi_B)^{-\gamma}+(1-\varphi_B)^{-\gamma})}^{\Omega_B}}{\mu_G^{-\gamma}(1+\xi)\underbrace{((1+\varphi_G)^{-\gamma}+(1-\varphi_G)^{-\gamma})}_{\Omega_G} + \mu_B^{-\gamma}(1-\xi)\underbrace{((1+\varphi_B)^{-\gamma}+(1-\varphi_B)^{-\gamma})}_{\Omega_B}} \\ &= \frac{M(\varphi_G,\varphi_B)}{N(\varphi_G,\varphi_B)}.\end{aligned}$$

where $M(\varphi_G,\varphi_B)$ is the numerator expressed as a function of φ_G and φ_B and $N(\varphi_G,\varphi_B)$ is the denominator expressed as a function of φ_G and φ_B

Effect of φ_B and φ_G on $\bar{r}^e$. How does $\bar{r}^e$ vary as a function of φ_B (holding all other parameters constant) in this economy? Differentiating $\bar{r}^e$ with respect to φ_B

$$\begin{aligned}\frac{\partial\bar{r}^e}{\partial\varphi_B} &= \frac{NM_B - MN_B}{N^2} > 0, \text{ since} \\ M_B &= \mu_B^{-\gamma}(-\gamma(1+\varphi_B)^{-\gamma-1} + \gamma(1-\varphi_B)^{-\gamma-1}) \\ &= \mu_B^{-\gamma}\gamma\underbrace{((1-\varphi_B)^{-\gamma-1}-(1+\varphi_B)^{-\gamma-1})}_{\Lambda_B}, \\ N_B &= \mu_B^{-\gamma}\gamma(1-\xi)\underbrace{((1-\varphi_B)^{-\gamma-1}-(1+\varphi_B)^{-\gamma-1})}_{\Lambda_B}, \qquad \text{and} \\ NM_B - MN_B &= \mu_G^{-\gamma}\mu_B^{-\gamma}\gamma\Omega_G\Lambda_B(1+\xi) + \mu_B^{-\gamma}\mu_B^{-\gamma}\gamma\Omega_B\Lambda_B(1-\xi) \\ &\quad - \mu_G^{-\gamma}\mu_B^{-\gamma}\gamma\Omega_G\Lambda_B(1-\xi) - \mu_B^{-\gamma}\mu_B^{-\gamma}\gamma\Omega_B\Lambda_B(1-\xi) \\ &= 2\xi\mu_G^{-\gamma}\mu_B^{-\gamma}\gamma\Omega_G\Lambda_B > 0.\end{aligned}$$

So the equity premium $\bar{r}^e$ is increasing in φ_B, which means that larger idiosyncratic shocks in the bad state can help explain the equity premium puzzle.

How does $\bar{r}^e$ vary as a function of φ_G (holding all other parameters constant) in this economy?

$$\begin{aligned}
\frac{\partial \bar{r}^e}{\partial \varphi_G} &= \frac{NM_G - MN_G}{N^2} < 0, \text{ since} \\
M_G &= \mu_g^{-\gamma} \gamma \Lambda_G, \\
N_G &= \mu_G^{-\gamma} \gamma (1+\xi) \Lambda_G, \qquad \text{and} \\
NM_G - MN_g &= \mu_G^{-\gamma}(1+\xi)\Omega_G \mu_G^{-\gamma} \gamma \Lambda_G + \mu_B^{-\gamma}(1-\xi)\Omega_B \mu_G^{-\gamma} \gamma \Lambda_G \\
&\quad - \mu_G^{-\gamma} \Omega_G \mu_G^{-\gamma} \gamma (1+\xi) \Lambda_G - \mu_B^{-\gamma} \Omega_B \mu_G^{-\gamma} \gamma (1+\xi) \Lambda_G \\
&= -2\xi \mu_B^{-\gamma} \mu_G^{-\gamma} \gamma \Lambda_G \Omega_B < 0.
\end{aligned}$$

So the equity premium $\bar{r}^e$ is decreasing in φ_G holding φ_B fixed, which means that a larger idiosyncratic shock in the good state, holding the bad-state idiosyncratic shock fixed, lead to an even larger equity premium puzzle.

Notice that if $\varphi_G = \varphi_B$, then

$$\bar{R}^e = \frac{\mu_G^{-\gamma} + \mu_B^{-\gamma}}{\mu_G^{-\gamma}(1+\xi) + \mu_G^{-\gamma}(1-\xi)}, \tag{77}$$

and $\bar{r}^e$ does not depend on φ_B. In other words, the equity premium $\bar{r}^e$ does not depend on the magnitude of the idiosyncratic shocks when $\varphi_G = \varphi_B$. That is, the equity premium $\bar{r}^e$ does not depend on the magnitude of idiosyncratic shocks that are the same irrespective of the aggregate state (or more generally that are distributed independently of the aggregate state).

Econometrician. Suppose an econometrician thought the economy had a representative agent with utility given by (76). What expression would this econometrician obtain for $\bar{r}^e$? This econometrician thinks $\varphi_G = \varphi_B = 0$, and so would obtain the expression given in (77) for $\bar{r}^e$. So using equation (77), it follows that idiosyncratic shocks with the same distribution across aggregate states do not affect the equity premium and cannot help explain the equity premium puzzle.

Importance of a continuum of agents. How important is the assumption of a continuum of agents for the results obtained above? The answer is not very important since it is possible to describe an economy with a finite number of agents and the same parameters, $\mu_G, \mu_B, \varphi_G, \varphi_B$ and γ, which has identical prices for equity and the riskless asset (as in the economy above) whenever the parameter values for $\mu_G, \mu_B, \varphi_G, \varphi_B$ and γ are the same (as in the economy above).

Consider an economy with two agents each with the same utility function as in a continuous economy. Each is endowed with consumption at time 0 normalized to 1. There are two possible states at time 1, G and B, that are equally likely. In state G, one agent receives $\mu_G(1+\varphi_G)$ and the other receives $\mu_G(1-\varphi_G)$. Each agent has a 50% chance of receiving $\mu_G(1+\varphi_G)$. In state B, one agent receives $\mu_B(1+\varphi_B)$ and the other receives $\mu_B(1-\varphi_B)$. Again, each agent has a 50% chance of receiving $\mu_B(1+\varphi_B)$. There are the same two assets as in the economy with the continuum of agents. Since the first order conditions of the agents and the aggregate quantities are the same as those in the economy with a continuum of agents, it follows that the prices in this economy are the same as those in that economy.

Implications for the equity premium puzzle. To summarize, the model developed in this subsection shows the following:

- idiosyncratic uninsurable labor income risk can help the equity premium puzzle if it's concentrated in bad states.
- idiosyncratic uninsurable labor income risk does not effect pricing if it's independent of the state of the economy.
- the model's assumption of a continuum of agents is likely not critical for the first two results.

13.3. Epstein–Zin preferences

CRRA utility. Consider the problem faced an infinitely-lived agent with CRRA preferences:

$$\max_{\{c_\tau,\boldsymbol{\alpha}_\tau\}_{\tau=t}^{\infty}} \mathrm{E}\left[\sum_{\tau=t}^{\infty}\delta^{\tau-t}(1-\delta)\frac{C_\tau^{1-\gamma}}{1-\gamma}\,\middle|\,\mathcal{F}_t\right],$$

subject to

$$C_\tau, \boldsymbol{\alpha}_\tau \in \mathcal{F}_\tau, W_{\tau+1} = (W_\tau - C_\tau)(\boldsymbol{\alpha}_\tau(\mathbf{R}_{\tau+1} - R_{f,\tau+1}\mathbf{i}_N) + R_{f,\tau+1}). \tag{78}$$

where $R_{f,\tau+1}$ is the riskfree rate from τ to $\tau+1$ and $\mathbf{R}_{\tau+1}$ is a $N \times 1$ vector of risky asset returns from τ to $\tau+1$.

The Bellman equation has the form

$$V(W_\tau, \mathcal{F}_\tau) = \max_{c_\tau,\boldsymbol{\alpha}_\tau}\left\{(1-\delta)\frac{C_\tau^{1-\gamma}}{1-\gamma} + \delta\,\mathrm{E}[V(W_{\tau+1}, \mathcal{F}_{\tau+1})\,|\,\mathcal{F}_\tau]\right\} \quad \text{s.t. (78).}$$

We know that the solution satisfies

$$C_\tau = \varphi(\mathcal{F}_\tau)W_\tau \quad \text{and} \quad \boldsymbol{\alpha}_\tau = \boldsymbol{\alpha}(\mathcal{F}_\tau), \tag{79}$$

and we know the form of the value function is

$$a(\mathcal{F}_\tau)W_\tau^{1-\gamma} = V(W_\tau, \mathcal{F}_\tau).$$

Define

$$J(W_\tau, \mathcal{F}_\tau) = [(1-\gamma)V(W_\tau, \mathcal{F}_\tau)]^{\frac{1}{1-\gamma}}.$$

Then it follows that

$$J(W_\tau, \mathcal{F}_\tau) = \phi(\mathcal{F}_\tau)W_\tau$$

and the Bellman equation can be rewritten:

$$J(W_\tau, \mathcal{F}_\tau) = \max_{c_\tau,\boldsymbol{\alpha}_\tau}\left[(1-\delta)C_\tau^{1-\gamma} + \delta\,\mathrm{E}[J(W_{t+1}, \mathcal{F}_{\tau+1})^{1-\gamma}\,|\,\mathcal{F}_\tau]\right]^{\frac{1}{1-\gamma}} \quad \text{s.t. (78).}$$

The first-order condition is

$$\forall i = f, 1, \ldots, N : 1 = \mathrm{E}_t\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} R_{i,t+1}\,\middle|\,\mathcal{F}_\tau\right].$$

A more general specification. Consider the following more general specification for the Bellman equation:

$$J(W_\tau,\mathcal{F}_\tau)=\max_{c_\tau,\boldsymbol{\alpha}_\tau}\left[(1-\delta)C_\tau^\rho+\delta(\mathrm{E}[J(W_{\tau+1},\mathcal{F}_{\tau+1})^\alpha\mid\mathcal{F}_\tau])^{\rho/\alpha}\right]^{1/\rho}\quad\text{s.t. (78).}\tag{80}$$

This is the Bellman equation for an infinitely-lived investor with Epstein-Zin preferences. Notice that when $\rho=\alpha=1-\gamma$, this specification collapses down to the rewritten Bellman equation for the CRRA case. So power preferences can be regarded as a special case of Epstein-Zin preferences. It can be shown that for Epstein-Zin prefernces, $\alpha=1-\gamma$, where γ is relative risk aversion, and $\rho=1-1/\psi$, where ψ is the elasticity of intertemporal substitution. So $\alpha=\rho\Leftrightarrow\gamma=1/\psi$ as expected.

Preferences for the timing of the resolution of uncertainty. An investor with Epstein-Zin preferences cares about the timing of the resolution of uncertainty:

- $\alpha<\rho$ or $\gamma>1/\psi$ means early resolution of uncertainty is preferred
- $\alpha>\rho$ or $\gamma<1/\psi$ means late resolution of uncertainty is preferred.

Here is an example to illustrate this. Consider an investor with Epstein-Zin preferences who consumes at three dates: 0, 1, and 2. Hold C_0 and C_1 constant and let

$$C_2=\begin{cases}C^u & \text{with probability } \frac{1}{2},\\ C^d & \text{with probability } \frac{1}{2}.\end{cases}$$

Suppose C_2 is not known until time 2. Then the agent's value function at times 1 and 2 can be written as

$$\begin{aligned}J_1&=\left[(1-\delta)C_1^\rho+\delta\left[\tfrac{1}{2}((C^u)^\alpha+(C^d)^\alpha)\right]^{\rho/\alpha}\right]^{1/\rho}\qquad\text{and}\\ J_0&=\left[(1-\delta)C_0^\rho+\delta(J_1^\alpha)^{\rho/\alpha}\right]^{1/\rho}\\ &=\left[(1-\delta)C_0^\rho+\delta\left[(1-\delta)C_1^\rho+\delta\big(\tfrac{1}{2}((C^u)^\alpha+(C^d)^\alpha)\big)^{\rho/\alpha}\right]\right]^{1/\rho},\end{aligned}$$

respectively.

Suppose instead that C_2 is known at time 1. Then the agent's value function at time 1 depends on consumption at time 2:

$$\begin{aligned}\hat{J}_1(C^u)&=\left[(1-\delta)C_1^\rho+\delta[(C^u)^\alpha]^{\rho/\alpha}\right]^{1/\rho}\\ &=\left[(1-\delta)C_1^\rho+\delta(C^u)^\rho\right]^{1/\rho}\qquad\text{and}\\ \hat{J}_1(C^d)&=\left[(1-\delta)C_1^\rho+\delta(C^d)^\rho\right]^{1/\rho},\end{aligned}$$

and so the agent's value function at time 0 is given by

$$\begin{aligned}\hat{J}_0&=\left[(1-\delta)C_0^\rho+\delta\big(\tfrac{1}{2}(\hat{J}_1(C^u)^\alpha+\hat{J}_1(C^d)^\alpha\big)^{\rho/\alpha}\right]^{1/\rho}\\ &=\left[(1-\delta)C_0^\rho+\delta\big(\tfrac{1}{2}\big([(1-\delta)C_1^\rho+\delta(C^u)^\rho]^{\alpha/\rho}+[(1-\delta)C_1^\rho+\delta(C^d)^\rho]^{\alpha/\rho}\big)\big)^{\rho/\alpha}\right]^{1/\rho}.\end{aligned}$$

Now $\hat{J}_0 > J_0$ iff

$$\frac{1}{\delta}(\hat{J}_0^{\rho} - (1-\delta)C_0^{\rho})\operatorname{sign}(\rho) > \frac{1}{\delta}(J_0^{\rho} - (1-\delta)C_0^{\rho})\operatorname{sign}(\rho)$$

$\Leftrightarrow$

$$\left(\frac{1}{2}^{\rho/\alpha}([(1-\delta)C_1^{\rho} + \delta(C^u)^{\rho}]^{\alpha/\rho} + [(1-\delta)C_1^{\rho} + \delta(C^d)^{\rho}]^{\alpha/\rho})^{\rho/\alpha}\right)\operatorname{sign}(\rho)$$
$$> \left((1-\delta)C_1^{\rho} + \delta\frac{1}{2}^{\rho/\alpha}((C^u)^{\alpha} + (C^d)^{\alpha})^{\rho/\alpha}\right)\operatorname{sign}(\rho)$$

$\Leftrightarrow$

$$\operatorname{sign}(\rho)\Big(\mathrm{E}\left[(k_1 + \delta C_2^{\rho})^{\alpha/\rho}\right]\Big)^{\rho/\alpha} > \big(k_1 + \delta(\mathrm{E}[C_2^{\alpha}])^{\rho/\alpha}\big)\operatorname{sign}(\rho), \quad k_1 > 0.$$

$\Leftrightarrow$

$$\operatorname{sign}(\rho)\operatorname{sign}\left(\frac{\rho}{\alpha}\right)\mathrm{E}[(k_1 + \delta(C_2^{\alpha})^{\rho/\alpha})^{\alpha/\rho}] > (k_1 + \delta\,\mathrm{E}[C_2^{\alpha}]^{\rho/\alpha})^{\alpha/\rho}\operatorname{sign}(\rho)\operatorname{sign}\left(\frac{\rho}{\alpha}\right).$$

Define

$$f(C_2^{\alpha}) = (k_1 + \delta(C_2^{\alpha})^{\rho/\alpha})^{\alpha/\rho}.$$

Now

$$\begin{aligned} f'(C_2^{\alpha}) &= (k_1 + \delta(C_2^{\alpha})^{\rho/\alpha})^{\alpha/\rho - 1}\frac{\alpha}{\rho}\delta(C_2^{\alpha})^{\rho/\alpha - 1}\frac{\rho}{\alpha} \\ &= (k_1(C_2^{\alpha})^{-\rho/\alpha} + \delta)^{\alpha/\rho - 1}, \qquad \text{and} \end{aligned}$$

$$f''(C_2^{\alpha}) = \left(\frac{\alpha}{\rho} - 1\right)(k_1(C_2^{\alpha})^{-\rho/\alpha} + \delta)^{\alpha/\rho - 2}k_1\Big(-\frac{\rho}{\alpha}\Big)(C_2^{\alpha})^{-\rho/\alpha - 1},$$

since

$$\frac{\frac{\rho}{\alpha} - 1}{\frac{\alpha}{\rho} - 1} = \frac{\frac{\rho - \alpha}{\alpha}}{\frac{\alpha - \rho}{\alpha}} = -\frac{\rho}{\alpha}.$$

Jensen's inequality says that for an arbitrary twice-continuously differentiable function $\hat{f}(.)$,

$$\mathrm{E}[\hat{f}(q)]\begin{Bmatrix} > \\ = \\ < \end{Bmatrix}\hat{f}(\mathrm{E}[q]) \quad \Leftrightarrow \quad \hat{f}''(\cdot)\begin{Bmatrix} > \\ = \\ < \end{Bmatrix}0.$$

Now define

$$\hat{f}(C_2^{\alpha}) = \operatorname{sign}(\rho)\operatorname{sign}\left(\frac{\rho}{\alpha}\right)f(C_2^{\alpha}).$$

Then

$$\hat{f}''(C_2^{\alpha}) =$$
$$\underbrace{\operatorname{sign}(\rho)\frac{1}{\rho}}_{+}(\alpha - \rho)\underbrace{\operatorname{sign}\left(\frac{\rho}{\alpha}\right)\Big(-\frac{\rho}{\alpha}\Big)}_{-}\underbrace{(k_1(C_2^{\alpha})^{-\rho/\alpha} + \delta)^{\alpha/\rho - 2}}_{+}\underbrace{k_1(C_2^{\alpha})^{-\rho/\alpha - 1}}_{+}.$$

So it follows that

$$\hat{f}''(\cdot)\begin{Bmatrix}>\\=\\<\end{Bmatrix}0 \Leftrightarrow \alpha\begin{Bmatrix}<\\=\\>\end{Bmatrix}\rho \Leftrightarrow \mathrm{E}[\hat{f}(C_2^{\alpha})]\begin{Bmatrix}>\\=\\<\end{Bmatrix}\hat{f}(\mathrm{E}[C_2^{\alpha}]) \Leftrightarrow \hat{J}_0\begin{Bmatrix}>\\=\\<\end{Bmatrix}J_0$$

$\Leftrightarrow$

$$\text{agent}\begin{cases}\text{prefers early resolution of uncertainty.}\\ \text{does not care about the timing of uncertainty resolution.}\\ \text{prefers late resolution of uncertainty.}\end{cases}$$

This example illustrates how the agent's preference for the timing of the resolution of uncertainty depends on the relative magnitude of α and ρ.

First-order conditions and solution. Suppose the value function can be written

$$J(W_{t+1},\mathcal{F}_{t+1}) = \phi(\mathcal{F}_{t+1})W_{t+1}$$

and the solution satisfies (79), then the Bellman equation for Epstein-Zin preferences (80) can be written

$$\begin{aligned}(81)\qquad & \phi(\mathcal{F}_t)W_t = \\ & \max_{\varphi(\mathcal{F}_t),\boldsymbol{\alpha}(\mathcal{F}_t)}\left[(1-\delta)\varphi(\mathcal{F}_t)^{\rho} + \delta(1-\varphi(\mathcal{F}_t))^{\rho}\underbrace{(E[(\phi(\mathcal{F}_{t+1})R_{W,t+1})^{\alpha}\,|\,\mathcal{F}_t])^{\rho/\alpha}}_{\Omega(\mathcal{F}_t;\boldsymbol{\alpha})^{\rho}}\right]^{1/\rho}W_t,\end{aligned}$$

since

$$\text{s.t.}\quad W_{t+1} = W_t(1-\varphi(\mathcal{F}_t))\underbrace{(\boldsymbol{\alpha}(\mathcal{F}_t)'(\mathbf{R}_{t+1}-R^f_{t+1}\mathbf{i}_N)+R^f_{t+1})}_{R_{W,t+1}}.$$

The first-order condition for this problem with respect to $\boldsymbol{\alpha}_t$ is given by

$$(82)\qquad \mathrm{E}[\varphi(\mathcal{F}_{t+1})^{\alpha}(R_{W,t+1})^{\alpha-1}(\mathbf{R}_{t+1}-R^f_{t+1}\mathbf{i}_N)\,|\,\mathcal{F}_t] = \mathbf{0},$$

which shows that $\boldsymbol{\alpha}_\tau$ does not depend on W_τ as conjectured. The first-order condition with respect to φ is

$$(83)\qquad (1-\delta)\rho\varphi(\mathcal{F}_t)^{\rho-1} - \delta\rho(1-\varphi(\mathcal{F}_t))^{\rho-1}\Omega(\mathcal{F}_t;\boldsymbol{\alpha}_t)^{\rho} = 0,$$

which can be rewritten

$$\begin{aligned}&(1-\delta)\varphi(\mathcal{F}_t)^{\rho-1} = \delta(1-\varphi(\mathcal{F}_t))^{\rho-1}\Omega(\mathcal{F}_t;\boldsymbol{\alpha}_t)^{\rho}\\ \Leftrightarrow\ &\left(\frac{1-\delta}{\delta}\right)^{\frac{1}{\rho-1}}\varphi(\mathcal{F}_t) = (1-\varphi(\mathcal{F}_t))\Omega(\mathcal{F}_t;\boldsymbol{\alpha}_t)^{\frac{\rho}{\rho-1}}\\ \Leftrightarrow\ &\left[\left(\frac{\delta}{1-\delta}\right)^{\frac{1}{1-\rho}} + \Omega(\mathcal{F}_t;\boldsymbol{\alpha})^{\frac{\rho}{\rho-1}}\right]\varphi(\mathcal{F}_t) = \Omega(\mathcal{F}_t;\boldsymbol{\alpha}_t)^{\frac{\rho}{\rho-1}}\end{aligned}$$

$\Leftrightarrow$

$$\varphi(\mathcal{F}_t) = \frac{1}{\left[\frac{\delta}{1-\delta}\Omega(\mathcal{F}_t;\boldsymbol{\alpha}_t)^\rho\right]^{\frac{1}{1-\rho}} + 1}. \tag{84}$$

This expression shows that $\varphi(\mathcal{F}_t)$ does not depend on wealth W_t as conjectured.

Now to verify the conjectured form of the value function we first substitute (84) into the Bellman equation (81) to obtain

$$\begin{aligned}
J(W_t, \mathcal{F}_t) &= \left[(1-\delta)\varphi(\mathcal{F}_t)^\rho W_t^\rho + \delta(1-\varphi(\mathcal{F}_t))^\rho W_t^\rho \Omega(\mathcal{F}_t;\boldsymbol{\alpha})^\rho\right]^{\frac{1}{\rho}} \\
&= \left[(1-\delta)\varphi(\mathcal{F}_t)^\rho + \delta(1-\varphi(\mathcal{F}_t))^\rho \Omega(\mathcal{F}_t;\boldsymbol{\alpha})^\rho\right]^{1/\rho} W_t \\
&= \left(\frac{(1-\delta) + \delta\frac{(1-\delta)}{\delta}\left[\frac{\delta}{1-\delta}\Omega(\mathcal{F}_t;\boldsymbol{\alpha})^\rho\right]^{\frac{\rho}{1-\rho}+1}}{\left(\left[\frac{\delta}{1-\delta}\Omega(\mathcal{F}_t;\boldsymbol{\alpha})^\rho\right]^{\frac{1}{1-\rho}} + 1\right)^\rho}\right)^{1/\rho} W_t \\
&= (1-\delta)^{1/\rho}\left(\frac{1 + \left(\frac{\delta}{1-\delta}\right)^{1+\frac{\rho}{1-\rho}}(\Omega(\mathcal{F}_t;\boldsymbol{\alpha})^\rho)^{\frac{\rho}{1-\rho}+1}}{\left(\left[\frac{\delta}{1-\delta}\Omega(\mathcal{F}_t;\boldsymbol{\alpha})^\rho\right]^{\frac{1}{1-\rho}} + 1\right)^\rho}\right)^{1/\rho} W_t \\
&= (1-\delta)^{1/\rho}\varphi(\mathcal{F}_t)^{\frac{\rho-1}{\rho}} W_t,
\end{aligned}$$

as conjectured, with

$$\phi(\mathcal{F}_t) = (1-\delta)^{1/\rho}\varphi(\mathcal{F}_t)^{\frac{\rho-1}{\rho}}. \tag{85}$$

We have just shown that the form of the value function and the solution are in fact as conjectured.

Next we show how to obtain a representation of the first order conditions for $\boldsymbol{\alpha}$ and c that is often used in the literature and that is used below to obtain expressions for conditional expected log asset returns. First, equation (83), which is the first order condition with respect to φ, can be written as

$$(1-\delta)\varphi(\mathcal{F}_t)^{\rho-1} = \delta(1-\varphi(\mathcal{F}_t))^{\rho-1}\Omega(\mathcal{F}_t;\boldsymbol{\alpha})^\rho,$$

which can be rewritten as

$$\varphi(\mathcal{F}_t)^{(\rho-1)\frac{\alpha}{\rho}} = \left(\frac{\delta}{1-\delta}\right)^{\alpha/\rho}(1-\varphi(\mathcal{F}_t))^{(\rho-1)\frac{\alpha}{\rho}}\,\mathrm{E}[\phi(\mathcal{F}_{t+1}^\alpha)(R_{W,t+1})^\alpha \,|\, \mathcal{F}_t]. \tag{86}$$

Then take equation (82), which is the first order condition for $\boldsymbol{\alpha}$, and premultiply by $\boldsymbol{\alpha}(\mathcal{F}_t)$ to obtain

$$\mathrm{E}[\phi(\mathcal{F}_{t+1})^\alpha(R_{W,t+1})^{\alpha-1}(R_{W,t+1} - R_{t+1}^f) \,|\, \mathcal{F}_t] = 0. \tag{87}$$

Now (82), (87) and (86) imply

$$\begin{aligned}
&\varphi(\mathcal{F}_t)^{(\rho-1)\frac{\alpha}{\rho}} \\
&\quad = \left(\frac{\delta}{1-\delta}\right)^{\frac{\alpha}{\rho}}(1-\varphi(\mathcal{F}_t))^{(\rho-1)\frac{\alpha}{\rho}}\,\mathrm{E}[\phi(\mathcal{F}_{t+1})^\alpha(R_{W,t+1})^{\alpha-1}R_{i,t+1} \,|\, \mathcal{F}_t], \quad i = f, 1, \ldots, N.
\end{aligned} \tag{88}$$

Finally substitute (85) into (88) to obtain

$$\underbrace{\varphi(\mathcal{F}_t)^{(\rho-1)\frac{\alpha}{\rho}}}_{C_t/W_t} = \left(\frac{\delta}{1-\delta}\right)^{\frac{\alpha}{\rho}} \mathrm{E}\Big[\underbrace{(1-\varphi(\mathcal{F}_t))^{(\rho-1)\frac{\alpha}{\rho}} \varphi(\mathcal{F}_{t+1})^{\frac{\rho-1}{\rho}\alpha} (R_{W,t+1})^{\frac{\rho-1}{\rho}\alpha}}_{C_{t+1}/W_t}$$
$$(1-\delta)^{\frac{\alpha}{\rho}} (R_{W,t+1})^{\frac{1-\rho}{\rho}\alpha} (R_{W,t+1})^{\alpha-1} R_{i,t+1} \,\Big|\, \mathcal{F}_t \Big]$$

$\Rightarrow$

$$1 = \delta^{\frac{\alpha}{\rho}} \mathrm{E}\left[\left(\frac{C_{t+1}}{C_t}\right)^{\frac{\rho-1}{\rho}\alpha} (R_{W,t+1})^{\frac{\alpha-\rho}{\rho}} R_{i,t+1} \,\middle|\, \mathcal{F}_t \right], \quad i = f, 1, \ldots, N. \tag{89}$$

Notice that this expression collapses to the CRRA first order condition when $\rho = \alpha$.

Log risk free rate and expected log risky asset return. Suppose

$$g_{t+1} \equiv \ln\left(\frac{C_{t+1}}{C_t}\right), \quad r_{W,t+1} \equiv \ln(R_{W,t+1}) \quad \text{and} \quad r_{i,t+1} \equiv \ln(R_{i,t+1})$$

are conditionally distributed

$$\text{Normal}\left(\begin{bmatrix} \mu_{g,t} \\ \mu_{W,t} \\ \mu_{i,t} \end{bmatrix}, \begin{bmatrix} \sigma^2_{g,t} & \sigma_{gW,t} & \sigma_{gi,t} \\ \sigma_{Wg,t} & \sigma^2_{W,t} & \sigma_{Wi,t} \\ \sigma_{ig,t} & \sigma_{iW,t} & \sigma^2_{i,t} \end{bmatrix} \right),$$

and let $r_{f,t+1} \equiv \ln(R_{f,t+1})$.

With these assumptions and definitions, it is possible to obtain closed-form expressions for $r_{f,t+1}$ and

$$\ln(\mathrm{E}_t[R_{i,t+1}]/R_{f,t+1}) = \mathrm{E}_t[r_{i,t+1} - r_{f,t+1}] + \tfrac{1}{2}\sigma^2_{i,t}.$$

Define

$$m_{t+1} = \theta \ln\delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{W,t+1},$$

where

$$\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}.$$

Using the first order condition (89) derived above, we know that $\exp\{m_{t+1}\}$ is the agent's marginal rate of substitution which can be used as an sdf. So

$$\mathrm{E}_t[\exp\{m_{t+1} + r_{i,t+1}\}] = 1,$$

$\Leftrightarrow$

$$\exp\left\{\mathrm{E}_t[m_{t+1} + r_{i,t+1}] + \tfrac{1}{2}\sigma^2_t[m_{t+1} + r_{i,t+1}]\right\} = 1,$$

$\Leftrightarrow$

$$\mathrm{E}_t[m_{t+1} + r_{i,t+1}] + \tfrac{1}{2}\sigma^2_t[m_{t+1}] + \tfrac{1}{2} - \sigma^2_t[r_{i,t+1}] + \sigma_t[m_{t+1}, r_{i,t+1}] = 0. \tag{90}$$

Since (90) implies that

$$-\mathrm{E}_t[m_{t+1}] - \tfrac{1}{2}\sigma_t^2[m_{t+1}] = r_{f,t+1},$$

it follows that

$$\mathrm{E}_t[r_{i,t+1} - r_{f,t+1}] + \tfrac{1}{2}\sigma_t^2[r_{i,t+1}] = -\sigma_t[m_{t+1}, r_{i,t+1}].$$

And finally, since the definition of m_{t+1} implies that

$$\sigma_t[m_{t+1}, r_{i,t+1}] = -\frac{\theta}{\psi}\sigma_{gi,t} + (\theta - 1)\sigma_{Wi,t},$$

it follows that

$$\mathrm{E}_t[r_{i,t+1} - r_{f,t+1}] + \tfrac{1}{2}\sigma_{i,t}^2 = \frac{\theta}{\psi}\sigma_{gi,t} + (1 - \theta)\sigma_{Wi,t}. \tag{91}$$

Setting $i = W$ in equation (91) gives us the following expression for the expected log return on the agent's wealth portfolio:

$$\mathrm{E}_t[r_{W,t+1} - r_{f,t+1}] + \tfrac{1}{2}\sigma_{W,t}^2 = \frac{\theta}{\psi}\sigma_{gW,t} + (1 - \theta)\sigma_{W,t}^2. \tag{92}$$

Using the definition of θ, it follows that

$$1 - \theta = \frac{\gamma - \frac{1}{\psi}}{1 - \frac{1}{\psi}}.$$

So if $\psi > 1$, then a preference for early resolution of uncertainty (i.e., $\gamma > \frac{1}{\psi}$) implies that the agent dislikes the volatility associated with the log excess return on her wealth portfolio and so requires a higher conditional expected log excess return on her wealth portfolio as this volatility increases. Moreover, consistent with intuition, expression (92) implies that the conditional expected log excess return on arbitrary asset i is increasing in the conditional covariance of its log excess return with the log excess return on the agent's wealth portfolio. Alternately, if $\psi < 1$, then a preference for early resolution of uncertainty (i.e., $\gamma > \frac{1}{\psi}$) implies that the agent likes the volatility associated with the log excess return on her wealth portfolio and so requires a lower conditional expected log excess return on her wealth portfolio as this volatility increases. Moreover, consistent with intuition, expression (92) implies that the conditional expected log excess return on arbitrary asset i is decreasing in the conditional covariance of its log excess return with the log excess return on the agent's wealth portfolio.

It is now possible to obtain an expression for $r_{f,t+1}$ in terms of utility parameters, variances, covariances and $\mu_{g,t}$. First, it is possible to express $r_{f,t+1}$ in terms of utility parameters, variances, covariances, $\mu_{g,t}$ and $\mathrm{E}_t[r_{W,t+1} - r_{f,t+1}]$. Using (90), we know

$$\begin{aligned}
r_{f,t+1} &= -\mathrm{E}_t[m_{t+1}] - \tfrac{1}{2}\sigma_t^2[m_{t+1}] \\
&= -\theta \ln \delta + \frac{\theta}{\psi}\mu_{g,t} + (1 - \theta)\mu_{W,t} \\
&\qquad - \frac{1}{2}\left(\frac{\theta^2}{\psi^2}\sigma_{g,t}^2 + (\theta - 1)^2\sigma_{W,t}^2 - 2\frac{\theta(\theta - 1)}{\psi}\sigma_{gW,t}^2\right).
\end{aligned}$$

Subtract $(1-\theta)r_{f,t+1}$ from both sides and divide by θ to obtain

$$r_{f,t+1} = -\ln\delta + \frac{1}{\psi}\mu_{g,t} + \frac{1-\theta}{\theta}\,\mathrm{E}_t[r_{W,t+1} - r_{f,t+1}] \\ - \frac{1}{2}\frac{\theta}{\psi^2}\sigma^2_{g,t} - \frac{1}{2}\frac{(\theta-1)^2}{\theta}\sigma^2_{W,t} + \frac{(\theta-1)}{\psi}\sigma^2_{gW,t}.$$

Now substitute in (92) to obtain

$$\begin{aligned} r_{f,t+1} &= -\ln\delta + \frac{1}{\psi}\mu_{g,t} - \frac{1}{2}\frac{\theta}{\psi^2}\sigma^2_{g,t} + \left(\frac{(1-\theta)(\frac{1}{2}-\theta)}{\theta} - \frac{1}{2}\frac{(\theta-1)^2}{\theta}\right)\sigma^2_{W,t} \\ &\quad + \left(\frac{1-\theta}{\theta}\frac{\theta}{\psi} + \frac{\theta-1}{\psi}\right)\sigma_{gW,t} \\ &= -\ln\delta + \frac{1}{\psi}\mu_{g,t} - \frac{1}{2}\frac{\theta}{\psi^2}\sigma^2_{g,t} + \frac{1}{2}\theta(1+2\theta^2-3\theta-\theta^2-1+2\theta)\sigma^2_{W,t} \\ &= -\ln\delta + \frac{1}{\psi}\mu_{g,t} - \frac{1}{2}\frac{\theta}{\psi^2}\sigma^2_{g,t} + \frac{1}{2}(\theta-1)\sigma^2_{W,t}. \end{aligned}$$

Equity premium and riskfree rate puzzles: CRRA versus Epstein-Zin. We compare how $\ln(\mathrm{E}_t\, R_{W,t+1}/R_{f,t+1})$ and $\ln(R_{f,t+1})$ change going from CRRA preferences with $\gamma = 10$ to Epstein–Zin preferences with $\gamma = 10$ and $\psi = 1.5$. Note that

$$\theta = \frac{1-\gamma}{1-\frac{1}{\psi}} = \frac{1-10}{1-\frac{1}{1.5}} = \frac{1.5(1-10)}{0.5} = -27.$$

for Epstein–Zin preferences. This comparison shows how Epstein-Zin preferences can help with the equity premium puzzle and the risk-free rate puzzle.

First, consider $\ln(\mathrm{E}_t[R_{W,t+1}]/R_{f,t+1})$:

$$\begin{aligned} \text{E–Z:}\qquad \mathrm{E}_t[r_{W,t+1} - r_{f,t+1}] + \tfrac{1}{2}\sigma^2_{W,t} &= \frac{\theta}{\psi}\sigma_{gW,t} + (1-\theta)\sigma^2_{W,t} \\ &= \frac{-27}{1.5}\sigma_{gW,t} + 28\sigma^2_{W,t} \\ &= -18\sigma_{gW,t} + 28\sigma^2_{W,t} \\ \text{CRRA:}\qquad \mathrm{E}_t[r_{W,t+1} - r_{f,t+1}] + \tfrac{1}{2}\sigma^2_{W,t} &= 10\sigma_{gW,t}. \end{aligned}$$

Recall that the equity premium puzzle says that the equity Sharpe ratio is too high using CRRA preferences and aggregate consumption data moments. Since $\sigma_{W,t} \gg \sigma_{gW}$, it follows that

$$\mathrm{E}_t[r_{W,t+1} - r_{f,t+1}] + \tfrac{1}{2}\sigma^2_{W,t}$$

is much higher for Epstein–Zin than power utility, which means Epstein–Zin preferences can help explain the equity premium puzzle.

Second, consider $\ln(R_{f,t+1})$:

$$\begin{aligned}
\text{E–Z:} \qquad r_{f,t+1} &= -\ln\delta + \frac{1}{\psi}\mu_{g,t} - \frac{1}{2}\frac{\theta}{\psi^2}\sigma^2_{g,t} + \frac{1}{2}(\theta-1)\sigma^2_{W,t} \\
&= -\ln\delta + \frac{2}{3}\mu_{g,t} + \frac{1}{2}\frac{27\times 4}{3\times 3}\sigma^2_{g,t} - \frac{1}{2}28\sigma^2_{W,t} \\
&= -\ln\delta + \frac{2}{3}\mu_{g,t} + 6\sigma^2_{g,t} - 14\sigma^2_{W,t} \\
\text{CRRA:} \qquad r_{f,t+1} &= -\ln\delta + 10\mu_{g,t} - 50\sigma^2_{g,t}.
\end{aligned}$$

Recall that the risk free rate puzzle says the risk free rate is too low using CRRA preferences and aggregate consumption data moments.. Since $\frac{2}{3}\mu_{g,t} \ll 10\mu_{g,t}$ and $\sigma^2_{W,t} \gg \sigma^2_{g,t}$, it follows that $r_{f,t+1}$ is lower for Epstein–Zin than power preferences, which means Epstein–Zin preferences can simultaneously help explain the risk free rate puzzle.

13.4. Long run risk model

Model setup. Consider an economy with a representative agent who has Epstein–Zin preferences, and define g_{t+1} to be log consumption growth and $g_{i,t+1}$ to be the log dividend growth of asset i. Suppose

$$\begin{aligned}
g_{t+1} &= \mu + x_t + \sigma\eta_{t+1}, \\
g_{i,t+1} &= \mu_i + \phi_i x_t + \varphi_i \sigma u_{t+1}, \\
x_{t+1} &= \rho x_t + \varphi\sigma e_{t+1},
\end{aligned}$$

where η_{t+1}, u_{t+1} and e_{t+1} are iid $N[0,1]$, with $0<\rho<1$. Note that $r_{W,t+1}$ is the return on the total wealth portfolio whose dividend process equals the aggregate consumption process C_{t+1}.

Approximate expression for $r_{W,t+1}$. Let P_t be the price of the aggregate consumption series and define $p_{t+1} = \ln P_{t+1}$, $c_{t+1} = \ln(C_{t+1})$ and $z_t = \ln(P_t/C_t)$. We can use a log linearization of $c_{t+1} - p_{t+1}$ around its unconditional mean $\bar{c} - \bar{p}$ to derive the approximate relation

$$r_{W,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1}, \tag{93}$$

where κ_0 and κ_1 are linearization constants.

To show this, first note that

$$f(x) \approx f(\bar{x}) + f'(\bar{x})(x-\bar{x}),$$

and so taking $f(x) = \ln(1+\exp x)$, it follows that

$$\ln(1+\exp x) \approx \ln(1+\exp(\bar{x})) + \frac{\exp\bar{x}}{1+\exp\bar{x}}(x-\bar{x}). \tag{94}$$

Now

$$
\begin{aligned}
r_{W,t+1} &= \ln\left(\frac{P_{t+1}+C_{t+1}}{P_t}\right)\\
&= \ln\left(\frac{P_{t+1}+C_{t+1}}{P_{t+1}}\right) + p_{t+1} - p_t\\
&= \ln(1+\exp(c_{t+1}-p_{t+1})) + p_{t+1} - p_t\\
&\approx p_{t+1} - p_t + \ln(1+\exp(-\bar{z})) + \underbrace{\frac{\exp(-\bar{z})}{1+\exp(-\bar{z})}}_{1-\rho}(c_{t+1}-p_{t+1}+\bar{z}) \qquad \text{using (94)}\\
&= \rho p_{t+1} - p_t + \ln\left(\frac{1}{\rho}\right) + (1-\rho)p_{t+1} + (1-\rho)(c_{t+1}-p_{t+1}) - (1-\rho)\underbrace{\ln(-\bar{z})}_{\frac{1}{\rho}-1}\\
&= \underbrace{-\ln(\rho) - (1-\rho)\ln\left(\frac{1}{\rho}-1\right)}_{\kappa_0} + \rho p_{t+1} + (1-\rho)c_{t+1} - p_t\\
&= \kappa_0 + \underbrace{\rho}_{\kappa_1}(p_{t+1}-c_{t+1}) + c_{t+1} - c_t - (p_t - c_t)\\
&= \kappa_0 + \kappa_1 z_{t+1} + g_{t+1} - z_t \qquad \text{as required.}
\end{aligned}
$$

Pricing the consumption claim. By substituting (93) into the Epstein–Zin first-order condition for $r_{W,t+1}$, we can show that

$$z_t = A_0 + A_1 x_t, \tag{95}$$

where A_0 and A_1 are constants. Conjecture that (95) holds. Using (89) for $i = W$, the first-order condition for $R_{W,t+1}$ is given by

$$\mathrm{E}_t\left[\exp\left\{\theta\ln\delta - \frac{\theta}{\psi}g_{t+1} + \theta r_{W,t+1}\right\}\right] = 1.$$

Substituting in (93), and using the conjectured expression for z_t in (95), gives

$$
\begin{aligned}
&\exp\left\{\theta\ln\delta - \frac{\theta}{\psi}\mathrm{E}_t[g_{t+1}] + \theta\,\mathrm{E}_t[r_{W,t+1}] + \frac{1}{2}\sigma_t^2\left[\frac{\theta}{\psi}g_{t+1} + \theta r_{W,t+1}\right]\right\} = 1,\\
&\Rightarrow \theta\ln\delta - \frac{\theta}{\psi}(\mu + x_t) + \theta(\kappa_1 E_t[z_{t+1}] - z_t + \mu + x_t)\\
&\qquad + \frac{1}{2}\sigma_t^2\left[\frac{\theta}{\psi}\sigma\eta_{t+1} + \theta(\kappa_1 z_{t+1} + \sigma\eta_{t+1})\right] = 0,
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
&\theta\ln\delta - \frac{\theta}{\psi}(\mu + x_t) + \theta\kappa_1(A_0 + A_1\rho x_t) - \theta(A_0 + A_1 x_t) + \theta(\mu + x_t)\\
&\qquad + \frac{1}{2}\sigma_t^2\left[\frac{\theta}{\psi}\rho\eta_{t+1} + \theta\kappa_1 A_1\varphi\sigma e_{t+1} + \theta\sigma\eta_{t+1}\right] = 0.
\end{aligned} \tag{96}
$$

Equation (96) holds as required if the constant terms sum to 0 and the coefficients on the x_t terms sum to 0. Thus, the conjectured expression for z_t in (95) holds. Collecting the x_t terms and setting the sum of the coefficients to 0 gives

$$
\begin{aligned}
-\frac{\theta}{\psi} + \theta\kappa_1 A_1\rho - \theta A_1 + \theta = 0 & \\
\Rightarrow 1 - \frac{1}{\psi} &= A_1(1-\kappa_1\rho), \\
\Rightarrow A_1 &= \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho},
\end{aligned}
$$

which gives an expression for A_1. An expression for A_0 can be obtained similarly setting the sum of the constant terms to 0.

Pricing an arbitrary asset *i*. Let $P_{i,t}$ be the price of the ith asset and $D_{i,t}$ be the dividend of the ith asset. Define $z_{i,t} = \ln(P_{i,t}/D_{i,t})$. Just as the approximate relation (93) can be shown to hold for $r_{W,t+1}$, the following approximate relation can be shown to hold for the return on asset i, $r_{i,t+1}$:

$$r_{i,t+1} = \kappa_{0i} + \kappa_{1i} z_{i,t+1} - z_{i,t} + g_{i,t+1} \tag{97}$$

By substituting (97) and (93) into the Epstein–Zin first-order condition for $r_{i,t+1}$ and using 95 for z_t, we can show that

$$z_{i,t} = A_{0i} + A_{1i} x_t. \tag{98}$$

Conjecture that (98) holds. The Epstein–Zin first-order condition for $r_{i,t+1}$ in (89) can be written

$$\mathrm{E}_t\left[\exp\left\{\theta\ln\delta - \frac{\theta}{\psi}g_{t+1} + (\theta-1)r_{W,t+1} + r_{i,t+1}\right\}\right] = 1.$$

Substituting in (93) and (97), and using the conjectured expression for z_t in (95) and the conjectured expression for $z_{i,t}$ in (98), gives

$$
\begin{aligned}
&\exp\left\{\theta\ln\delta - \frac{\theta}{\psi}\,\mathrm{E}_t\, g_{t+1} + (\theta-1)\,\mathrm{E}_t\, r_{W,t+1} + \mathrm{E}_t\, r_{i,t+1}\right. \\
&\qquad\qquad \left. + \frac{1}{2}\sigma_t^2\left[\frac{\theta}{\psi}g_{t+1} + (\theta-1)r_{W,t+1} + r_{i,t+1}\right]\right\} = 1,
\end{aligned}
$$

$$\Rightarrow$$

$$
\begin{aligned}
&\theta\ln\delta - \frac{\theta}{\psi}(\mu + x_t) + (\theta-1)\kappa_1(A_0 + A_1\rho x_t) - (\theta-1)(A_0 + A_1 x_t) \\
&+(\theta-1)(\mu+x_t) + \kappa_{1i}(A_{0i} + A_{1i}\rho x_t) - A_{0i} - A_{1i}x_t + \mu_i + \phi^i x_t \\
&\qquad + \frac{1}{2}\sigma_t^2\left[\frac{\theta}{\psi}\rho\eta_{t+1} + (\theta-1)\kappa_1 A_1\varphi\sigma e_{t+1} + (\theta-1)\sigma\eta_{t+1}\right. \\
&\qquad\qquad \left. + \kappa_{10}A_{10}\varphi\sigma e_{t+1} + \varphi_i\sigma u_{t+1}\right] = 0.
\end{aligned}
\tag{99}
$$

Equation (99) holds as required if the constant terms sum to 0 and the coefficients on the x_t terms sum to 0. Thus, the conjectured expression for $z_{i,t}$ in (98) holds. Collecting the x_t terms and setting the sum of the coefficients to 0 gives

$$\underbrace{\frac{1}{\psi}-\frac{\theta}{\psi}+(\theta-1)\kappa_1 A_1\rho-(\theta-1)A_1+(\theta-1)}_{=0}+\kappa A_{1i}\rho-A_{1i}+\phi^i-\frac{1}{\psi}=0,$$

$$\begin{aligned}\Rightarrow \phi-\frac{1}{\psi}&=A_{1i}(1-\rho\kappa_{1i}),\\ \Rightarrow A_{1i}&=\frac{\phi^i-\frac{1}{\psi}}{1-\rho\kappa_{1i}}.\end{aligned}$$

which gives an expression for A_{1i}. An expression for A_{0i} can be obtained similarly setting the sum of the constant terms to 0.

Log risk premium for an arbitrary asset i. We know that

$$\mathrm{E}_t[r_{i,t+1}-r_{f,t+1}]+\frac{1}{2}\sigma_{i,t}^2=\frac{\theta}{\psi}\sigma_{gi,t}+(1-\theta)\sigma_{Wi,t}.$$

Now

$$\begin{aligned}r_{i,t+1}&=\kappa_{0i}+\kappa_{1i}z_{i,t+1}-z_{i,t}+g_{i,t+1}\\ &=\kappa_{0i}-z_{i,t}+\kappa_{1i}A_{0i}+\kappa_{1i}A_{1i}\rho x_t+\mu_i+\phi_i x_t+\kappa_{1i}\varphi\sigma A_{1i}e_{t+1}+\varphi_i\sigma u_{t+1},\end{aligned}$$

and,

$$\begin{aligned}r_{W,t+1}&=\kappa_0+\kappa_1 z_{t+1}-z_t+g_{t+1}\\ &=\kappa_0-z_t+\kappa_1(A_0+A_1x_{t+1})+g_{t+1}\\ &=\kappa_0-z_t+\kappa_1(A_0+A_1(\rho x_t+\varphi\sigma e_{t+1}))+\mu+x_t+\sigma\eta_{t+1}\\ &=\kappa_0-z_t+\kappa_1A_0+\kappa_1A_1\rho x_t+\mu+x_t+\kappa_1\varphi\sigma A_1e_{t+1}+\sigma\eta_{t+1}.\end{aligned}$$

It follows that

$$\sigma_{gi,t}=0$$

and

$$\sigma_{Wi,t}=\kappa_1\varphi\sigma A_1\kappa_{1i}\varphi\sigma A_{1i},$$

and thus

$$\begin{aligned}\mathrm{E}_t[r_{i,t+1}-r_{f,t+1}]+\frac{1}{2}\sigma_{i,t}^2&=(1-\theta)\varphi^2\kappa_1\kappa_{1i}A_1A_{1i}\sigma^2\\ &=(1-\theta)\varphi^2\kappa_1\kappa_{1i}\left(\frac{1-\frac{1}{\psi}}{1-\kappa_1\rho}\right)\left(\frac{\phi^i-\frac{1}{\psi}}{1-\rho\kappa_{1i}}\right)\sigma^2.\end{aligned}\tag{100}$$

Log risk premium and Sharpe ratio for market equity. Consider the market equity portfolio such that $i = m$, and assume $\phi^m > 1$, as would be expected if equity is a levered claim to consumption. Assume that the elasticity of substitution $\psi > 1$ and relative risk aversion $\gamma > 1$, which imply that $\theta < 0$. Given these assumptions, (100) can be used to show that the conditional expected log excess market return

$$\mathrm{E}_t[r_{m,t+1} - r_{f,t+1}] + \tfrac{1}{2}\sigma_{mt}^2$$

is a constant that is increasing in ρ, the persistence parameter for the state variable, x_t. Further, the conditional market Sharpe ratio

$$\frac{\mathrm{E}_t[r_{m,t+1} - r_{f,t+1}] + \frac{1}{2}\sigma_{mt}^2}{\sigma_{mt}^2}$$

is also a constant that does *not* vary with x_t since

$$\begin{aligned}\sigma_{mt}^2 &= \sigma^2[\kappa_{1m}\varphi\sigma A_{1m}e_{t+1} + \varphi_m\sigma u_{t+1}]\\ &= [\varphi^2\kappa_{1m}^2 A_{1m}^2 + \varphi_m^2]\sigma^2\end{aligned}$$

is a constant. Thus, expected market equity return and the market Sharpe ratio are not time varying in this economy, which is counterfactual.

Model with long-run risk in consumption growth volatility. Suppose instead that

$$\begin{aligned}g_{t+1} &= \mu + x_t + \sigma_t\eta_{t+1},\\ g_{i,t+1} &= \mu_i + \phi_i x_t + \varphi_t\sigma_t u_{t+1},\\ x_{t+1} &= \rho x_t + \varphi\sigma_t e_{t+1},\\ \sigma_{t+1}^2 &= \sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1},\end{aligned}$$

where

$$\begin{bmatrix} e_{t+1}\\ u_{t+1}\\ \eta_{t+1}\\ w_{t+1}\end{bmatrix} \sim N(0, I) \quad \text{i.i.d.}$$

The relevant state variables for the economy are now x_t and σ_t^2.

The approximate solution for the price-consumption ratio is

$$z_t = A_0 + A_1 x_t + A_2\sigma_t^2,$$

where A_1 is the same as the previous case and

$$A_2 = \frac{0.5\theta\left[\left(1 - \frac{1}{\psi}\right)^2 + A_1\kappa_1\varphi\right]}{(1 - \kappa_1\nu_1)}.$$

The approximate solution for the price-dividend ratio for asset i is

$$z_{i,t} = A_{0i} + A_{1i}x_t + A_{2i}\sigma_t^2,$$

where A_{1i} is the same as the previous case and A_{2i} is calculated similarly to A_{1i} in the previous case.

Consider the market equity portfolio such that $i = m$, in this economy. It can be shown that

$$\mathrm{E}_t[r_{m,t+1} - r_{f,t+1}] + \frac{1}{2}\sigma_t^2[r_{m,t+1}] = \beta_{m,e}\lambda_{m,e}\sigma_t^2 + \beta_{m,w}\lambda_{m,w}\sigma_w^2$$

and

$$\sigma_t^2[r_{m,t+1}] = \beta_{m,e}^2\sigma_t^2 + \varphi_m^2\sigma_t^2 + \beta_{m,w}^2\sigma_w^2,$$

where

$$\begin{aligned}
\beta_{m,w} &= \kappa_{1m}A_{2m}, \\
\beta_{m,e} &= \kappa_{1m}A_{1m}\varphi, \\
\lambda_{m,w} &= (1-\theta)A_2\kappa_1, \\
\lambda_{m,e} &= (1-\theta)A_1\kappa_1\varphi.
\end{aligned}$$

Notice that the conditional expected log excess market return

$$\mathrm{E}_t[r_{m,t+1} - r_{f,t+1}] + \frac{1}{2}\sigma_{mt}^2$$

and the conditional market Sharpe ratio

$$\frac{\mathrm{E}_t[r_{m,t+1} - r_{f,t+1}] + \frac{1}{2}\sigma_{mt}^2}{\sigma_{mt}^2}$$

are now both functions of σ_t^2 and time-varying. Thus, a version of the model with time-varying and highly persistent volatility for consumption growth can deliver the market return predictability found empirically.

13.5. Frictions: Euler equation inequalities

Consider an economy populated by many investors at time t indexed by j in which $N+1$ assets are available. The ith asset offers a return of $R_{\tau+1}^i$ from τ to $\tau+1$, while the $(N+1)$th asset pays a riskless return of $R_{\tau+1}^f$ from τ to $\tau+1$.

The jth investor has wealth W_t^j at time t and is endowed with labor income y_τ^j at time τ for all τ from $t+1$ to T^j when she dies. If $T^j > t$ then the jth investor solves the following problem at time t:

$$\max_{\{c_\tau,\boldsymbol{\alpha}_\tau\}_{\tau=t}^{T^j-1}} \mathrm{E}_t\left[\Sigma_{\tau=t}^{T^j}\delta^{\tau-t}u(c_\tau^j)\right]$$

subject to

$$W_{\tau+1}^j = y_{\tau+1}^j + (W_\tau^j - c_\tau^j)((\boldsymbol{\alpha}^j)'\boldsymbol{R}_{\tau+1} + (1-(\boldsymbol{\alpha}^j)'\mathbf{i}_N)R_{\tau+1}^f)$$

and $c_\tau^j, \boldsymbol{\alpha}_\tau^j \in \mathcal{F}_\tau$ for $\tau = t, \ldots, T^j - 1$. where $\boldsymbol{\alpha}_\tau^j$ is a $N \times 1$ vector of portfolio weights with ith element, $\alpha_{i,\tau}^j$, the portfolio weight in the ith risky asset. Let $\{c_\tau^{j*}\}_{\tau=t}^{T^j-1}$ be the solution to the jth investor's problem.

No frictions. Suppose the jth investor reduces c_t^{j*} by $\Delta_t \in \mathcal{F}_t$, where $\Delta_t > 0$, invests Δ_t more in asset i, and increases c_{t+1}^{j*} by $\Delta_t R_{t+1}^i$. Consider

$$L(\Delta_t) = u(c_t^{j*} - \Delta_t) + \mathrm{E}_t[\delta u(c_{t+1}^{j*} + \Delta_t R_{t+1}^i)]$$

and

$$\frac{dL(\Delta_t)}{d\Delta_t} = -u'(c_t^{j*} - \Delta_t) + \mathrm{E}_t[\delta u'(c_{t+1}^{j*} + \Delta_t R_{t+1}^i) R_{t+1}^i].$$

Since $\{c_\tau^{j*}\}_{\tau=t}^{T^j-1}$ is the solution to the jth investor's problem, we know that

$$\frac{dL(0)}{d\Delta_t} = -u'(c_t^{j*}) + \mathrm{E}_t[\delta u'(c_{t+1}^{j*}) R_{t+1}^i] \leq 0. \tag{101}$$

Suppose the jth investor increases c_t^{j*} by $\Delta_t \in \mathcal{F}_t$, where $\Delta_t > 0$, invests Δ_t less in asset i, and reduces c_{t+1}^{j*} by $\Delta_t R_{t+1}^i$. Consider

$$L(\Delta_t) = u(c_t^{j*} + \Delta_t) + \mathrm{E}_t[\delta u(c_{t+1}^{j*} - \Delta_t R_{t+1}^i].$$

and

$$\frac{dL(\Delta_t)}{d\Delta_t} = u'(c_t^{j*} + \Delta_t) - \mathrm{E}_t[\delta u'(c_{t+1}^{j*} - \Delta_t R_{t+1}^i) R_{t+1}^i].$$

Since $\{c_\tau^{j*}\}_{\tau=t}^{T^j-1}$ is the solution to the jth investor's problem, we know that

$$\frac{dL(0)}{d\Delta_t} = u'(*c_t^{j*}) - \mathrm{E}_t[\delta u'(c_{t+1}^{j*}) R_{t+1}^i] \leq 0. \tag{102}$$

Putting (101) and (102) together, we obtain

$$u'(c_t^{j*}) - \mathrm{E}_t[\delta u'(c_{t+1}^{j*}) R_{t+1}^I] = 0 \Leftrightarrow \mathrm{E}_t\left[\frac{\delta u'(c_{t+1}^{j*})}{u'(c_t^{j*})} R_{t+1}^i\right] = 1. \tag{103}$$

Note that in equilibrium (103) holds for all j with $T^j > t$, and is the standard Euler equation that is derived in earlier chapters. The implication of (103) is that any agent j's marginal rate of substitution works as a conditional sdf in the frictionless case and can be used as a one-factor Beta pricing model that pins down the conditional expected return of asset i.

Short-sale constraints. Suppose there are short-sale constraints in the economy of the following form: for $i \in A$, $\alpha_{i,\tau}^j \geq 0$ for all j, and for $i \notin A$, $\alpha_{i,\tau}^j \in \mathbb{R}$ for all j. For $i \notin A$, the frictionless analysis goes through and (103) holds. For $i \in A$, the jth investor can always reduce c_t^{j*} by $\Delta_t \in \mathcal{F}_t$, where $\Delta_t > 0$, and invest Δ_t more in asset i. But if $\alpha_{i,t}^j = 0$, she won't be able to increase c_t^{j*} by $\Delta_t \in \mathcal{F}_t$, where $\Delta_t > 0$, and invest Δ_t less in asset i. So (101) continues holds but (102) doesn't, and so for all $i \notin A$

$$\mathrm{E}_t\left[\frac{\delta u'(c_{t+1}^{j*})}{u'(c_t^{j*})} R_{t+1}^i\right] \leq 1.$$

It follows that

$$\mathrm{E}_t\left[\frac{\delta u'(c_{t+1}^{j*})}{u'(c_t^{j*})}R_{t+1}^i\right]\begin{cases}=1 & i\notin A\\ \leq 1 & i\in A.\end{cases} \tag{104}$$

Note that (104) holds for all j with $T^j > t$ in equilibrium. Moreover, it implies that the marginal rate of substitution of any agent j can be used to pin down the conditional expected return on any asset $i \notin A$, and to put an upper bound on the conditional expected return on any asset $i \in A$.

Proportional transaction costs. Let ρ_b^i be the proportional cost to buy asset i and ρ_s^i be the proportional cost to sell asset i. Suppose the jth investor reduces c_t^{j*} by $\Delta_t \in \mathcal{F}_t$, where $\Delta_t > 0$, to buy $\Delta_t/(1+\rho_b^i)$ more of asset i at t and sells this incremental holding at $t+1$, increasing c_{t+1}^{j*} by

$$\frac{\Delta_t}{(1+\rho_b^i)}R_{t+1}^i(1-\rho_s^i).$$

Consider

$$L(\Delta_t) = u(c_t^{j*}-\Delta_t) + \mathrm{E}_t\left[\delta u\left(c_{t+1}^{j*} + \frac{\Delta_t}{1+\rho_b^i}R_{t+1}^i(1-\rho_s^i)\right)\right]$$

and

$$\frac{dL(\Delta_t)}{d\Delta_t} = -u'(c_t^{j*}-\Delta_t) + \mathrm{E}_t\left[\delta u'\left(c_{t+1}^{j*} + \frac{\Delta_t}{1+\rho_b^i}R_{t+1}^i(1-\rho_s^i)\right)R_{t+1}^i\frac{1-\rho_s^i}{1+\rho_b^i}\right]$$

Since $\{c_\tau^{j*}\}_{\tau=t}^{T^j-1}$ is the solution to the jth investor's problem, we know that

$$\frac{dL(0)}{d\Delta_t} = -u'(c_t^{j*}) + \mathrm{E}_t\left[\delta u'\left(c_{t+1}^{j*}\right)R_{t+1}^i\frac{1-\rho_s^i}{1+\rho_b^i}\right] \leq 0$$

$\Leftrightarrow$

$$\mathrm{E}_t\left[\frac{\delta u'(c_{t+1}^{j*})}{u'(c_t^{j*})}R_{t+1}^i\right] \leq \frac{1+\rho_b^i}{1-\rho_s^i}. \tag{105}$$

Suppose the jth investor increases c_t^{j*} by $\Delta_t \in \mathcal{F}_t$, where $\Delta_t > 0$, by selling $\Delta_t/(1-\rho_s^i)$ of asset i and then buys it back at $t+1$, reducing c_{t+1}^{j*} by

$$\frac{\Delta_t}{(1-\rho_s^i)}R_{t+1}^i(1+\rho_b^i).$$

Consider

$$L(\Delta_t) = u(c_t^{j*}+\Delta_t) + \mathrm{E}_t\left[\delta u\left(c_{t+1}^{j*} - \frac{\Delta_t}{1-\rho_s^i}R_{t+1}^i(1+\rho_b^i)\right)\right]$$

and

$$\frac{dL(\Delta_t)}{d\Delta_t} = u'(c_t^{j*}+\Delta_t) - \mathrm{E}_t\left[\delta u'\left(c_{t+1}^{j*} - \frac{\Delta_t}{1-\rho_s^i}R_{t+1}^i(1+\rho_b^i)\right)R_{t+1}^i\frac{1+\rho_b^i}{1-\rho_s^i}\right].$$

Since $\{c_\tau^{j*}\}_{\tau=t}^{T^j-1}$ is the solution to the jth investor's problem, we know that

$$\frac{dL(0)}{d\Delta_t} = u'(c_t^{j*}) - \mathrm{E}_t\left[\delta u'(c_{t+1}^{j*})R_{t+1}^i \frac{1+\rho_b^i}{1-\rho_s^i}\right] \le 0$$

$\Leftrightarrow$

$$\mathrm{E}_t\left[\frac{\delta u'(c_{t+1}^{j*})}{u'(c_t^{j*})}R_{t+1}^i\right] \ge \frac{1-\rho_s^i}{1+\rho_b^i}. \tag{106}$$

Combining (105) and (106) gives

$$\frac{1-\rho_s^i}{1+\rho_b^i} \le \mathrm{E}_t\left[\frac{\delta u'(c_{t+1}^{j*})}{u'(c_t^{j*})}R_{t+1}^i\right] \le \frac{1+\rho_b^i}{1-\rho_s^i}. \tag{107}$$

Note that (107) holds for all j with $T^j > t$ in equilibrium, and can be used to put upper and lower bounds on the conditional expected return for asset i given the marginal rate of substitution of any agent j.

Discussion. First, note that frictions cause Euler equation equalities to become inequalities. The implication is that the marginal rates of substitution of agents need not be sdfs in economies with frictions. Second, the perturbation argument used to obtain the Euler equation inequalities is applied to a holding period equal to 1 period, but the same arguments and resulting inequalities can be obtained for any holding period.

13.6. An equilibrium pricing model with illiquidity

Model setup. The setup is an overlapping generations economy in which a new generation of N agents indexed by n is born at each time $t \in \{\ldots, -2, 1, 0, 1, 2, \ldots\}$. Agent n born at t receives an endowment at t of e_t^n, trades at t and $t+1$ and derives CARA utility from W_{t+1}^n, the wealth of agent n available for consumption at time $t+1$: i.e, the agent cares about $-\mathrm{E}_t[\exp\{-A^n W_{t+1}^n\}]$ where A^n is agent n's constant abolute risk aversion coefficient.

There are I securities indexed by i, with S^i shares of security i. Security i pays a dividend of D_t^i at t, has an ex-dividend price at t of P_t^i and has an illiquidity cost at t of C_t^i, where C_t^i is the per-share cost of selling security i at time t; i.e. agents can buy at P_t^i, but must sell at $P_t^i - C_t^i$. Agents can borrow and lend at a riskless rate $r^f > 1$.

Let

$$\boldsymbol{D}_t = \begin{bmatrix} D_t^1 \\ \vdots \\ D_t^I \end{bmatrix}, \quad \boldsymbol{C}_t = \begin{bmatrix} C_t^1 \\ \vdots \\ C_t^I \end{bmatrix}, \quad \boldsymbol{P}_t = \begin{bmatrix} P_t^i \\ \vdots \\ P_t^I \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} S^i \\ \vdots \\ S^i \end{bmatrix}.$$

Assume $\boldsymbol{D}_t$ and $\boldsymbol{C}_t$ are AR(1) processes,

$$\begin{aligned} \boldsymbol{D}_t &= \bar{\boldsymbol{D}} + \rho^D(\boldsymbol{D}_{t-1} - \bar{\boldsymbol{D}}) + \boldsymbol{\varepsilon}_t, \\ \boldsymbol{C}_t &= \bar{\boldsymbol{C}} + \rho^C(\boldsymbol{C}_{t-1} - \bar{\boldsymbol{C}}) + \boldsymbol{\eta}_t, \end{aligned} \tag{108}$$

where $\bar{\boldsymbol{D}}$ and $\bar{\boldsymbol{C}}$ are positive $I \times 1$ vectors, $\rho^D, \rho^C \in [0, 1]$, and $\boldsymbol{\varepsilon}_t, \boldsymbol{\eta}_t \sim$ Normal with $\mathrm{E}[\boldsymbol{\varepsilon}_t] = \mathrm{E}[\boldsymbol{\eta}_t] = 0$, $\mathrm{var}[\boldsymbol{\varepsilon}_t] = \Sigma^D$, $\mathrm{var}[\boldsymbol{\eta}_t] = \Sigma^C$ and $\mathrm{E}[\boldsymbol{\varepsilon}_t \boldsymbol{\eta}_t'] = \Sigma^{DC}$.

Discussion.

- It is restrictive that agents must sell after one period: this magnifies the effect of the illiquidity cost on expected returns. Agents who live for many periods would hold assets for more than one period to avoid paying the illiquidity cost every period.
- The illiquidity cost can go negative which is undesirable.
- Perhaps we can generalize scalar ρ^D to matrix $\underset{I\times I}{\boldsymbol{\rho}^D}$ and scalar ρ^C to matrix $\underset{I\times I}{\boldsymbol{\rho}^C}$.

Competitive equilibrium. Conjecture that

$$\boldsymbol{P}_t = \boldsymbol{Y} + b_D \boldsymbol{D}_t - b_C \boldsymbol{C}_t \tag{109}$$

for scalars b_D and b_C, which means $\boldsymbol{P}_{t+1}$ is normally distributed conditional on $\boldsymbol{D}_t$ and $\boldsymbol{C}_t$. It follows that the problem of agent n born at time t can be written as

$$\max_{\underset{N\times 1}{\boldsymbol{y}_t^n}} \left\{ \mathrm{E}_t[W_{t+1}^n] - \tfrac{1}{2} A^n \sigma^2 [W_{t+1}^n] \right\}$$

subject to

$$W_{t+1}^n = (\boldsymbol{P}_{t+1} + \boldsymbol{D}_{t+1} - \boldsymbol{C}_{t+1})' \boldsymbol{y}_t^n + r^f (e_t^n - \boldsymbol{P}_t' \boldsymbol{y}_t^n).$$

The first-order condition is

$$\boldsymbol{y}_t^n = \frac{1}{A^n} (\sigma_t^2 [\boldsymbol{P}_{t+1} + \boldsymbol{D}_{t+1} - \boldsymbol{C}_{t+1}])^{-1} (\mathrm{E}_t[\boldsymbol{P}_{t+1} + \boldsymbol{D}_{t+1} - \boldsymbol{C}_{t+1}] - r^f \boldsymbol{P}_t).$$

In equilibrium,

$$\sum_{n=1}^{N} \boldsymbol{y}_t^n = \boldsymbol{S},$$

which implies

$$\boldsymbol{P}_t = \frac{1}{r^f} \left[\mathrm{E}_t[\boldsymbol{P}_{t+1} + \boldsymbol{D}_{t+1} - \boldsymbol{C}_{t+1}] - A \sigma_t^2 [\boldsymbol{P}_{t+1} + \boldsymbol{D}_{t+1} - \boldsymbol{C}_{t+1}] \boldsymbol{S} \right], \tag{110}$$

where

$$A = \left(\sum_{n=1}^{n} \frac{1}{A^n} \right)^{-1}.$$

Substituting (108) and (109) into (110), we obtain

$$\begin{aligned} \boldsymbol{Y} + b_D \boldsymbol{D}_t - b_C \boldsymbol{C}_t = \frac{1}{r^f} \big[& \boldsymbol{Y} + b_D (\bar{\boldsymbol{D}} + \rho^D (\boldsymbol{D}_t - \bar{\boldsymbol{D}})) + \bar{\boldsymbol{D}} + \rho^D (\boldsymbol{D}_t - \bar{\boldsymbol{D}}) \\ & - b_C (\bar{\boldsymbol{C}} + \rho^C (\boldsymbol{C}_t - \bar{\boldsymbol{C}})) - \bar{\boldsymbol{C}} - \rho^C (\boldsymbol{C}_t - \bar{\boldsymbol{C}}) - A \sigma_t^2 [b_D \boldsymbol{\varepsilon}_t - b_C \boldsymbol{\eta}_t + \boldsymbol{\varepsilon}_t - \boldsymbol{\eta}_t] \boldsymbol{S} \big]. \end{aligned}$$

Collecting terms, we find

$$b_D = \frac{1}{r^f}[b_D \rho^D + \rho^D] \Rightarrow b_D = \frac{\rho^D}{r^f - \rho^D},$$
$$b_C = \frac{1}{r^f}[b_C \rho^C + \rho^C] \Rightarrow b_C = \frac{\rho^C}{r^f - \rho^C},$$
$$\boldsymbol{Y} = \frac{1}{r^f - 1}\left(\frac{r^f(1-\rho^D)}{r^f - \rho^D}\bar{\boldsymbol{D}} - \frac{r^f(1-\rho^C)}{r^f - \rho^C}\bar{\boldsymbol{C}} - A\sigma^2\left[\frac{r^f}{r^f - \rho^D}\boldsymbol{\varepsilon}_t + \frac{r^f}{r^f - \rho^C}\boldsymbol{\eta}_t\right]\boldsymbol{S}\right),$$

which confirms the conjecture. Any investor n born at t holds a fraction $A/A^n > 0$ of the market portfolio $\boldsymbol{S}$, so all investors hold the market portfolio in equilibrium, which does not require short selling.

Expected returns. Let

$$r^i_{t+1} = \frac{D^i_{t+1} + P^i_{t+1}}{P^i_t}$$

be security i's gross return from t to $t+1$,

$$c^i_{t+1} = \frac{C^i_{t+1}}{P^i_t}$$

be the relative illiquidity cost of holding security i from t to $t+1$,

$$r^M_{t+1} = \frac{\sum_{i=1}^I S^i(D^i_{t+1} + P^i_{t+1})}{\sum_{i=1}^I S^i P^i_t}$$

be the market portfolio's gross return from t to $t+1$, and

$$c^M_{t+1} = \frac{\sum_{i=1}^I S^i C^I_{t+1}}{\sum_{i=1}^I S^i P^i_t}$$

be the relative illiquidity of holding the market portfolio from t to $t+1$.

All agents are holding mean-variance efficient portfolios net of illiquidity costs, so it follows that the conditional expected net return of security i is given by

$$\mathrm{E}_t[r^i_{t+1} - c^i_{t+1}] = r^f + \lambda_t \frac{\mathrm{cov}_t[r^i_{t+1} - c^i_{t+1}, r^M_{t+1} - c_t]}{\mathrm{var}_t[r^M_{t+1} - c^M_{t+1}]},$$

where $\lambda_t = \mathrm{E}_t[r^M_{t+1} - c^M_{t+1} - r^f]$ is the risk premium, which is > 0, since the market is mean-variance efficient. The conditional expected gross return of security i is given by

$$\begin{aligned}\mathrm{E}_t[r^i_{t+1}] = r^f + \mathrm{E}_t[c^i_{t+1}] &+ \hat{\lambda}_t \mathrm{cov}_t[r^i_{t+1}, r^M_{t+1}] + \hat{\lambda}_t \mathrm{cov}_t[c^i_{t+1}, c^M_{t+1}] \\ &- \hat{\lambda}_t \mathrm{cov}_t[r^i_{t+1}, c^M_{t+1}] - \hat{\lambda}_t \mathrm{cov}_t[c^i_{t+1}, r^M_{t+1}],\end{aligned}$$

where

$$\hat{\lambda}_t = \frac{\lambda_t}{\mathrm{var}_t[r^M_{t+1} - c^M_{t+1}]} > 0.$$

There are three sources of illiquidity risk:

- $\text{cov}_t[c^i_{t+1}, c^M_{t+1}]$ which investors dislike, so the risk premium is > 0.
- $\text{cov}_t[r^i_{t+1}, c^M_{t+1}]$ which investors like, so the risk premium is < 0.
- $\text{cov}_t[c^i_{t+1}, r^M_{t+1}]$ which investors like, so the risk premium is < 0.