Lecture Notes 8

Index Models

Buzz Words: ***Return Generating Model,***
Zero Correlation Component of Securities Returns,
Statistical Decomposition of Systematic and Nonsystematic Risks,
Regression, Security Characteristic Line,
Historical (Raw and Adjusted) Beta.

I. Readings and Suggested Practice Problems

BKM, Chapter 10, Section 1 (Skim Section 4)
Suggested Problems, Chapter 10: 5-13

II. A Single Index Model

An Index Model is a *Statistical* model of security returns (as opposed to an economic, equilibrium-based model).

A Single Index Model (SIM) specifies two sources of uncertainty for a security's return:

1. *Systematic* (macroeconomic) uncertainty (which is *assumed* to be well represented by a single index of stock returns)

2. *Unique* (microeconomic) uncertainty (which is represented by a security-specific random component)

A. Model's Components

1. The Basic Idea

***A Casual Observation*:** Stocks tend to move together, driven by the same economic forces.

Based on this observation, as an ad-hoc approach to represent securities' returns, we can model the way returns are generated by a simple equation.

2. *Formalizing the Basic Idea: The Return Generating Model*

Can always write the return of asset *j* as related linearly to a single common underlying factor (typically chosen to be a stock index):

$$r_j = \alpha_j + \beta_j r_I + e_j$$

where

r_I is the *random* return on the index (the common factor),
e_j is the *random* firm-specific component of the return,
where $E[e_j] = 0$, $Cov[e_j, r_I] = 0$,
α_j is the expected return if $E[r_I]=0$ (when the index is neutral),
β_j is the sensitivity of r_j to r_I , $\beta_j=Cov[r_j, r_I]/Var[r_I]$.

Also, *assume* (and this is the only, but crucial assumption) that:

$$Cov[e_j, e_i] = 0.$$

So, $\alpha_j + e_j$ is the return part independent of the index return,
$\beta_j r_I$ is the return part due to index fluctuations.

B. Expressing the First and Second Moments using the Model's Components

1. Mean return of security *j*: $E[r_j] = \alpha_j + \beta_j E[r_I]$

2. Variance of security *j*: $\sigma_j^2 = \beta_j^2 \sigma_I^2 + \sigma^2[e_j]$

3. Covariance between return of security *j* and return of security *i*

$$\sigma_{ji} = \beta_j \beta_i \sigma_I^2$$

C. Systematic & Unique Risk of an Asset according to the SIM

1. ***Expected Return,*** $E[r_j] = \alpha_j + \beta_j E[r_I]$, ***has 2 parts*:**

a. Unique (asset specific): α_j
b. Systematic (index driven): $\beta_j E[r_I]$

2. ***Variance,*** $\sigma_j^2 = \beta_j^2 \sigma_I^2 + \sigma^2[e_j]$, ***has similarly 2 parts*:**

a. Unique risk (asset specific): $\sigma^2[e_j]$
b. Systematic risk (index driven): $\beta_j^2 \sigma_I^2$

3. ***Covariance between securities' returns is due to only the systematic source of risk*:**

$$
\begin{aligned}
\text{Cov}[r_j, r_i] &= \text{Cov}[\alpha_j + \beta_j r_I + e_j, \alpha_i + \beta_i r_I + e_i] \\
&= \text{Cov}[\beta_j r_I , \beta_i r_I] = \beta_j \beta_i \text{Cov}[r_I , r_I] \\
&= \beta_j \beta_i \sigma_I^2
\end{aligned}
$$

Covariance ($\beta_j \beta_i \sigma_I^2$) depends (by assumption) only on single-index risk and sensitivities of returns to that single index.

D. *Typically, the chosen index is a "Market Index"*

You need to choose an index so that e_j and e_i are *indeed uncorrelated* for any two assets.

It "makes sense" to choose the entire stock market (a value-weighted portfolio) as a proxy to capture all macroeconomic fluctuations.

In practice, take a portfolio, i.e., index, which proxies for the market. A popular choice is for the S&P500 index to be the index in the SIM.

Then, the model states that

$$r_j = \alpha_j + \beta_j r_M + e_j$$

where r_M is the *random* return on the market proxy.
This SIM is often referred to as the "Market Model."

Example

You choose the S&P500 as your market proxy. You analyze the stock of General Electric (GE), and find (see later in the notes) that, using weekly returns, α_j = -0.07%, β_j = 1.44.

If you expect the S&P500 to increase by 5% next week, then according to the market model, you expect the return on GE next week to be:

$$\text{E}[r_{GE}] = \alpha_{GE} + \beta_{GE}\,\text{E}[r_M] = -0.07\% + 1.44 \times 5\% = 7.13\%.$$

III. Why the Single Index Model is Useful?

A. The SIM Provides the Most Simple Tool to Quantify the Forces Driving Assets' Returns

This is what we discussed above.

However, note that the SIM does not fully characterize the determinants of *expected* returns -- we don't know how α_j varies across assets.

Also, remember that the SIM assumes that the correlation structure across assets depends on a single factor (but more factors may be needed in practice).

B. The SIM Helps us to Derive the Optimal Portfolio for Asset Allocation (the Tangent Portfolio T) by Reducing the Necessary Inputs to the Markowitz Portfolio Selection Procedure

We identified the portfolio P, used for the asset allocation, with the tangency portfolio T.

To compute the weights of T , we need to describe all the risky assets in the portfolio selection model. This requires a large number of parameters.

Usually these parameters are unknown, and have to be estimated.

1. **With n risky assets, we need $2n + (n^2 - n)/2$ parameters:**

n	expected returns $E[r_j]$
n	return standard deviations σ_j
$n(n-1)/2$	correlations (or covariances)

Example

$n = 2$	number of parameters = 2 + 2 + 1 = 5
$n = 8$	number of parameters = 8 + 8 + 28 = 44
$n = 100$	number of parameters = 100 + 100 + 4950 = 5150
$n = 1000$	number of parameters = 1000+ 1000 + 499500= 501500

With large n:

Large estimation error,
Large data requirements (for monthly estimates, with n=1000, need at least 1000 months, i.e., more than 83 years of data)

2. **Assuming the SIM is correctly specified, we only need the following parameters:**

n	α_j parameters
n	β_j parameters
n	$\sigma^2[e_j]$ parameters
1	$E[r_I]$
1	$\sigma^2\ [r_I]$

These $3n$+2 parameters generate all the $E[r_j]$, σ_j, and σ_{ji}.
We get the parameters by estimating the index model for each of the n securities.

Example
With 100 stocks need 302 parameters. With 1000 need 3002.

IV. A Detailed Example

A. SIM for GE

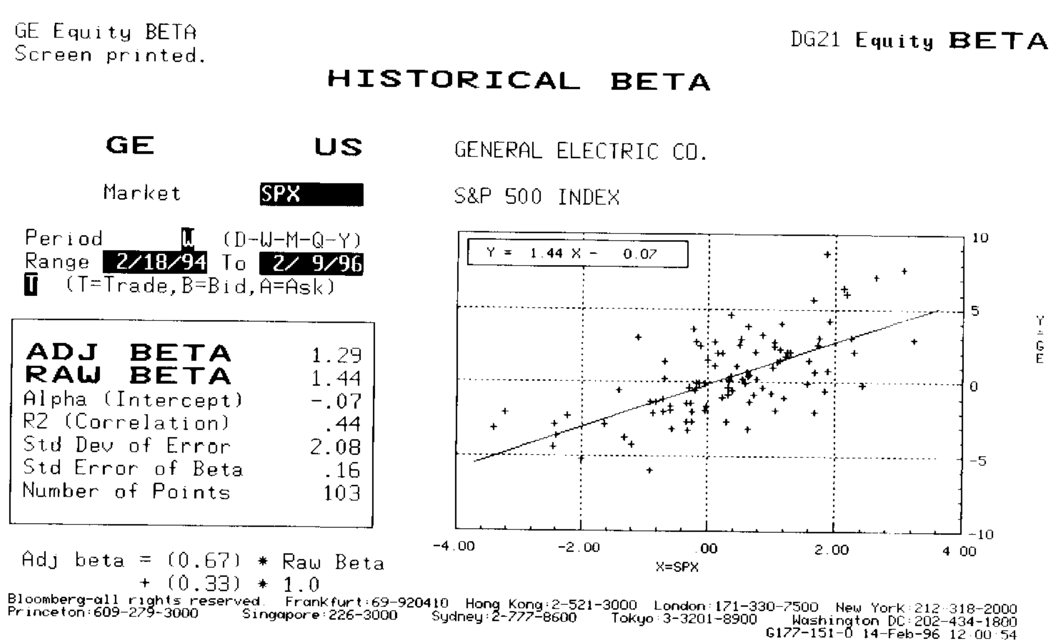
How is the return on an individual stock (GE) driven by the return on an overall market index, *M*, measured by the S&P500 index?

To answer this:
- Collect historical data on r_{GE} and r_M
- Run a simple linear regression of r_{GE} against r_M:

$$r_{GE} = \alpha_{GE} + \beta_{GE}\, r_M + e_{GE}$$

where α_{GE} ("alpha") is the intercept,
β_{GE} ("beta") is the slope,
e_{GE} is the regression error.

The fitted regression line is called the ***Security Characteristic Line*** (SCL).



B. Using the SIM to Interpret the Risk of Investing in GE

The variance of GE is $\sigma_{GE}^2 = \beta_{GE}^2\, \sigma_M^2 + \sigma^2[e_{GE}]$
Where $\sigma[e_{GE}]$ is the "Std. Dev. Of Error."

Over the sample, $\sigma_M = 1.28\%$ (from other data):

$$\sigma_{GE}^2 = \beta_{GE}^2\, \sigma_M^2 + \sigma^2[e_{GE}]$$
$$= (1.44)^2\, (1.28)^2 + (2.08)^2 = 3.40 + 4.33 = 7.73$$

$$R^2 = \frac{\beta_{GE}^2 \sigma_M^2}{\sigma_{GE}^2} = \frac{3.40}{7.73} = .44$$

R^2 is called ***the coefficient of determination***, and it gives the fraction of the variance of the dependent variable (the return on GE) that is explained by movements in the independent variable (the return on the Market portfolio).

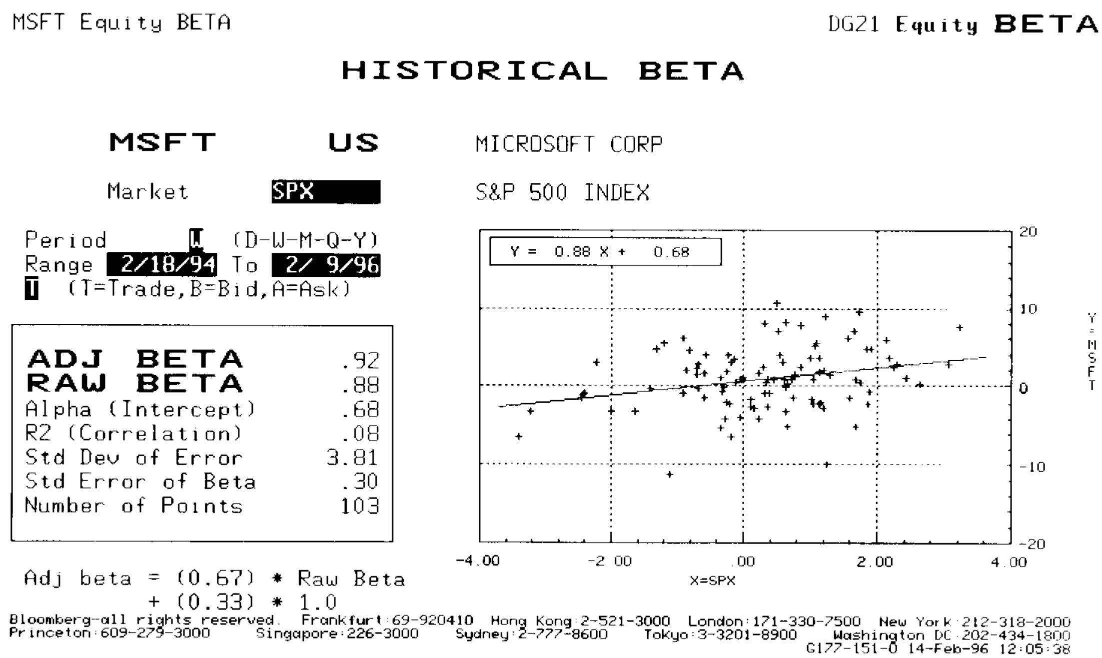
Note that for portfolios, the coefficient of determination from a regression estimation can be used as a measure of diversification (0 min, 1 max).

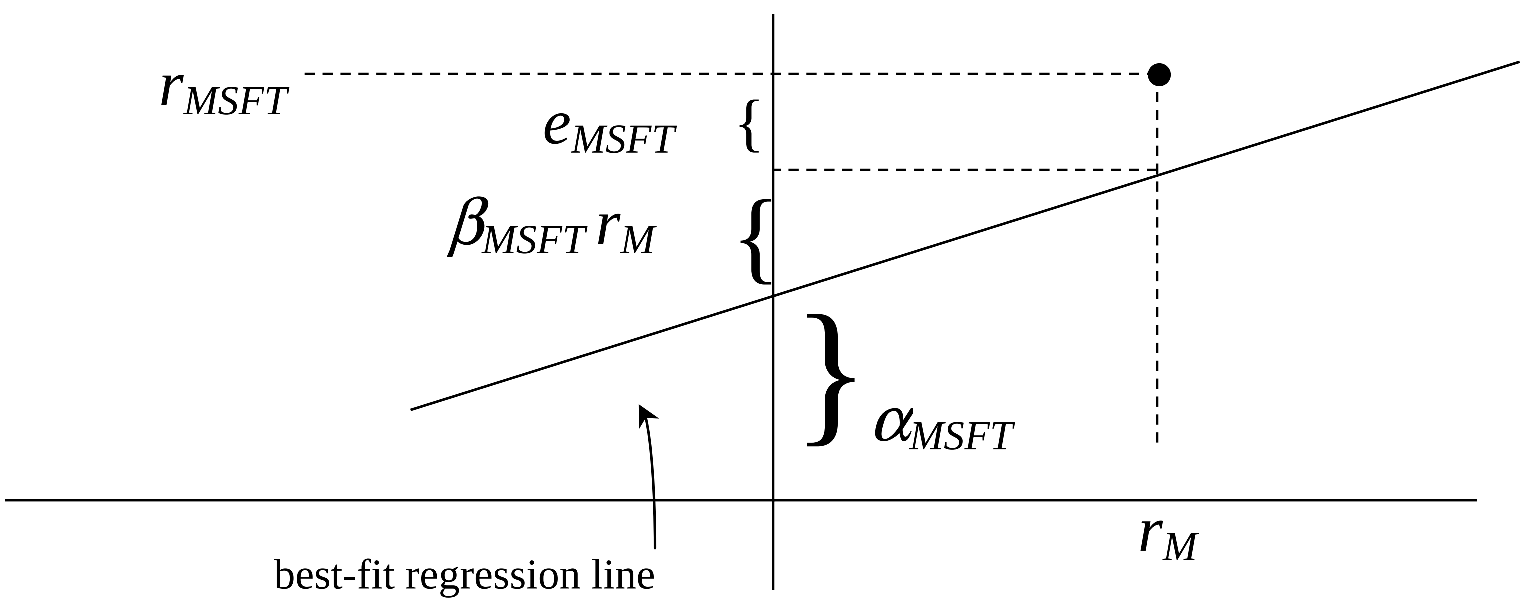
$$r_{GE} = \underbrace{\alpha_{GE}}_{\text{constant}} + \underbrace{\beta_{GE}\, r_M}_{\substack{\text{market-driven,}\\ \text{systematic}}} + \underbrace{e_{GE}}_{\substack{\text{non-market,}\\ \text{firm-specific}}}$$

$$\underbrace{\sigma_{GE}^2}_{\text{Total "risk"}} = \underbrace{\beta_{GE}^2\, \sigma_M^2}_{\substack{\text{Market or}\\ \text{systematic risk}}} + \underbrace{\sigma^2(e_{GE})}_{\substack{\text{Non-market,}\\ \text{firm-specific risk}}}$$

$$\underbrace{7.73}_{\text{Total "risk"}} = \underbrace{3.40}_{\substack{\text{Market or}\\ \text{systematic risk}}} + \underbrace{4.33}_{\substack{\text{Non-market,}\\ \text{firm-specific risk}}}$$

C. SIM for Microsoft





$$\text{Over the sample,} \sigma_M = 1.28\% \text{ (from other data)}$$

$$\sigma^2_{MSFT} = \beta^2_{MSFT}\sigma^2_M + \sigma^2(e_{MSFT})$$
$$= (0.88)^2(1.28)^2 + (3.81)^2$$
$$= 1.27 + 14.52 = 15.79$$

$$R^2 = \frac{\beta^2_{MSFT}\sigma^2_M}{\sigma^2_{MSFT}} = \frac{1.27}{15.79} = 0.08$$

Unlike GE, firm-specific risk dominates for MSFT (which is indeed more single-industry focused than GE).

D. The SIM and the Covariance between r_{GE} and r_{MSFT}

$$r_{GE} = \alpha_{GE} + \beta_{GE} r_M + e_{GE}$$

constant

firm-specific

$$r_{MSFT} = \alpha_{MSFT} + \beta_{MSFT} r_M + e_{MSFT}$$

The *only common influence* driving GE and MSFT is the market return r_M, so can easily calculate the covariance and correlation:

$$Cov(r_{GE}, r_{MSFT}) = \beta_{GE}\beta_{MSFT}\sigma^2_M = (.88)(1.44)(1.28)^2 = 2.08$$
$$Corr(r_{GE}, r_{MSFT}) = 2.08/\sqrt{7.33 \times 15.79} = 0.19$$

E. More Realistic Models

A more realistic model than the SIM, should allow the systematic risk to be driven by *several* factors.

We assume that more than one common factor affect securities' returns, and try to identify the "macro-non-market" forces that cause all stocks (or large groups of stocks) to move together.

For example, a ***two-factor*** model may account for interest-rate effects in addition to market effects:

$$r_j = \alpha_j + \beta_{jM}\, r_M + \beta_{jIR}\, r_{IR} + e_j$$

where r_M is a return on a market index, and
r_{IR} is a return on a portfolio of bonds.

Clearly, such models will better capture the correlation between GE and MSFT (but in our course we focus only on the basic single index model, and you should only remember that multi-index models exist as well).

V. Two Approaches for Specifying Index Models

We often use the index model given in these notes:

$$(*) \quad r_{GM} = \alpha_{GM} + \beta_{GM} r_M + e_{GM}$$

A common variation is the "excess return" form of the model:

$$(**) \quad r_{GM} - r_f = \alpha_{GM} + \beta_{GM}\left(r_M - r_f\right) + e_{GM}$$

or,

$$R_{GM} = \alpha_{GM} + \beta_{GM} R_M + e_{GM}$$

where $R_{GM} = r_{GM} - r_f$ and $R_M = r_M - r_f$

- BKM use the "excess return" form.
- Note: the α's are not the same in (*) and (**),
  and β's are not the same in (*) and (**) if r_f varies over time.

Unless stated otherwise, in all the problems in this course we will assume a constant, non-random riskfree rate, r_f , and then the β's above are the same in the raw return form as in (*) and the excess return form as in (**). The α's then differ just by a constant: $\alpha^{(*)}=\alpha^{(**)}+r_f(1-\beta)$