Lecture Notes 15

Options: Valuation and (No) Arbitrage

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Buzz Words: ***Continuously Compounded Returns, Adjusted Intrinsic Value, Hedge Ratio, Implied Volatility, Option's Greeks, Put Call Parity, Synthetic Portfolio Insurance, Implicit Options, Real Options***

I. Readings and Suggested Practice Problems

BKM, Chapter 21.1-21.5

Suggested Problems, Chapter 21: 2, 5, 12-15, 22

II. Introduction: Objectives and Notation

- In the previous lecture we have been mainly concerned with understanding the payoffs of put and call options (and portfolios thereof) at maturity (i.e., expiration).

  Our objectives now are to understand:

  1. The value of a call or put option prior to maturity.

  2. The applications of option theory for valuation of financial assets that embed option-like payoffs, and for providing incentives at the work place.

- The results in this handout refer to ***non-dividend paying stocks*** (underlying assets) unless otherwise stated.

- Notation

S, or S_0	the value of the stock at time 0.
C, or C_0	the value of a call option with exercise price X and expiration date T
P or P_0	the value of a put option with exercise price X and expiration date T
H	Hedge ratio: the number of shares to buy for each option sold in order to create a safe position (i.e., in order to hedge the option).
r_f	EAR of a safe asset (a money market instrument) with maturity T.
r	annualized continuously compounded risk free rate of a safe asset with maturity T. $r = \ln[1+r_f]$
σ	standard deviation of the annualized continuously compounded rate of return of the stock. Continuously compounded rate of return is calculated by $\ln[S_{t+1}/S_t]$, and it is the continuously compounded analog to the simple return $(S_{t+1}-S_t)/S_t$.

III. No Arbitrage Pricing Bound

The general approach to option pricing is first to assume that prices do not provide arbitrage opportunities.

Then, the derivation of the option prices (or pricing bounds) is obtained by replicating the payoffs provided by the option using the underlying asset (stock) and risk-free borrowing/lending.

Illustration with a Call Option

Consider a call option on a stock with exercise price X. (Assume that the stock pays no dividends.)

At time 0 (today):

Intrinsic Value = Max[S-X, 0],

The intrinsic value sets a *lower bound* for the call value:

C > Max[S-X, 0]

In fact, considering the payoff at time T, Max[S_T-X, 0] we can make a stronger statement:

C > Max[S-$PV(X)$, 0] ≥ Max[S-X, 0]

where $PV(X)$ is the present value of X (computed using a borrowing rate).

If the above price restriction is violated we can arbitrage.

Example

Suppose S = 100, X = 80, r_f = 10% and T = 1 year.

Then $S\text{-}PV(X)$ = 100 - 80/1.10 = 27.27.

Suppose that the market price of the call is C = 25.
(Note that C > intrinsic value = 20, but C < 27.27, which is the *adjusted intrinsic value*).

Today, we can . . .

	CF
Buy the call	-25.00
Sell short the stock	+100.00
Invest PV(X) at r_f	-72.72
Total	+2.27

At maturity, our cash flows depend on whether S_T exceeds X:

Position	CF	
	$S_T < 80$	$S_T \geq 80$
long call	0	S_T–80
short stock	-S_T	-S_T
investment	80	80
Total	80-S_T>0	0

We have an initial cash inflow (of 2.27) and a guaranteed no-loss position at expiration.

Note that for this in-the-money call,
only when C > 27.27 = Max[$S\text{-}PV(X)$, 0],
the arbitrage opportunity is eliminated.

It is important to understand that when $S_T \geq 80$, the CF generated at T with long call is the same as with long stock and borrowing at $t = 0$ $PV(X)$ until T. When $S_T < 80$, the CF generated with long call is more than that of a long stock and borrowing $PV(X)$.

So to prevent arbitrage, must have: $C > S - PV(X)$.

Note that we only found a bound. That is useful to get a general idea about the option price range, but our next step is to actually find the option price

IV. The Binomial Pricing Model

A. The basic model

We restrict the final stock price S_T to two possible outcomes:

$$
\begin{tikzcd}[row sep=small]
 & S^{+} = 130 \\
S_0 = 100 \arrow[ur] \arrow[dr] & \\
 & S^{-} = 50
\end{tikzcd}
$$

Consider a call option with $X = 110$. What is it worth today?

$$
\begin{tikzcd}[row sep=small]
 & C^{+} = 20 \\
C_0 = ? \arrow[ur] \arrow[dr] & \\
 & C^{-} = 0
\end{tikzcd}
$$

Definitions

1. The *hedge portfolio* is short one call and long H shares of stock.

2. H, the *hedge ratio*, is chosen so that the portfolio is risk-free: it replicates a bond.

Example

S_0 = \$100, r_f = 10%, X = \$110, and T = 1 year.
What is the call price C_0?

First step, construct the hedge portfolio:

The initial and time-T values of the hedge portfolio are given by

$$\begin{tikzcd}[column sep=small] & HS^{+} - C^{+} = 130H - 20 \\ HS_0 - C_0 = ? \arrow[ur] \arrow[dr] & \\ & HS^{-} - C^{-} = 50H - 0 \end{tikzcd}$$

For this portfolio to be risk-free, means that it must have same final value in either up or down cases:

$$50H - 0 = 130H - 20$$

$$\Rightarrow \quad H = \frac{C^{+} - C^{-}}{S^{+} - S^{-}} = 20/80 = 0.25 \text{ shares.}$$

So, a portfolio that is long 0.25 shares of stock and short one call is risk-free:

$$\begin{tikzcd}[column sep=small] & 0.25S^{+} - C^{+} = 0.25\times130 - 20 = 12.5 \\ 0.25S_0 - C_0 = ? \arrow[ur] \arrow[dr] & \\ & 0.25S^{-} - C^{-} = 0.25\times50 - 0 = 12.5 \end{tikzcd}$$

It pays \$12.5 in either case.

$$\begin{tikzcd}[column sep=small] & 12.5 \\ 0.25S_0 - C_0 = ? \arrow[ur] \arrow[dr] & \\ & 12.5 \end{tikzcd}$$

Second step, note that the hedge portfolio replicates a bond:

Since r_f = 10% and T = 1 year, a bond that pays 12.5 will be worth today B_0 = 12.5 / 1.10 = 11.36.

This bond is *equivalent* to the portfolio $0.25S_0 - C_0$.

In other words, you can either pay $11.36 for the bond, or use the $11.36 plus the proceeds of writing a call to buy 0.25 share of stock, because the payoff of the latter hedge portfolio is the same as the bond's ($12.5 next year).

Therefore, the bond and the hedge portfolio must have the same market value:

$$0.25\, S_0 - C_0 = 0.25\times 100 - C_0 = 11.36$$

$$\Rightarrow C_0 = 25 - 11.36 = 13.64$$

Remark

Notice that in the above example the "bond" was constructed using knowledge of the current stock price S_0, the "up/down volatility" of the stock, the exercise price X, and the riskless rate r_f.

Since 1 bond, priced B_0, is replicated by the portfolio $0.25S_0 - C_0$, 1 call is replicated, or "synthesized" by $0.25S_0 - B_0$, i.e., HS_0=0.25×100 = $25 long in the stock combined with borrowing $11.36 at the riskless rate, and hence the call price is $13.64.

$$
\begin{tikzcd}
 & 0.25\times130 - 12.5 = 20 \\
C_0 = HS_0 - B_0 \arrow[ur] \arrow[dr] & \\
 & 0.25\times50 - 12.5 = 0
\end{tikzcd}
$$

You will be willing to spend exactly \$13.64 to get the payoff of a call, by directly buying the call

or

by borrowing \$11.36 and adding it to \$13.64 to buy \$25 worth of stock.

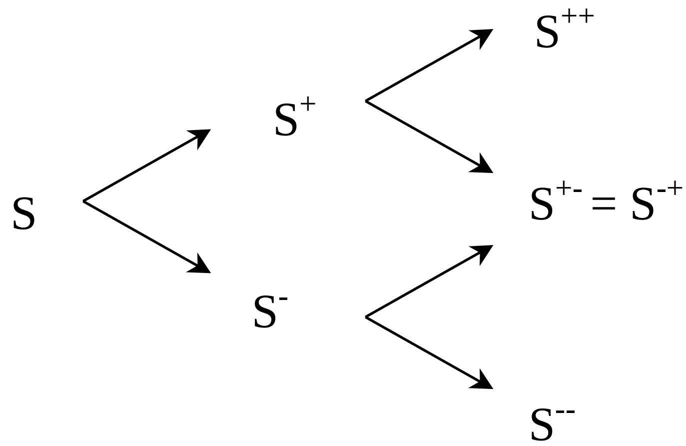
This is so because the latter position will give you a payoff of \$20 as the good outcome, and payoff of \$0 as the bad outcome, exactly like the payoffs of a call. Since you are indifferent between synthesizing the call or paying for it directly, the price of the synthesizing position must be the same as the call price (or else of course there would be an arbitrage opportunity).

A Call Option is equivalent to a leveraged, appropriately constructed stock portfolio.

1 call can be "synthesized" by borrowing B_0 and buying H shares of stock priced S_0, which costs HS_0-B_0, and thus must be the price of the call.

B. Extending the binomial model

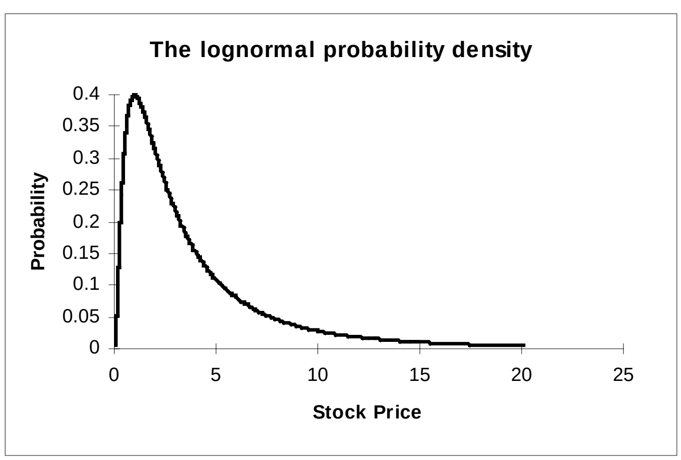
The binomial model can be made more realistic by adding more branch points (the up/down steps in the added branch points are as in the basic model):



At each branch point ("node"), there will be a different value for C (*and a different hedge ratio*)

C. The Limiting Case

With some additional assumptions, as the number of nodes gets large, the logarithm of the stock price at maturity is normally distributed: S_T is said to be lognormally distributed.



V. The Black-Scholes Model

A. The Black-Scholes (B-S) Call Value

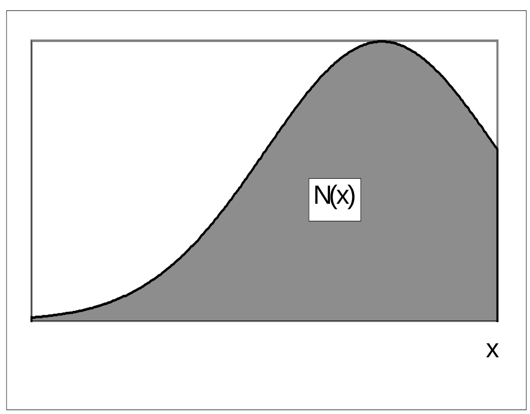
As the number of nodes (in the extended binomial model) goes to infinity, C approaches the Black-Scholes value:

$$C = S\,N(d_1) - Xe^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

$N(d)$ is the area under the standard normal density:



Assumptions:

- Yield curve is flat through time at the same interest rate. (So there is no interest rate uncertainty.)
- Underlying asset return is lognormally distributed with constant volatility and does not pay dividends.
- Continuous trading is possible.
- No transaction costs, taxes or other market imperfections.

Variables affecting the value of the call:

S: C is monotonically ***increasing*** in S as would be expected.

X: C is monotonically ***decreasing*** in X as would be expected.

σ: C is monotonically ***increasing*** in σ. Why?

(1) Option feature of the call truncates the payoff at 0 when the underlying's value is less than the strike price.
(2) When σ increases, the volatility of $S(T)$ increases.
(3) The call option holder benefits from the greater upside potential of $S(T)$ but does not bear the greater downside potential due to the truncation of the option payoff at 0.
(4) So, given S, the value of the call increases.

T: C is monotonically ***increasing*** in T. Why?

(1) The exercise price does not have to be paid until time T. When T increases, the current value of X paid at T decreases making the option more valuable for given S.
(2) Second, with a longer time to maturity the volatility of S_T increases for given σ. So the value of the call today increases for the same reason that an increase in σ increases the call's value today.
(3) Both effects are acting in the same direction.

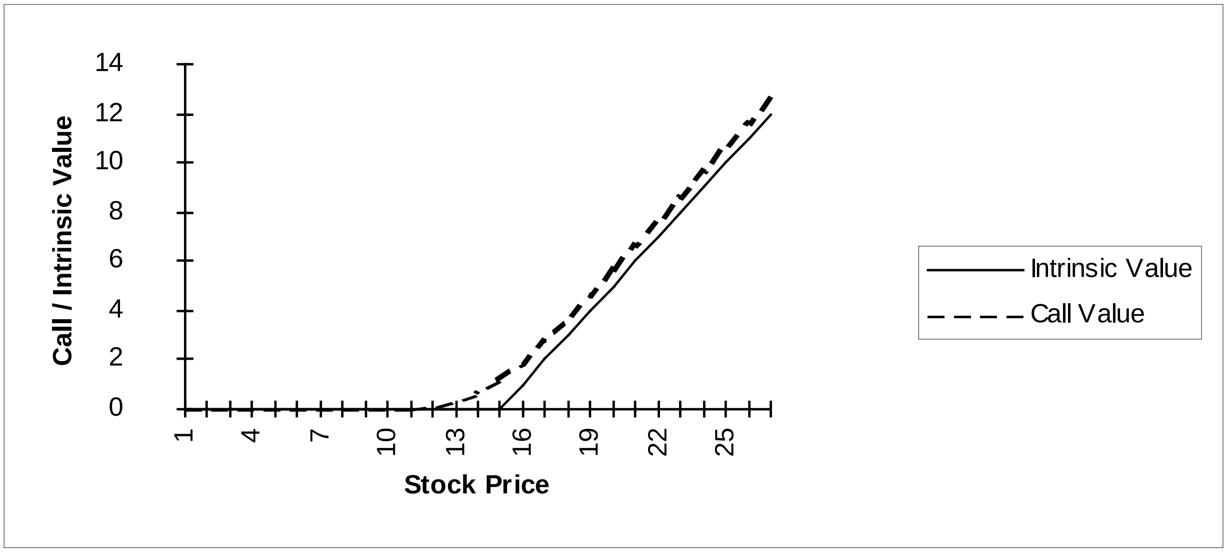
r: C is monotonically ***increasing*** in r. Why?

The exercise price does not have to be paid until time T. When r increases, the current value of X paid at T decreases making the option more valuable for given S.

Variables not affecting the value of the call:

The expected return on the underlying asset. Why?

B. The B-S Call price (C) vs. Stock price (S)



C. The B-S Hedge Ratio

In the Black-Scholes formula, the hedge ratio is H=$N(d_1)$. (This is also called the option's "delta".)

***Example*: *H* for Microsoft Call**

What is H for the MSFT 85 Oct 21 call we discussed in class?

S = 90.875, X = 85, T=201/365.
If r_f = 6%, σ = 48.52% (see below), then H = 0.675.
(The risk-free hedge portfolio is short one call and long 0.675 shares of stock.)

This hedge ratio changes as S changes or with the passage of time: the hedge is *dynamic*.

In the above example, can verify *H* by a direct calculation:

$$d_1 = \frac{\ln\left(\frac{S}{X}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{90.875}{85}\right)+\left(0.05827+\frac{0.48527^2}{2}\right)(201/365)}{0.48527\sqrt{201/365}} = 0.45475$$

$H = N(0.45475) = 0.\ 675.$

Or use an appropriate software, as illustrated below.

The *option calculator* used below is a Java applet available at the Chicago Board Option Exchange's Web site (http://www.cboe.com/TradTool/OptionCalculator.asp).

D. *What volatility to use in the B-S formula?*

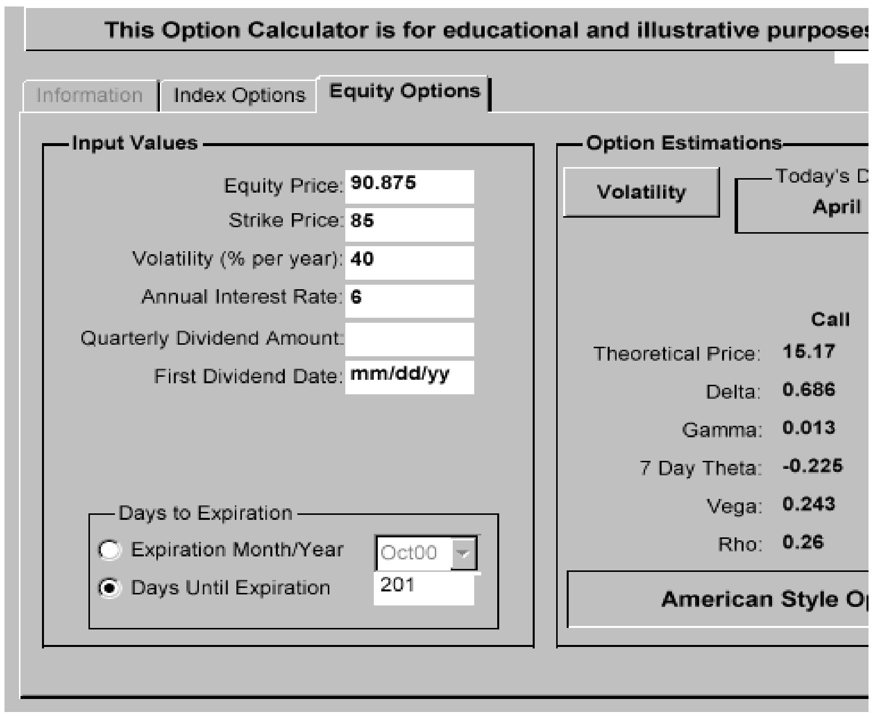
Future volatility means the annualized standard deviation of daily returns during some future period, typically between now and an option expiration. And it is future volatility that the option pricing formula needs as an input in order to calculate the theoretical value of an option. Unfortunately, future volatility is unknown. Consequently, the volatility numbers used in option pricing formulas are only estimates of future volatility, e.g., using historic volatility.

Historical volatility is a measure of actual price changes during a specific time period in the past. Historical volatility is the annualized standard deviation of daily returns during a specific period (such as 30 days).

Example: Microsoft

If we examine MSFT over the past year, we find that historical volatility is about 40%.

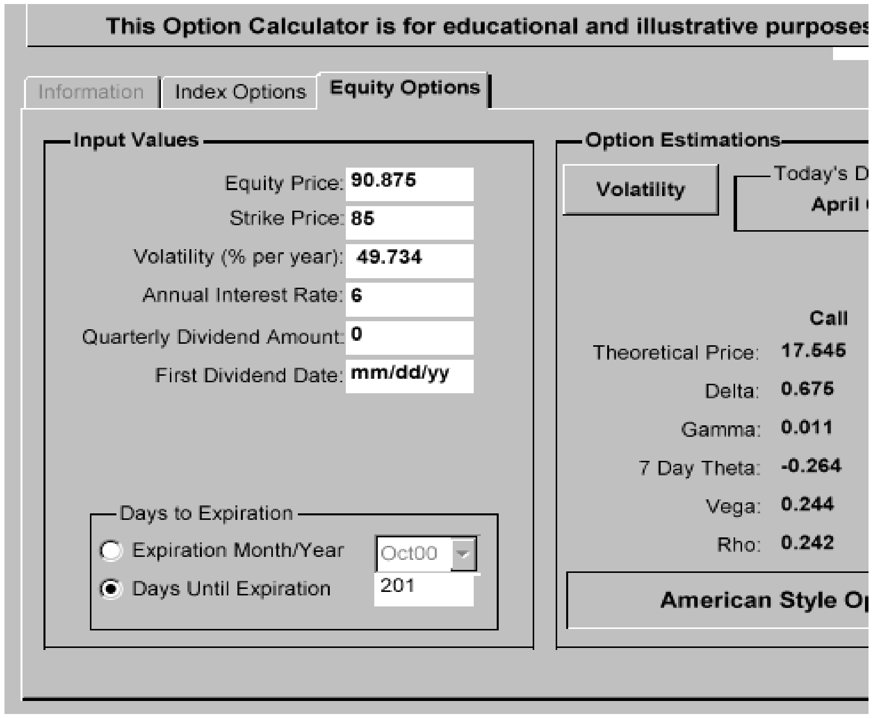
Using this volatility, the B-S price for the call on 4/3/2000 is:



However, It can be argued that the more recent historical volatility is the better estimate of the future volatility.

Then historical volatility based on, e.g., February 2000 is reported by the CBOE to be: 49.734%
(See http://www.cboe.com/mktdata/historicalvolatility.asp)

Using this higher volatility, the B-S price for the call on 4/3/2000 is:



Assuming on 4/3 that the relatively higher volatility will persist (which is not unreasonable, given Justice Jackson's ruling against the "abusive monopoly" of MSFT), that is assuming the recent historical volatility is a good estimate of future volatility, the B-S model is quite precise (17.54 vs. 17.25)

E. Implied volatility

- Black Scholes call value: $C(S, X, r, \sigma, T)$

  Among the inputs, there is most uncertainty about σ.

  To compute C, we usually take a historical estimate of σ.

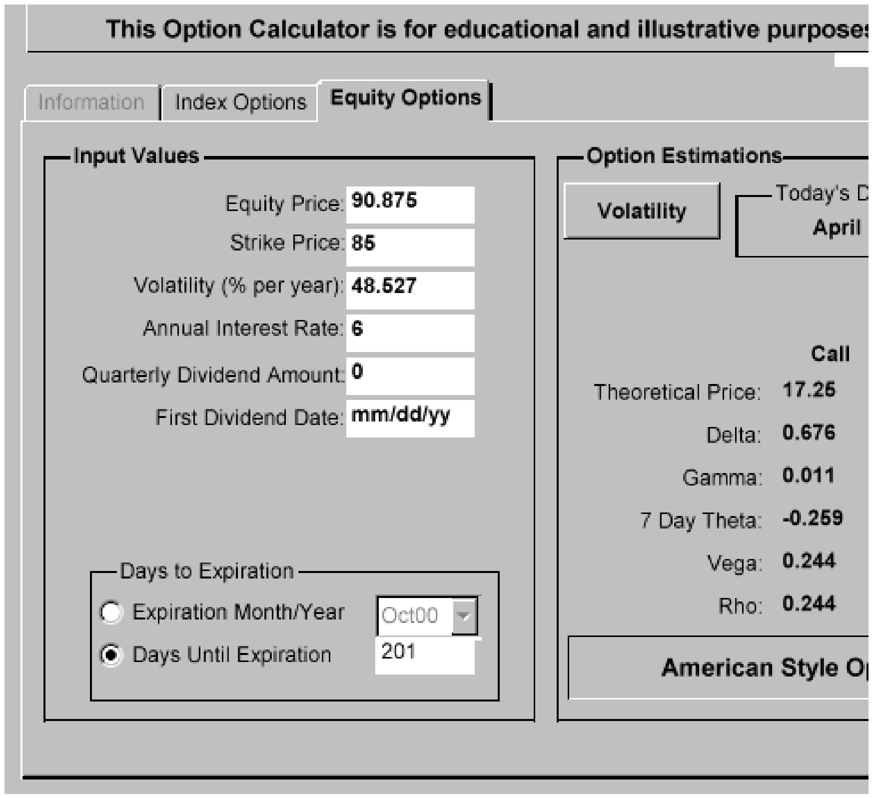
- Alternatively, take the observed call price, C^*, as *given*, and solve for the value of σ that makes the Black-Scholes equation correct.

  The *implied* volatility is the value σ^* that makes

  $$C^* = C(S, X, r, \sigma^*, T)$$

  (When we do this we have "used up" the data in the call price: we can't then use the implied volatility to see if the option is correctly valued.)

***Example*:** Microsoft's implied volatility



F. *Put-call parity for European options on a non-dividend paying stock*

When discussing portfolios of options, we have seen that a portfolio consisting of

Bond (par = X) + Call(X)

has the same payoff as a portfolio consisting of

Stock + Put(par = X)

Since these have the same payoffs, they must have the same market values: The following equality is called the *Put-Call Parity*

$$B + C = S + P$$

Therefore,

If C is the Black-Scholes price of the call, then the Black-Scholes price of a European put is:

$$P = B + C - S$$

VI. Dynamic Hedging

A. Dynamic hedging in the Black-Scholes model

In the Binomial Model:

$$HS - C = Bond \quad \Leftrightarrow \quad C = SH - Bond$$

The Black-Scholes Model:

$$C = \underbrace{S\,\underbrace{N(d_1)}_{H}}_{Stocks} - \underbrace{\underbrace{Xe^{-rT}}_{PV(X)} N(d_2)}_{Bonds}$$

By "analogy" $N(d_1) = H$, the hedge ratio.

But since

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$

$H = N(d_1)$ is not constant: we have to continuously ("dynamically") adjust the hedge.

B. Synthetic portfolio insurance using dynamic hedging

Let X be the "floor" we want on our portfolio.

Let B be a bond with par value = X.

Get a portfolio insurance payoff with Bond + Call:

$$B(\text{par } X) + C(\text{strike}=X) \quad = \quad B + [\, N(d_1)\, S - B\, N(d_2)\,]$$

$$= \quad N(d_1)\, S + (1 - N(d_2))\, B$$

In other words, we can get portfolio insurance by mixing stocks and bonds.

- This strategy is attractive for portfolio managers because
  - don't have to sell all your stock
  - don't have to actually buy the call option

BUT, you have to continuously adjust your portfolio mix.

Example

Call option with X = 90, σ = 20%, r = 5%, T=1 year.

S	C	Delta
100	16.80	0.81
95	12.92	0.73
90	9.48	0.64

As S↓, Delta ↓: we need to sell stock. As S ↑, we buy stock.

(This is a *destabilizing* strategy, but its implications are still debated!)

C. Practical Considerations

- Dynamic hedging is valid only if the price changes are "small". (Crash of 1987! LTCM's failure in 1998!)

- Most over-the-counter options are dynamically hedged.

  Example

  An investment bank writes a pension fund a Nikkei put option. The bank will dynamically hedge the put that it has written.

- For listed options, we know how many have been written (open interest).

  For over-the-counter options, we don't know how many have been written. Many investors may believe that they can effectively dynamically hedge when in fact they can't.

VII. Applications

A. Employee stock options

- Often issued to key technical or managerial (executive) personnel to align their objectives with that of the corporation (see articles in the additional readings section of the previous lecture)

- Accounting issues: FAS 123 (October, 1996).
  - On income statement, must expense intrinsic value of options (hence, to avoid expenses, options are typically granted at-the-money).
  - In footnotes, must disclose Black-Scholes value of options granted.

- There is an ongoing debate of whether and how to expense options.

- Implicit employee stock options
  - Bonus schemes
  - Compensation and motivation of traders.

B. Levered Equity: Equity of a levered firm as a call option

Example:

Assume that we have a single-project firm.

The debt in this firm consists of a single zero coupon bond with par value 120, due in 1 year.

- The equity in this firm is a call option!

Why?

If V is the value of the firm's assets, then the equity in the firm is a European call option on V with exercise price 120:

In one year, the stockholders get Max[V-120, 0]

If the $V<120$, the firm is bankrupt.

Implications

- The equity holders can make their option more valuable by increasing the risk of the assets.
- They may take on undesirable investments simply because they have the chance of a high payoff.

VIII. Appendix

The "Greeks" in the option calculator output.

In practice, option values change on account of a number of factors: Movements in the price of the underlying asset, Passage of time, Changes in volatility of the underlying asset, Changes in the rate of interest. Corresponding to these parameters, are five sensitivity measures (formally: partial derivatives):

Delta (Δ): The hedge ratio is also the sensitivity of option value to unit change in the underlying. Delta indicates a percentage change. For example, a delta of 0.40 indicates the option's theoretical value will change by 40% of the change in the price of the underlying. Be aware, however, that delta changes as the price of the underlying changes. Therefore, option values can change more or less than the amount indicated by the delta. Delta is used to measure an impact of a *small* change in the underlying.

Gamma (Γ): Sensitivity of Delta to unit change in the underlying. Gamma indicates an absolute change in delta. For example, a Gamma change of 0.150 indicates the delta will increase by 0.150 if the underlying price increased or decreases by 1.0. Due to options' convexity, Γ (combined with Δ) is used to measure an impact of a *large* change in the underlying.

Theta (Θ): Sensitivity of option value to change in time. Theta indicates an absolute change in the option value for a 'one unit' reduction in time to expiration. The Option Calculator assumes 'one unit' of time is 7 days. For example, a theta of –0.250 indicates the option's theoretical value will change by -0.250 if the days to expiration is reduced by 7.
NOTE: 7 day Theta changes to 1 day Theta if days to expiration is 7 or less.

Vega (V): Sensitivity of option value to change in volatility. Vega indicates an absolute change in option value for a one percent change in volatility. For example, a Vega of 0.090 indicates an absolute change in the option's theoretical value will increase (decrease) by 0.090 if the volatility percentage is increased (decreased) by 1.0.

Rho (ρ): Sensitivity of option value to change in interest rate. Rho indicates the absolute change in option value for a one percent change in the interest rate. For example, a Rho of 0.060 indicates the option's theoretical value will increase by 0.060 if the interest rate is decreased by 1.0.