

Value-at-Risk-Based Risk Management: Optimal Policies and Asset Prices

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This article analyzes optimal, dynamic portfolio and wealth/consumption policies of utility maximizing investors who must also manage market-risk exposure using Value-at-Risk (VaR). We find that VaR risk managers often optimally choose a larger exposure to risky assets than non-risk managers and consequently incur larger losses when losses occur. We suggest an alternative risk-management model, based on the expectation of a loss, to remedy the shortcomings of VaR. A general-equilibrium analysis reveals that the presence of VaR risk managers amplifies the stock-market volatility at times of down markets and attenuates the volatility at times of up markets.

In recent years, we have witnessed an unprecedented surge in the usage of risk management practices, with the Value-at-Risk (VaR)-based risk management emerging as the industry standard by choice or by regulation [Jorion (1997), Dowd (1998), Saunders (1999)]. VaR describes the loss that can occur over a given period, at a given confidence level, due to exposure to market risk. The wide usage of the VaR-based risk management (VaR-RM) by financial as well as nonfinancial firms [Bodnar et al. (1998)] stems from the fact that VaR is an easily interpretable summary measure of risk¹ and also has an appealing rationale, as it allows its users to focus attention on “normal market conditions” in their routine operations. However, evidence abounds that in practice VaR estimates not only serve as summary statistics for decision makers but are also used as a tool to manage and control risk—where economic agents struggle to maintain the VaR of their market

We would like to thank Jakša Cvitanić, Sanjiv Das, Francis Diebold, Bernard Dumas (the editor), Philip Dybvig, Philippe Jorion, Maureen O'Hara (the executive editor), Anthony Saunders, Suresh Sundaresan, Lucie Teplá, Tan Wang, the seminar participants at INSEAD, London Business School, MIT, Rutgers University, University of Chicago, Wharton School, the Spring 1999 NYU-Columbia Finance Workshop, the 1999 CIRANO Conference on Intertemporal Asset Pricing, the 1999 INFORMS, the 2000 Carnegie Mellon University Conference on Quantitative Risk Management in Finance, the 1999 European Finance Association, the 1999 Western Finance Association, the 2000 American Finance Association Meetings, and three anonymous referees for their comments. All errors are solely our responsibility. This article was written while the first author was at the Wharton School of the University of Pennsylvania. Address correspondence to Suleyman Basak, Institute of Finance and Accounting, London Business School, Regents Park, London NW1 4SA, UK or e-mail sbasak@london.edu.

¹ Regulators also view VaR as a useful summary measure; since 1997, the Securities and Exchange Commission has required banks and other large-capitalization registrants to quantify and report their market-risk exposure (Regulation S-K, Item 305), with VaR disclosure being one way to comply.

exposure at a prespecified level.² Surprisingly, the academic literature has largely overlooked this fact; at present, we lack rigorous understanding of its economic implications, and, in particular, little is known about optimal behavior consistent with the VaR-RM.

Our goal is to undertake a comprehensive analysis of the VaR-RM while retaining the standard financial economics paradigms of rational expectations, utility maximization, and market clearing. In particular, we study the implications of the VaR-RM for optimal portfolio policies, (horizon) wealth choice, and equilibrium prices. To the best of our knowledge, ours is the first attempt to directly embed risk management objectives into a utility-maximizing framework. Recognizing that risk management is typically not an economic agent's primary objective, we focus on portfolio choice within the familiar (continuous time) complete markets setting, where the novel feature of our analysis is the assumption that agents may limit their risks while maximizing expected utility. In particular, we assume that a risk-managing agent is constrained to maintain the VaR of horizon wealth at a prespecified level; in other words, he is constrained to maintain, below some prespecified level α , the probability of his wealth falling below some "floor." Our setting has the convenient property that it nests ($\alpha = 1$) the benchmark agent [who does not limit losses; Karatzas et al. (1987), Cox and Huang (1989)] and ($\alpha = 0$) the portfolio insurer [who maintains his horizon wealth above the floor in all states; Grossman and Vila (1989), Basak (1995), Grossman and Zhou (1996)].

Our main results are as follows: First, under general security price uncertainty and general state-independent preferences, we show that an agent, with his VaR capped, optimally chooses to insure against intermediate-loss states, while incurring losses in the worst states of the world. The somewhat surprising feature of the solution is that the uninsured states are chosen independently of preferences and endowments; they are simply the worst states up to a probability of exactly α . The intuition is that the VaR risk manager is willing to incur losses in compliance with the VaR constraint, and it is optimal for him to incur losses in those states against which it is most expensive to insure. We exhibit a problematic feature of the derived optimal behavior, in that although the probability of a loss is fixed, when a large loss occurs, it is *larger* than when not engaging in the VaR-RM.

Second, under constant relative risk aversion (CRRA) preferences and lognormal state prices, we show the VaR risk manager's dynamic portfolio choice to deviate considerably from that of a portfolio insurer and a benchmark agent. The deviation is most pronounced in "transitional" states, where

² See, for example, the lead article of the *Economist* (October 17, 1998), Smith et al. (1995), and Jorion (1999). The risk-monitoring facet of VaR is encouraged by regulators, and to that end, the Basle Committee on Banking Supervision (and the Federal Reserve, in particular) decided, effective January 1998, to allow large banks the option to use a VaR measure to set the capital reserves necessary to cover their market-risk exposure. Regulators expect social benefits, assuming the VaR-RM to reduce the likelihood of large-scale financial failures.

there is the highest uncertainty regarding whether losses will occur. Then, the risk manager takes on large equity positions to finance a high wealth level, should economic conditions turn favorable at the horizon, while allowing for large losses in unfavorable conditions.

Third, recognizing the shortcomings of the VaR-RM to stem from its focus on the probability of a loss, regardless of the magnitude, we propose and evaluate an alternative form of risk management that maintains limited expected losses (LEL) when losses occur. In contrast to the VaR-RM, under the LEL-based risk management (LEL-RM), when losses occur, they are *lower* than those when not engaging in the LEL-RM. For reasonable parameter values, expected losses under the VaR-RM may be several times larger than those under the LEL-RM. Our model abstracts away imperfections and externalities that lead regulators to encourage risk-management practices. However, our analysis predicts that if regulators, and hence risk managers, are concerned with disclosing and monitoring *expected* losses (instead of VaR), then agents' optimal behavior should be consistent with reducing losses in *any* of the most adverse states of the world.

Finally, to investigate the impact of extensive usage of the VaR-RM, we move from the partial equilibrium analysis to a general equilibrium setting. We allow agents to consume continuously, while keeping the VaR of their horizon wealth at a prespecified level. For tractability and realism, we do not require the VaR horizon to coincide with the investment horizon. We work in a familiar Lucas (1978)-type pure exchange economy populated by a representative VaR risk manager and a representative non-risk manager, both long-lived beyond the VaR horizon. Our focus is on the implications for stock market price dynamics. We find that when the economy contains VaR risk managers, the stock market volatility (and risk premium) increases relative to the benchmark case in down markets and decreases in up markets. The highest departure from the benchmark occurs as a response to VaR risk managers' aggressive bidding for stocks in the "transitional" states.

Our results may shed some light on the controversy surrounding the large losses incurred by some banks and hedge funds during the recent (August 1998) stock market downturn. If indeed, as it appears, there was a prevalent use of VaR-based models of risk management by these institutions (*Economist*, October 17, 1998), then assuming deteriorating fundamentals, our model offers a rational explanation for their large losses. It is also interesting to note that the recent downturn was associated with high stock market volatility, consistent with our general equilibrium results. According to our model, when the fundamentals are deteriorating, it is then, in the transition from the good states of the world to the bad states, that the presence of VaR risk managers in the economy should cause the stock volatility to increase relative to the benchmark.

The extant VaR-related academic literature focuses mainly on measuring VaR [see, for example, Linsmeier and Pearson (1996), Duffie and Pan (1997),

Engle and Manganelli (1999)], or on theoretically evaluating properties of VaR and other risk measures [Artzner et al. (1999), Cvitanic and Karatzas (1999), Wang (1999)]. Closer to our work is the line of research that analyzes what may be broadly referred to as mean-VaR optimization. This analysis was initiated by the early studies on shortfall constraints [see the safety-first approach of, for example, Roy (1952), Telser (1956), Kataoka (1963)], and is extended in the recent studies that explicitly address the VaR-RM [see, for example, Klüppelberg and Korn (1998), Alexander and Baptista (1999), Embrechts et al. (1999), Kast et al. (1999)]. However, these mean-VaR studies do not actually embed the VaR-RM into a mean-variance preference-based optimization, but instead compare the two approaches and, in particular, link between mean-variance and mean-VaR efficient frontiers.³ We study a more general preference structure and, most important, do not treat expected utility maximization and risk management as mutually exclusive activities, but merge the two into one optimization problem.

A different approach is presented by Luciano (1998) who, as we do, focuses on optimal portfolio policies of a utility-maximizing agent and also maps the VaR regulatory requirements into a constraint similar to ours. Rather than explicitly applying the constraint to the agent's optimization problem, she analyzes deviations from the constraint, having solved the unconstrained optimization (with and without bid-ask spreads). Such an analysis can be viewed as complementary to ours, as it allows one to examine whether an optimizing agent would automatically comply with the VaR regulation (or with what probability he would do so). In contrast, we apply the VaR constraint *directly* to the utility maximization problem, which allows us to analyze the impact of the VaR-RM on endogenously determined economic quantities. Moreover, ours is the only work to address VaR-related issues in a dynamic general equilibrium setting.

The remainder of the article is organized as follows. Section 1 describes the economy. Section 2 solves the individual's optimization problem under VaR-RM, and Section 3 analyzes the optimization under LEL-RM. Section 4 provides the equilibrium analysis. Section 5 concludes the article. The Appendix contains the proofs.

1. The Economic Setting

1.1 The economy

We consider a finite-horizon, $[0, T]$, economy with a single consumption good (the numeraire). Uncertainty is represented by a filtered probability

³ Ahn et al. (1999) also explicitly acknowledge economic agents' wish to limit the VaR of their market exposure, and they address the question of how to design a put option to minimize the VaR of a position in a stock and options, given a cost constraint on hedging. In the context of developing a model of international portfolio choice, Das and Uppal (1999) constrain the distribution of an agent's portfolio return, imposing an upper bound on the portfolio's excess kurtosis. They interpret this constraint as an implicit limit the agent imposes on the portfolio's VaR.

space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, on which is defined an N -dimensional Brownian motion $w(t) = (w_1(t), \dots, w_N(t))^\top$, $t \in [0, T]$. All stochastic processes are assumed adapted to $\{\mathcal{F}_t; t \in [0, T]\}$, the augmented filtration generated by w . All stated (in)equalities involving random variables hold P -almost surely. In what follows, given our focus is on characterization, we assume all stated processes to be well defined, without explicitly stating the regularity conditions ensuring this.⁴

Investment opportunities are represented by $N + 1$ securities; an instantaneously riskless bond in zero net supply, and N risky stocks, each in constant net supply of 1 and paying dividends at rate δ_j , $j = 1, \dots, N$. The bond price, B , and stock prices, S_j , are assumed to follow

$$dB(t) = B(t)r(t)dt, \quad (1)$$

$$dS_j(t) + \delta_j(t)dt = S_j(t)[\mu_j(t)dt + \sigma_j(t)dw(t)], \quad j = 1, \dots, N, \quad (2)$$

where the interest rate r , the drift coefficients $\mu \equiv (\mu_1, \dots, \mu_N)^\top$, and the volatility matrix $\sigma \equiv \{\sigma_{jk}, j = 1, \dots, N; k = 1, \dots, N\}$ are possibly path-dependent.

Dynamic market completeness (under no arbitrage) implies the existence of a unique state price density process, ξ , given by

$$d\xi(t) = -\xi(t)[r(t)dt + \kappa(t)^\top dw(t)], \quad (3)$$

where $\kappa(t) \equiv \sigma(t)^{-1}(\mu(t) - r(t)\bar{1})$ is the market price of risk (or the Sharpe ratio) process, and $\bar{1} \equiv (1, \dots, 1)^\top$. The quantity $\xi(T, \omega)$ is interpreted as the Arrow-Debreu price per unit probability P of one unit of consumption good in state $\omega \in \Omega$ at time T .

Each agent i in the economy is endowed at time 0 with e_{ij} shares of the risky security j , providing him with an initial wealth of $W_i(0) = e_i^\top S(0)$. (As our focus until Section 4 is on the optimal behavior of a single risk-managing agent, we drop, for now, the subscript i .) Each agent chooses a nonnegative, terminal-horizon wealth $W(T)$ and a portfolio process θ , where $\theta(t) \equiv (\theta_1(t), \dots, \theta_N(t))^\top$ denotes the vector of fractions of wealth invested in each stock. The agent's prehorizon wealth process W then follows

$$\begin{aligned} dW(t) = & W(t)[r(t) + \theta(t)^\top(\mu(t) - r(t)\bar{1})]dt \\ & + W(t)\theta(t)^\top\sigma(t)dw(t). \end{aligned} \quad (4)$$

Each agent is assumed to derive state-independent utility $u(W(T))$ over terminal wealth. The function $u(\cdot)$ is assumed twice continuously differentiable,

⁴ Anticipating the quantities to be introduced in this section and in Section 4, see, for example, Karatzas and Shreve (1998) for the required integrability conditions on consumption policies, prices, and portfolio holdings, as well as the associated Novikov's condition. In the equilibrium constructed in Section 4, these conditions [which, in particular, guarantee nonsingularity of σ in Equation (2)] can be shown to be satisfied.

strictly increasing, strictly concave, and to satisfy $\lim_{x \rightarrow 0} u'(x) = \infty$ and $\lim_{x \rightarrow \infty} u'(x) = 0$.

1.2 Modeling the VaR-RM

The financial industry has standardized on the following definition of $VaR(\alpha)$ [see, for example, Duffie and Pan (1997), Jorion (1997)]: *It is the loss, which is exceeded with some given probability, α , over a given horizon.* Assuming the VaR horizon to coincide with the investment horizon, this definition translates into our setting as

$$P(W(0) - W(T) \leq VaR(\alpha)) \equiv 1 - \alpha, \quad \alpha \in [0, 1]. \quad (5)$$

Note that VaR can be interpreted as the worst loss over a given time interval, under “normal market conditions.”

Our objective is to embed the VaR-RM strategy into a utility maximizing framework. This could be interpreted either as an agent himself managing risk or as an intermediary managing risk on the agent’s behalf, using the VaR approach. The most convenient and natural way to embed the VaR-RM is to assume that an additional constraint is imposed on the agent’s optimization problem, requiring the $VaR(\alpha)$ to be maintained below some prespecified level, that is,

$$VaR(\alpha) \leq W(0) - \underline{W}, \quad (6)$$

where the “floor” $\underline{W}$ is specified exogenously. Equations (5)–(6) can be combined to yield the “VaR constraint”:

$$P(W(T) \geq \underline{W}) \geq 1 - \alpha. \quad (7)$$

Constraint (7) requires of an agent that only with probability α , or less, will he lose more than $W(0) - \underline{W}$. Clearly, if $P(W^B(T) \geq \underline{W}) > 1 - \alpha$ for the wealth in the benchmark (B) case of no constraints, then the VaR constraint never binds, $VaR(\alpha) < W(0) - \underline{W}$; otherwise, $VaR(\alpha) = W(0) - \underline{W}$.

Note that the formulation in (7) nests the B-case; specifically, when $\alpha = 1$ the VaR constraint is never binding. More interestingly, when $\alpha = 0$, our formulation reduces to the case of portfolio insurance (PI), which constrains the horizon wealth to be above the floor $\underline{W}$ in all states [see, for example, Grossman and Vila (1989), Basak (1995), Grossman and Zhou (1996)]. One can thus view the VaR constraint as a “softer” portfolio-insurance constraint, permitting the portfolio value to deteriorate below the floor of $\underline{W}$ with a prespecified probability.

2. Optimization under VaR-RM

In this section, we solve the optimization problem of a VaR risk manager and then analyze the properties of the solution.

2.1 Agent's optimization

We solve the dynamic optimization problem of the VaR agent using the martingale representation approach [Karatzas et al. (1987), Cox and Huang (1989)], which allows the problem to be restated as the following static variational problem:

$$\begin{aligned} & \max_{W(T)} E[u(W(T))] \\ \text{subject to } & E[\xi(T)W(T)] \leq \xi(0)W(0), \\ & P(W(T) \geq \underline{W}) \geq 1 - \alpha. \end{aligned} \quad (8)$$

We note that the VaR constraint complicates the maximization by introducing nonconcavity into the problem. Proposition 1 characterizes the optimal solution, assuming it exists.⁵

Proposition 1. *The time- T optimal wealth of the VaR agent is*

$$W^{VaR}(T) = \begin{cases} I(y\xi(T)) & \text{if } \xi(T) < \underline{\xi}, \\ \underline{W} & \text{if } \underline{\xi} \leq \xi(T) < \bar{\xi}, \\ I(y\xi(T)) & \text{if } \bar{\xi} \leq \xi(T), \end{cases} \quad (9)$$

where $I(\cdot)$ is the inverse function of $u'(\cdot)$, $\underline{\xi} \equiv u'(\underline{W})/y$, $\bar{\xi}$ is such that $P(\xi(T) > \bar{\xi}) \equiv \alpha$, and $y \geq 0$ solves $E[\xi(T)W^{VaR}(T; y)] = \xi(0)W(0)$. The VaR constraint in (7) is binding if, and only if, $\underline{\xi} < \bar{\xi}$. Moreover, the Lagrange multiplier y is decreasing in α , so that $y \in [y^B, y^{PI}]$.

Figure 1 depicts the optimal terminal wealth of a VaR agent [$\alpha \in (0, 1)$], a benchmark agent ($\alpha = 1$) and a portfolio insurer ($\alpha = 0$). Here, $\underline{W}$ is defined by

$$\underline{W} \equiv \begin{cases} I(y\bar{\xi}) & \text{if } \bar{\xi} < \underline{\xi}, \\ \underline{W} & \text{otherwise.} \end{cases}$$

In “good states” [low $\xi(T)$], the portfolio insurer behaves like a B-agent, but then he must insure against all unfavorable [high $\xi(T)$] states. In contrast, Figure 1 reveals the VaR agent to endogenously classify unfavorable states into two subsets: the “bad states” [$\xi(T) \geq \bar{\xi}$], which he leaves fully uninsured, and the “intermediate states” [$\underline{\xi} \leq \xi(T) < \bar{\xi}$], which he fully insures against.⁶ Because he is only concerned with the probability (and not the magnitude) of a loss, the VaR agent chooses to leave the worst states

⁵ We prove that if a terminal wealth satisfies (9) then it is the optimal policy for the VaR agent. As we note in the proof, to keep our focus, we do not provide general conditions for existence. However, we will provide explicit numerical solutions for a variety of parameter values. From (9), a feasibility bound on $\underline{W}$ for a solution is $\underline{W} \leq W(0)\xi(0)/E[\xi(T)1_{\{\xi(T) < \bar{\xi}\}}]$. Our method of proof is applicable to other problems, such as those with nonstandard preferences.

⁶ In the equilibrium analyzed in Section 4, we will verify that “good states,” low price of consumption $\xi(T)$, are associated with a high equity-market value, and vice versa for “bad states,” high $\xi(T)$.

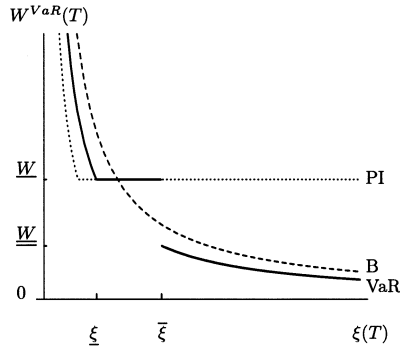


Figure 1

Optimal horizon wealth of the VaR risk manager

The figure plots the optimal horizon wealth of the VaR risk manager (Proposition 1) as a function of the horizon state price density $\xi(T)$. The dashed plot is for the unconstrained benchmark agent B, the dotted plot is for the portfolio insurer PI. The VaR risk manager's optimal horizon wealth falls into three distinct regions, where he exhibits distinct economic behavior. In the good states [low $\xi(T)$], the VaR agent behaves like a B agent. In the intermediate states [$\underline{\xi} \leq \xi(T) < \bar{\xi}$] he insures himself against losses, behaving like a PI agent, and in the bad states [high $\xi(T)$] he is completely uninsured, incurring all losses.

uninsured because they are the most expensive ones to insure against. The measure of these bad states is chosen to comply exactly with the VaR constraint. Consequently, $\bar{\xi}$ depends solely on α and the distribution of $\xi(T)$ and is independent of the agent's preferences and endowment. The agent can be thought of as "ignoring" losses in this upper tail of the $\xi(T)$ distribution, where consumption is the most costly.

Inspection of Figure 1 allows us to summarize the dependence of the solution on the parameters $\underline{W}$ and α . As the floor is increased, more states need to be insured against, and the intermediate region grows at the expense of the good-states region. Accordingly, the wealth in both good and bad regions must decrease to meet the higher floor in the intermediate region. As α increases, that is, the agent is allowed to make a loss with higher probability, the intermediate, insured region can shrink, and the good and bad regions both grow. The agent's horizon wealth can increase in both the good and bad states because he is not required to insure against as large a state space. Consequently, in the bad-states region $W^{VaR}(T) < W^B(T) < W^{PI}(T)$. This may be a source of concern for regulators and real-world risk managers. The VaR-RM is viewed by many as a tool to shield economic agents from large losses, which, when they occur, could cause credit and solvency problems. But our solution reveals that when a large loss occurs, it is an even larger loss under the VaR-RM and hence more likely to lead to credit problems, defeating the very purpose of using the VaR-RM. Proposition 2 later amplifies on this point.

Figure 2 depicts the shape of the probability density function of terminal wealth in the B, PI, and VaR solutions. There is a probability mass build-up in the VaR agent's horizon wealth, at the floor $\underline{W}$, as for the portfolio insurer.

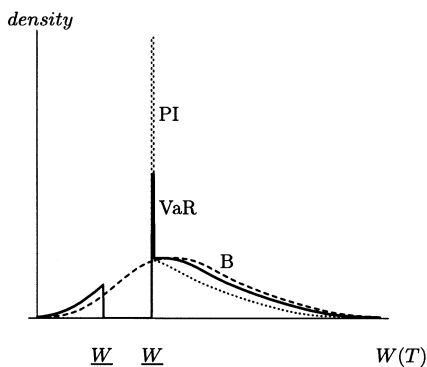


Figure 2

Probability density of the VaR risk manager's optimal horizon wealth

The figure plots the shape of the probability density function of the VaR risk manager's optimal horizon wealth. The dashed plot is for the unconstrained benchmark agent B, the dotted plot is for the portfolio insurer PI.

The VaR agent then has a discontinuity, with no states having wealth between $\underline{W}$ and $\bar{W}$, whereas states with wealth below $\underline{W}$ have probability α . Note that relative to the benchmark, the distribution in these bad states is shifted to the left, meaning more loss with higher probability.

It has been commonly observed [e.g., Basak(1995), Grossman and Zhou (1996)] that the optimal PI horizon wealth can be expressed as the B wealth plus a put option thereon, that is, $W^{PI}(T; y^{PI}) = W^B(T; y^{PI}) + \max[\underline{W} - W^B(T; y^{PI}), 0]$. Analogously, the VaR optimal wealth plan in (9) can be expressed as

$$\begin{aligned} W^{VaR}(T; y(W(0))) &= W^{PI}(T; y^B(W_*)) - (\underline{W} - W^B(T; y^B(W_*)))1_{\{\bar{\xi} \leq \xi(T)\}} \\ &= W^B(T; y^B(W_*)) + (\underline{W} - W^B(T; y^B(W_*)))1_{\{\bar{\xi} \leq \xi(T) < \bar{\xi}\}}, \end{aligned}$$

where W_* is set so that $y^B(W_*) = y(W(0))$. In other words, adjusting for the initial endowment, W^{VaR} is equivalent to a PI solution plus a short position in “binary” options, or to a B solution plus an appropriate position in “corridor” options.⁷ More precisely, because

$$\begin{aligned} W_* &= W(0) - E \left[\frac{\xi(T)}{\xi(0)} \max(\underline{W} - W^B(T; y^B(W_*)), 0) \right] \\ &\quad + E \left[\frac{\xi(T)}{\xi(0)} (\underline{W} - W^B(T; y^B(W_*)))1_{\{\bar{\xi} \leq \xi(T)\}} \right], \end{aligned}$$

⁷ For details on binary and corridor options see, for example, Briys et al. (1998). Browne (1999) provides an example where buying a binary option is the optimal policy to maximize the probability of reaching a given value of wealth by a fixed terminal time.

$W^B(T; y^B(W_*))$ is the optimal policy of an unconstrained agent, whose initial endowment is simply $W(0)$ decreased by the price of a put (needed to implement the PI component) and increased by the proceeds of short selling the binary options.

2.2 Properties of the VaR-RM strategy

To perform a detailed analysis of the optimal behavior under the VaR-RM strategy, we specialize the setting to CRRA preferences, $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, and to lognormal state prices with constant interest rate and market price of risk. Figures 1 and 2 appear to indicate higher losses in the bad-states region under the VaR-RM than without risk management. However, because the bad-states region itself shifts, the figures do not directly imply lower expected losses. Proposition 2 shows explicitly that under the VaR-RM the expected extreme losses are indeed higher than those incurred by an agent who does not concern himself with (7).

Proposition 2. Assume $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, and r and κ are constant. For a given terminal-wealth plan $W(T)$, define the following two measures of loss: $L_1(W) = E[(\underline{W} - W(T))1_{\{W(T) \leq \underline{W}\}}]$ and $L_2(W) = E[\frac{\xi(T)}{\xi(0)}(\underline{W} - W(T))1_{\{W(T) \leq \underline{W}\}}]$. Then, (i) $L_1(W^{VaR}) \geq L_1(W^B)$, and (ii) $L_2(W^{VaR}) \geq L_2(W^B)$.

In Proposition 2, we focus on the bad states, that is on the states where large losses occur. $L_1(W)$ measures the expected future value of a loss, when there is a large loss, while $L_2(W)$ measures its present value. Proposition 2 highlights further the undesirable features of VaR-RM, when viewed from a regulator's perspective. A regulatory requirement to manage risk using the VaR approach is designed to prevent large, frequent losses that may drive economic agents out of business. It is true that under the VaR-RM losses are not frequent, however, the largest losses are more severe than without the VaR-RM.

Proposition 3 presents explicit expressions for (and properties of) the VaR agent's optimal wealth and portfolio strategies before the horizon.

Proposition 3. Assume $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, and r and κ are constant. Then:

(i) The time- t optimal wealth is given by

$$W^{VaR}(t) = \frac{e^{\Gamma(t)}}{(y\xi(t))^{\frac{1}{\gamma}}} - \left[\frac{e^{\Gamma(t)}}{(y\xi(t))^{\frac{1}{\gamma}}} \mathcal{N}(-d_1(\underline{\xi})) - \underline{W}e^{-r(T-t)} \mathcal{N}(-d_2(\underline{\xi})) \right] + \left[\frac{e^{\Gamma(t)}}{(y\xi(t))^{\frac{1}{\gamma}}} \mathcal{N}(-d_1(\bar{\xi})) - \underline{W}e^{-r(T-t)} \mathcal{N}(-d_2(\bar{\xi})) \right], \quad (10)$$

where $\mathcal{N}(\cdot)$ is the standard-normal cumulative distribution function, y is as in Proposition 1, and

$$\begin{aligned}\underline{\xi} &= \frac{1}{y\underline{W}^\gamma}, \\ \Gamma(t) &\equiv \frac{1-\gamma}{\gamma} \left(r + \frac{\|\kappa\|^2}{2} \right) (T-t) + \left(\frac{1-\gamma}{\gamma} \right)^2 \frac{\|\kappa\|^2}{2} (T-t), \\ d_2(x) &\equiv \frac{\ln \frac{x}{\underline{\xi}(t)} + (r - \frac{\|\kappa\|^2}{2})(T-t)}{\|\kappa\| \sqrt{T-t}}, \\ d_1(x) &\equiv d_2(x) + \frac{1}{\gamma} \|\kappa\| \sqrt{T-t}.\end{aligned}$$

(ii) The fraction of wealth invested in stocks is

$$\theta^{VaR}(t) = q^{VaR}(t) \theta^B(t),$$

where the benchmark value, θ^B , and the exposure to risky assets relative to the benchmark, q^{VaR} , are

$$\begin{aligned}\theta^B(t) &= \frac{1}{\gamma} [\sigma(t)^\top]^{-1} \kappa, \\ q^{VaR}(t) &= 1 - \frac{\underline{W} e^{-r(T-t)} (\mathcal{N}(-d_2(\underline{\xi})) - \mathcal{N}(-d_2(\bar{\xi})))}{W^{VaR}(t)} \\ &\quad + \frac{\gamma(\underline{W} - \bar{W}) e^{-r(T-t)} \phi(d_2(\bar{\xi}))}{W^{VaR}(t) \|\kappa\| \sqrt{T-t}},\end{aligned}\tag{11}$$

respectively, and $\phi(\cdot)$ is the standard-normal probability distribution function.

(iii) The exposure to risky assets relative to the benchmark is bounded below: $q^{VaR}(t) \geq 0$, and

$$\lim_{\xi(t) \rightarrow 0} q^{VaR}(t) = \lim_{\xi(t) \rightarrow \infty} q^{VaR}(t) = 1.$$

(iv) When the VaR constraint is binding ($\underline{\xi} < \bar{\xi}$), then $q^{VaR}(t) > 1$ if, and only if, $\xi(t) > \xi^*(t)$, where $\xi^*(t)$ is deterministic and bounded:

$$\sqrt{\bar{\xi} \underline{\xi}} e^{(r - \|\kappa\|^2/2)(T-t)} \leq \xi^*(t) \leq \bar{\xi} e^{(r - \|\kappa\|^2/2)(T-t)} e^{(\|\kappa\|^2/\gamma)(T-t)}.$$

The option-based interpretation in Section 2.1 clarifies the expression of the time- t optimal wealth in Equation (10). The first term takes the form of the optimal wealth of a non-risk manager, while the remaining terms

represent the “insurance package” for keeping the time- T wealth at $\underline{W}$ in the intermediate states. The second and third terms represent the cost of a Black and Scholes (1973)–type put option on the B-wealth with strike price $\underline{W}$; the fourth and fifth terms are the proceeds from shorting a portfolio of binary options. Consequently, when the fraction invested in risky assets is expressed as a multiple of the B-policy, the second and third terms in Equation (11) correspond, respectively, to the positions in the long put and the short binary options.

Figure 3 compares graphically the optimal time- t wealth and the relative stock exposure in the B, PI, and VaR cases. Figure 3a reveals that the prehorizon wealth of the VaR agent behaves similarly to that of a portfolio insurer in the good states, whereas in the upper tail of the $\xi(t)$ distribution he behaves similarly to the B case. In the intermediate region, the VaR agent’s wealth exhibits concavity in $\xi(t)$, and it is easy to visualize how this concavity will increase as time approaches the horizon, and tend to the discontinuous shape in Figure 1. In these intermediate states, the VaR agent is beginning to insure himself.

Figure 3b illustrates the typical shape of the VaR agent’s optimal asset allocation, exhibiting some surprising features. We may characterize five segments in the $\xi(t)$ space. In the two extremes, the benchmark behavior prevails. In between, however, there are three distinct patterns: First, in the relatively cheap states, the VaR agent acts similarly to a portfolio insurer

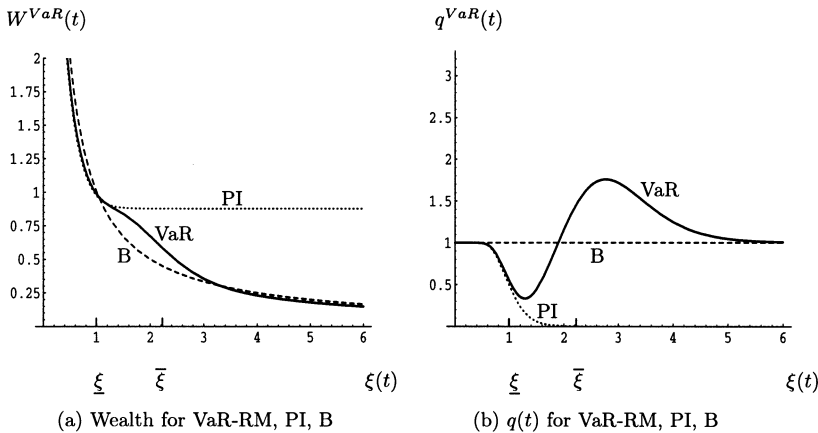


Figure 3

Optimal prehorizon wealth and risk exposure of the VaR risk manager

The figure plots the VaR risk manager’s (a) optimal prehorizon wealth, $W(t)$, and (b) exposure to risky assets relative to the benchmark, $q(t)$ (Proposition 3), as a function of the concurrent state price density $\xi(t)$. The dashed plots are for the benchmark agent B, the dotted plots are for the portfolio insurer PI. The VaR risk manager’s exposure to risky assets relative to the benchmark is given by $q^{VaR}(t) = \theta_j^{VaR}(t)/\theta_j^B(t)$, where θ_j denotes the fraction of wealth invested in stock j , for all j . The plots assume CRRA preferences and log-normal state price density. The fixed parameter values are: $\gamma = 1$, $\alpha = 0.01$, $W(0) = 1$, $\underline{W} = 0.9$, $r = 0.05$, $\|\kappa\| = 0.4$, $T = 1$, $t = 0.5$, $\xi(0) = 1$. Then, $\underline{\xi} = 0.99$, $\bar{\xi} = 2.23$.

investing a higher fraction of his wealth in the bond. Second, as $\xi(t)$ rises, instead of moving further out of the equity market the VaR agent begins to increase his equity exposure, tending back toward his B policy, then surpassing it considerably so that in the relatively expensive consumption states he invests a *higher* fraction of his wealth in stocks compared to the B case.⁸ The third segment occurs when $\xi(t)$ is high enough to deter the agent from further risk taking, and he converges to his benchmark policy. Formally, this nonmonotonic behavior across the state space is linked to the replication of a portfolio of binary options. Intuitively, the asset allocation is driven by the agent's desire to insure the intermediate-states region. When $\xi(t)$ is already very high, then it is very likely that the agent will end up in the bad-states region and it is too costly for him to bet on a favorable realization of a large equity investment. Hence, the VaR agent behaves similarly to the B case. On the other hand, when $\xi(t)$ is in the proximity of $\bar{\xi}$, not all hope is lost, and the agent attempts, via a relatively large exposure to equity, to reach the $\underline{W}$ level of wealth, under favorable time- T economic conditions.

Figure 4 displays a sensitivity analysis of $q^{VaR}(t)$ to α , $\underline{W}$, and time. In general terms, (a) decreasing α , (b) increasing $\underline{W}$, or (c) decreasing the time-to-horizon, all cause the agent to deviate more from the B behavior as the VaR constraint exerts more influence. As α decreases, the deviation from the benchmark also spreads to a larger region of $\xi(t)$, and as the time-to-horizon decreases, the deviation shrinks to a smaller region of $\xi(t)$.

Figure 5 displays the sensitivity to γ and κ of the risky asset holdings of the VaR agent, for a market with one risky stock. The deviation from the benchmark holdings becomes more pronounced for both lower γ (less risk averse agent) and higher κ (higher market price of risk). This behavior is fairly intuitive; as an agent becomes less risk averse, or as the stock's Sharpe ratio increases, he responds more aggressively to changes in the state variable ξ that affect his likelihood to end up with $W^{VaR}(T) \geq \underline{W}$, as opposed to $W^{VaR}(T) \leq \underline{W}$. Note that, contrary to the B case (but similarly to the PI case), the more risk-averse agent takes on more risk than the less risk averse in the "better" intermediate states; the more risk-averse agent invests more in the stock, preparing to end up with $W^{VaR}(T) > \underline{W}$, as opposed to $W^{VaR}(T) = \underline{W}$. Somewhat more surprising is that, contrary to the B case (and the PI case), in the "worse" intermediate states a higher Sharpe ratio does not necessarily cause the VaR agent to allocate more wealth to the stock. To understand why, note that a change in κ affects the dynamics of $\xi(t)$; in particular, the boundary into the bad-states region, $\bar{\xi}$, is increasing in κ . Hence, at some given $\xi(t)$, such as 2 (in this example), the lower the κ ,

⁸ For the parameters used in Figure 3, using the bounds in item (iv) of Proposition 3, $q^{VaR}(t)$, as a function of $\xi(t)$, must rise above 1 while $\xi(t)$ takes values in the (1.46, 2.38) interval. The bounds in (iv) identify, analytically, a transition from an underexposure to overexposure, relative to the B case, for all parameters' values, and Figure 3b confirms this for the chosen parameters. In addition, Figure 3b illustrates that the VaR agent deviates considerably from the B and the PI cases when $\xi(t)$ takes values within these bounds.

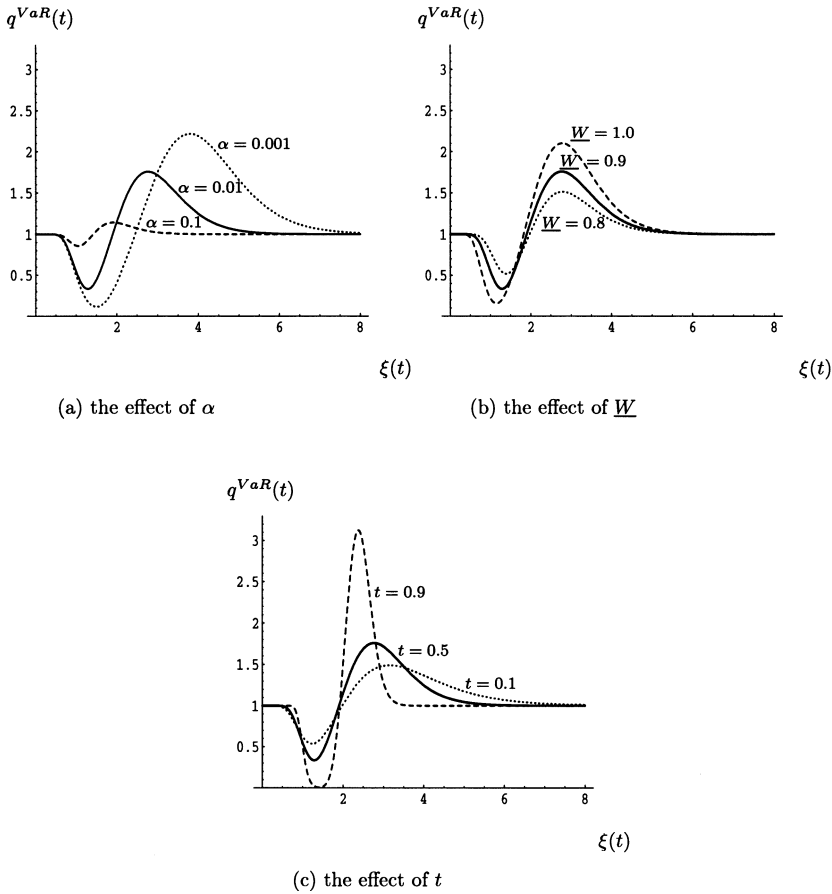


Figure 4
Behavior of the VaR risk manager's risk exposure with respect to VaR parameters

The figure plots the VaR risk manager's exposure to risky assets relative to the benchmark, $q(t)$, as a function of the concurrent state price density $\xi(t)$ for varying levels of (a) the VaR probability $\alpha \in \{0.001, 0.01, 0.1\}$, (b) the floor $\underline{W} \in \{0.8, 0.9, 1\}$, and (c) time $t \in \{0.1, 0.5, 0.9\}$. The VaR risk manager's exposure to risky assets relative to the benchmark is given by $q^{VaR}(t) = \theta_j^{VaR}(t)/\theta_j^B(t)$, where θ_j denotes the fraction of wealth invested in stock j , for all j . The plots assume CRR preferences and log-normal state price density. The solid line in all three charts represents the following case of fixed parameter values: $\gamma = 1$, $\alpha = 0.01$, $W(0) = 1$, $\underline{W} = 0.9$, $r = 0.05$, $\|\kappa\| = 0.4$, $t = 0.5$, $T = 1$, $\xi(0) = 1$. Then $\underline{\xi} = 0.99$, $\bar{\xi} = 2.23$.

the closer the agent is to the transition into the bad-states region so the more heavily he invests in the stock, targeting to finance $W^{VaR}(T) = \underline{W}$ should the bad states not occur.

3. Optimization under LEL-RM

In this section, we introduce the LEL-RM (limited-expected-losses-based risk management) strategy as an alternative to the VaR-RM strategy. We then

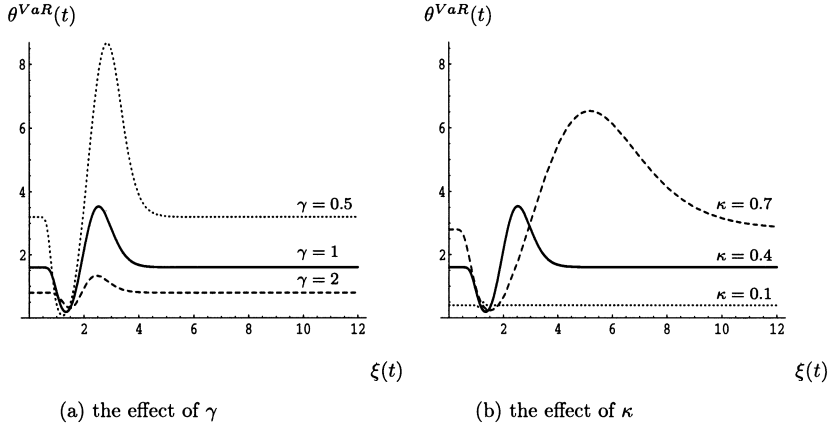


Figure 5
Behavior of the VaR risk manager's risky asset holdings with respect to risk aversion and market price of risk

The figure plots the VaR risk manager's optimal fraction of wealth invested in a stock, $\theta^{VaR}(t)$, as a function of the concurrent state price density $\xi(t)$ for varying levels of (a) the VaR risk manager's risk aversion $\gamma \in \{0.5, 1, 2\}$ and (b) the market price of risk $\kappa \in \{0.1, 0.4, 0.7\}$. The plots are for the case of a single risky stock, CRRA preferences, and log-normal state price density. The solid line in both charts represents the following case of fixed parameter values: $\gamma = 1$, $\alpha = 0.01$, $W(0) = 1$, $\underline{W} = 0.9$, $r = 0.05$, $||\kappa|| = 0.4$, $t = 0.5$, $T = 1$, $\xi(0) = 1$. The benchmark agent's corresponding optimal fraction of wealth in the stock, $\theta^B(t)$, in each chart is (a) $\theta^B(t) \in \{3.2, 1.6, 0.8\}$, (b) $\theta^B(t) \in \{0.4, 1.6, 2.8\}$.

solve the optimization problem of a LEL risk manager and analyze the properties of the solution.

3.1 LEL-RM

The shortcomings of VaR-RM, highlighted in the previous section, stem from the fact that the VaR agent is concerned with controlling the probability of a loss rather than its magnitude. It turns out that the expected losses, in the states where there are large losses, are higher than those the agent would have incurred if he had not engaged in VaR-RM in the first place. Ideally, to control the magnitude of losses, one ought to control all moments of the loss distribution. As a first step, in this section, we focus on controlling the first moment and examine how one can remedy the shortcomings of VaR-RM. We leave the analysis of higher moments for future work.

We define an LEL-RM strategy as one under which the present value of the agent's losses are constrained:

$$E[\xi(T)(\underline{W} - W(T))1_{\{W(T) \leq \underline{W}\}}] \leq \epsilon, \quad (12)$$

where $\epsilon \geq 0$ is a constant. Observe that, because $E[\xi(T)(\underline{W} - W(T))1_{\{W(T) \leq \underline{W}\}}] = E[\xi(T)(\underline{W} - W(T))|W(T) \leq \underline{W}]P(W(T) \leq \underline{W})$, this constraint penalizes both a high probability of a loss and a high expected loss given there is a loss. The constrained quantity in (12) can be interpreted

as a risk measure of time- T losses. We may note that this measure satisfies the subadditivity, positive homogeneity, and monotonicity axioms (but not the translation-invariance axiom) defined by Artzner et al. (1999) and hence avoids their criticism of the VaR measure of risk. In contrast to our endogenous losses-based criticism, their objection is that VaR fails to display subadditivity when combining the risk of two or more portfolios (the VaR of the whole may be greater than the sum of the VaRs of the individual parts).⁹

Analogously to the treatment of (7), we impose (12) as a constraint on the agent's optimization problem, thereby incorporating LEL-RM directly into the optimization. The formulation again nests the B case ($\epsilon = \infty$) and the PI case ($\epsilon = 0$). As we show next, when $0 < \epsilon < \infty$, the LEL strategy has the appealing property that it indeed yields results consistent with the stated goal of "managing risk" in the following sense: The LEL risk manager optimally chooses a wealth level, which in the low-wealth states is above the benchmark wealth.

3.2 Agent's optimization

Using the martingale representation approach, the dynamic optimization problem of the LEL risk manager (henceforth, the LEL agent) is restated as the following variational problem:

$$\begin{aligned} \max_{W(T)} E[u(W(T))] \quad \text{s.t.} \quad & E[\xi(T)W(T)] \leq \xi(0)W(0), \\ & E[\xi(T)(\underline{W} - W(T))1_{\{W(T) \leq \underline{W}\}}] \leq \epsilon. \end{aligned} \quad (13)$$

Proposition 4 characterizes the optimal solution, assuming it exists.¹⁰

Proposition 4. *The time- T optimal wealth of the LEL agent is*

$$W^{LEL}(T) = \begin{cases} I(z_1 \xi(T)) & \text{if } \xi(T) < \underline{\xi}_\epsilon, \\ \underline{W} & \text{if } \underline{\xi}_\epsilon \leq \xi(T) < \bar{\xi}_\epsilon, \\ I((z_1 - z_2)\xi(T)) & \text{if } \bar{\xi}_\epsilon \leq \xi(T), \end{cases} \quad (14)$$

⁹ Artzner et al. (1999) call a risk measure "coherent" if it satisfies the aforementioned four axioms, and hence our measure is not classified as coherent. However, because we model an agent as limiting the risk of his total position, we abstract from the idea of adding extra funds or adjusting margin levels (cases where translation invariance is applicable), and consequently in our setting monotonicity is in fact the only critical property of a risk measure, so that risks can be ranked. Artzner et al. (1999) discuss a leading example of a coherent measure, the tail conditional expectation (TCE), which measures expected losses (not deflated by state prices) conditional on the losses falling below a quantile of probability α . Unlike our LEL measure, the TCE does not then fully disentangle the notions of quantiles and expectations; we therefore chose LEL to more clearly illustrate the differences between the quantiles-based and the expectations-based approaches.

¹⁰ From (14), the feasibility bound on $\underline{W}$ for a solution is $\underline{W} \leq (W(0)\xi(0) + \epsilon)/E[\xi(T)]$. Note that if an agent wishes instead to limit expected future losses, $E[(\underline{W} - W(T))1_{\{W(T) \leq \underline{W}\}}] \leq \epsilon'$, his optimal wealth will have a structure similar to (14). The only changes are that $\bar{\xi}_{\epsilon'} \equiv (u'(\underline{W}) + z_2)/z_1$ and that in the $\bar{\xi}_{\epsilon'} \leq \xi(T)$ region his wealth is set to $I(z_1 \xi(T) - z_2)$. The nature of the implications discussed in this section are robust to this modeling change. Moreover, using expected future losses in Table 1 results in only minor quantitative adjustments to the reported values.

where $\xi_\epsilon \equiv \frac{u'(W)}{z_1}$, $\bar{\xi}_\epsilon \equiv \frac{u'(W)}{(z_1 - z_2)}$, and $(z_1 \geq 0, z_2 \geq 0)$ solve the following system:

$$\begin{cases} E[\xi(T)W^{LEL}(T; z_1, z_2)] = \xi(0)W(0), \\ E[\xi(T)(\underline{W} - W^{LEL}(T; z_1, z_2))1_{\{W^{LEL}(T; z_1, z_2) \leq \underline{W}\}}] = \epsilon \quad \text{or} \quad z_2 = 0. \end{cases}$$

The LEL constraint in (12) is binding if, and only if, $\xi_\epsilon < \bar{\xi}_\epsilon$. Moreover, the Lagrange multiplier z_1 is decreasing in ϵ , so that $z_1 \in [z_1^B, z_1^{PI}]$. Also, $z_1 - z_2 \leq z_1^B$.

Figure 6 depicts the optimal terminal wealth of an LEL agent [$\epsilon \in (0, \infty)$], a benchmark agent ($\epsilon = \infty$), and a portfolio insurer ($\epsilon = 0$). Figure 6 reveals that in contrast to the findings in the VaR case, now in the bad-states region, $W^B(T) < W^{LEL}(T) < W^{PI}(T)$. This highlights the most surprising, but also encouraging feature of the optimal behavior of the LEL agent; although in some states he is willing to settle for a wealth lower than $\underline{W}$, he does so while endogenously choosing a higher $W^{LEL}(T)$ than $W^B(T)$. The LEL agent endogenously decides to classify unfavorable states into two subsets: the bad states, against which he partially insures, and the intermediate states, against which he fully insures. Again, he chooses the worst states in which to maintain a loss, because these are the most expensive states to insure against, but maintains some level of insurance. Insuring a terminal wealth at the $\underline{W}$ level is too costly, so he settles for less, but enough to comply with the LEL constraint. Note that the LEL agent not only chooses ξ_ϵ endogenously but

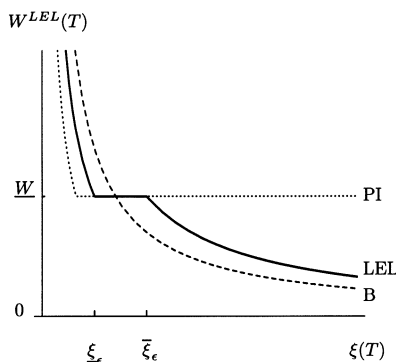


Figure 6
Optimal horizon wealth of the LEL risk manager

The figure plots the optimal horizon wealth of the LEL risk manager (Proposition 4) as a function of the horizon state price density $\xi(T)$. The dashed plot is for the unconstrained benchmark agent B, the dotted plot is for the portfolio insurer PI. The LEL risk manager's optimal horizon wealth falls into three distinct regions, where he exhibits distinct economic behavior. In the good states [low $\xi(T)$], the LEL agent behaves like a B agent. In the intermediate states [$\xi_\epsilon \leq \xi(T) < \bar{\xi}_\epsilon$] he insures himself against losses, behaving like a PI agent, and in the bad states [high $\xi(T)$] he partially insures himself, incurring partial losses in contrast to the VaR risk manager.

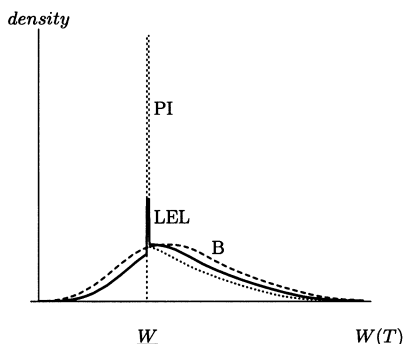


Figure 7

Probability density of the LEL risk manager's optimal horizon wealth

The figure plots the shape of the probability density function of the LEL risk manager's optimal horizon wealth. The dashed plot is for the unconstrained benchmark agent B, the dotted plot is for the portfolio insurer PI.

also endogenously determines the value of $\bar{\xi}_\epsilon$; unlike $\bar{\xi}$, $\bar{\xi}_\epsilon$ does depend on the agent's preferences and endowment. A further distinction with VaR-RM is that the terminal wealth policy under LEL-RM is continuous across the states of the world.

Figure 7 depicts the shape of the probability density function of terminal wealth in the B, PI, and LEL solutions. Similarly to Figure 2, there is a probability mass build-up in the LEL agent's horizon wealth, at the floor $\underline{W}$. However, LEL has no discontinuities across states. Also, relative to the benchmark, the distribution in the bad states is shifted to the right, meaning less loss with higher probability.

The optimal wealth plan in (14) can be expressed as

$$\begin{aligned} W^{LEL}(T; z_1(W(0)), z_2(W(0))) \\ &= \min[W^{PI}(T; y^B(W_\epsilon)), W^B(T; y^B(W_\epsilon) - z_2(W(0)))] \\ &= \max[W^B(T; y^B(W_\epsilon)), \min[\underline{W}, W^B(T; y^B(W_\epsilon) - z_2(W(0)))]], \end{aligned}$$

where we set W_ϵ so that $y^B(W_\epsilon) = z_1(W(0))$. Hence, adjusting for the initial endowment, W^{LEL} is equivalent to an option on a minimum of two "securities" (one being riskless), where the nonstandard feature of the option is that the strike price is stochastic.¹¹ The wealth adjustment, which equates the strike price to the wealth of a fictitious unconstrained agent, is obtained

¹¹ See Stulz (1982) for the analysis and applications of an option on a minimum of two assets (both risky), where the strike price is fixed.

by valuing this nonstandard option at the initial date:

$$W_\epsilon = W(0) - E \left[\frac{\xi(T)}{\xi(0)} \max \left[\min \left[\underline{W}, W^B(T; y^B(W_\epsilon) - z_2(W(0))) \right] - W^B(T; y^B(W_\epsilon)), 0 \right] \right].$$

3.3 Properties of the LEL-RM strategy

We now specialize the setting to CRRA preferences, $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, and to lognormal state prices with constant interest rate and market price of risk, analogous to the VaR analysis in Section 2. Using the notation defined in Proposition 3, Proposition 5 summarizes the wealth dynamics and the portfolio choice of the LEL agent.

Proposition 5. Assume $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, and r and κ are constant. Then:

(i) The time- t optimal wealth is given by

$$W^{LEL}(t) = \frac{e^{\Gamma(t)}}{(z_1 \xi(t))^{\frac{1}{\gamma}}} - \left[\frac{e^{\Gamma(t)}}{(z_1 \xi(t))^{\frac{1}{\gamma}}} \mathcal{N}(-d_1(\underline{\xi}_\epsilon)) - \underline{W} e^{-r(T-t)} \mathcal{N}(-d_2(\underline{\xi}_\epsilon)) \right] + \left[\frac{e^{\Gamma(t)}}{(z_1 - z_2)^{\frac{1}{\gamma}} \xi(t)^{\frac{1}{\gamma}}} \mathcal{N}(-d_1(\bar{\xi}_\epsilon)) - \underline{W} e^{-r(T-t)} \mathcal{N}(-d_2(\bar{\xi}_\epsilon)) \right], \quad (15)$$

where $\Gamma(t)$, $d_1(x)$, $d_2(x)$ are as given in Proposition 3, (z_1, z_2) are as given in Proposition 4,

$$\underline{\xi}_\epsilon = \frac{1}{z_1 \underline{W}^\gamma} \quad \text{and} \quad \bar{\xi}_\epsilon = \frac{1}{(z_1 - z_2) \underline{W}^\gamma}.$$

(ii) The fraction of wealth invested in stocks is

$$\theta^{LEL}(t) = q^{LEL}(t) \theta^B(t),$$

where the exposure to risky assets relative to the benchmark, $q^{LEL}(t)$ is

$$q^{LEL}(t) = 1 - \frac{\underline{W} e^{-r(T-t)} (\mathcal{N}(-d_2(\underline{\xi}_\epsilon)) - \mathcal{N}(-d_2(\bar{\xi}_\epsilon)))}{W^{LEL}(t)}.$$

(iii) The exposure to risky assets relative to the benchmark is bounded below and above: $0 \leq q^{LEL}(t) \leq 1$, and

$$\lim_{\xi(t) \rightarrow 0} q^{LEL}(t) = \lim_{\xi(t) \rightarrow \infty} q^{LEL}(t) = 1.$$

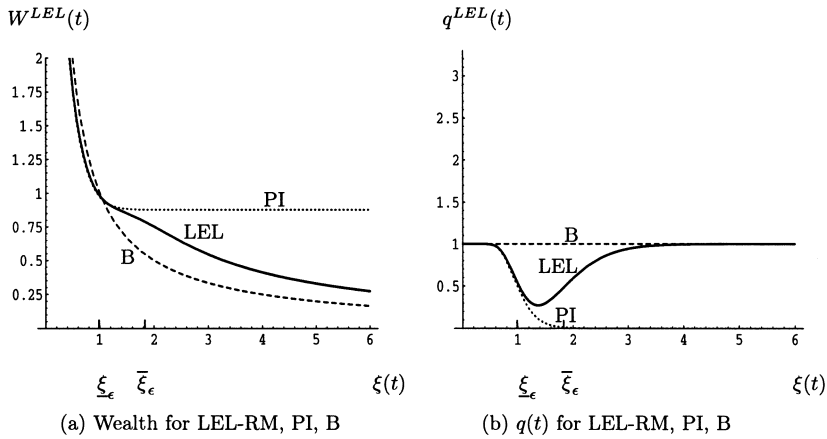


Figure 8

Optimal prehorizon wealth and risk exposure of the LEL risk manager

The figure plots the LEL risk manager's (a) optimal prehorizon wealth, $W(t)$, and (b) exposure to risky assets relative to the benchmark, $q(t)$ (Proposition 5), as a function of the concurrent state price density $\xi(t)$. The dashed plots are for the benchmark agent B, the dotted plots are for the portfolio insurer PI. The LEL risk manager's exposure to risky assets relative to the benchmark is given by $q^{LEL}(t) = \theta_j^{LEL}(t)/\theta_j^B(t)$, where θ_j denotes the fraction of wealth invested in stock j , for all j . The plots assume CRRA preferences and log-normal state price density. The fixed parameter values are: $\gamma = 1$, $\epsilon = 0.01$, $W(0) = 1$, $\underline{W} = 0.9$, $r = 0.05$, $\|\kappa\| = 0.4$, $T = 1$, $t = 0.5$, $\xi(0) = 1$. Then $\underline{\xi}_\epsilon = 0.98$, $\bar{\xi}_\epsilon = 1.83$.

Figure 8 compares graphically the optimal wealth and the stock exposure in the B, PI, and LEL cases. Figure 8a illustrates that, as in the VaR case, for low and intermediate values of $\xi(t)$ the agent's prehorizon wealth behaves more similarly to a portfolio insurer's wealth than to the benchmark one. In the intermediate range, the LEL agent attempts to insure as many states as he can afford, but in the higher tail of the $\xi(t)$ distribution, he reverts to a behavior similar to the B behavior. However, unlike in the VaR case, in this upper tail of the distribution the LEL agent maintains a higher wealth than in the B case. Again, one can easily visualize how the wealth in the intermediate states approaches the shape in Figure 6 as the time approaches the horizon.

In Figure 8b, we clearly see the properties of $q^{LEL}(t)$ stated in item (iii) of Proposition 5. The LEL agent maneuvers between his behavior in the B and the PI cases, never investing a higher fraction of his wealth in stocks compared to the B case. The agent's asset allocation has four distinct patterns over the $\xi(t)$ space. In the two extremes, the benchmark behavior prevails. But in between there are now only two distinct patterns: First, the LEL agent acts as a portfolio insurer; then, as $\xi(t)$ rises, instead of moving further into the riskless asset the agent increases his equity exposure, tending back toward his B-policy but never surpassing it in terms of the exposure to equity. Intuitively, the asset allocation of the LEL agent differs from that of the VaR agent because W^{LEL} is continuous across states. In the VaR case, if $\xi(t)$

is close to $\bar{\xi}$ as he approaches the horizon, the VaR agent must allow for the need to finance highly distinct wealths: $\underline{W}$ or $\underline{W}$. For the LEL agent, however, a slight change in $\xi(t)$ as t approaches T does not necessitate the financing of a very different level of wealth. Therefore, LEL-RM never leads risk managers to take extreme leveraged positions compared to the positions they would have taken as non-risk managers.¹²

3.4 Losses under the LEL-RM and the VaR-RM strategies

Proposition 1 revealed that in the bad states the losses under the VaR-RM exceed (or equal to) those under the B policy, whereas Proposition 4 illustrated that the losses under the LEL-RM are lower than (or equal to) those in the B case, as depicted in Figures 1 and 6, respectively. To quantify the economic significance of the loss reduction under LEL-RM versus VaR-RM, we examine the following loss ratio:

$$E[\xi(T)(\underline{W} - W^{VaR}(T))1_{\{\xi(T) \geq \bar{\xi}\}}] / E[\xi(T)(\underline{W} - W^{LEL}(T))1_{\{\xi(T) \geq \bar{\xi}\}}]. \quad (16)$$

This loss ratio employs the loss measure in (12), which uses the reference point $\underline{W}$, but redefines the measure to hold over those states considered as bad in the context of the VaR-RM. These bad states represent the abnormally adverse conditions against which the VaR risk manager chooses not to seek insurance, in contrast to the LEL risk manager who partially insures these states. For all parameter values examined below, this loss ratio is a conservative one in the sense of being a lower bound on the analogous ratio that uses the loss measure of Proposition 2.

Table 1 presents the loss ratio for varying levels of relative risk aversion γ , the VaR probability α , the LEL cap relative to the initial endowment $\epsilon/W(0)$, and the horizon T , under the maintained assumptions of CRRA preferences and log-normal state price density. The parameter values are chosen so as to capture reasonably realistic combinations, and thereby provide empirically relevant assessments of the loss difference between the two alternative risk-management practices.¹³ Inspection of the results in Table 1 establishes that

¹² As an aside, we may calculate the probability, $\alpha(\underline{W})$, of making a loss larger than $W(0) - \underline{W}$ for the benchmark and the LEL agents. We have

$$\alpha^{LEL}(\underline{W}) = P\left(\xi(T) > \frac{1}{(z_1 - z_2)\underline{W}^\gamma}\right) \leq P\left(\xi(T) > \frac{1}{z_1^B \underline{W}^\gamma}\right) = \alpha^B(\underline{W}).$$

Hence, the probability of a loss is also lowered by the LEL-RM strategy; to some extent, the LEL agent also manages his VaR.

¹³ The values of $\epsilon/W(0)$ used in Table 1 (0.5%, 1.0%, 2.0%) are chosen not only because they seem acceptable but also because they are meaningful if interpreted as being calibrated to the losses under VaR-RM. For example, with the fixed parameters in Table 1, relative risk aversion of 1.0, a three-month horizon, and VaR probability of 1.0%, 2.5%, and 5.0%, the present value of the loss under VaR-RM [i.e., the numerator in (16)] is 0.5%, 1.1%, and 1.7% of the initial endowment, respectively. Table 1 is therefore informative because it considers scenarios where the LEL parameters are set so that, when all states are considered, the LEL risk manager faces total expected losses of an order of magnitude similar to that when using VaR. For these parameter values, the table presents the loss ratio over the most adverse states, for which one may argue the regulators are attempting to prevent a financial meltdown.

Table 1
The ratio of losses under VaR-RM and LEL-RM

Horizon	Loss Ratio (%)								
	$\alpha\% = 1.0\%$			$\alpha\% = 2.5\%$			$\alpha\% = 5.0\%$		
	$\epsilon / W(0)\%$			$\epsilon / W(0)\%$			$\epsilon / W(0)\%$		
	0.5	1.0	2.0	0.5	1.0	2.0	0.5	1.0	2.0
Panel A: Relative risk aversion $\gamma = 0.5$									
$T = 0.25$	269	201	161	428	256	183	734	367	216
$T = 0.50$	316	219	171	620	318	207	1065	533	272
$T = 1.00$	419	243	177	883	442	238	1605	803	401
Panel B: Relative risk aversion $\gamma = 1.0$									
$T = 0.25$	196	152	123	244	157	127	340	192	131
$T = 0.50$	244	181	144	351	217	156	559	281	172
$T = 1.00$	311	213	162	559	289	188	993	496	255
Panel C: Relative risk aversion $\gamma = 2.0$									
$T = 0.25$	106	103	103	107	102	102	108	101	101
$T = 0.50$	146	113	104	164	115	104	198	118	102
$T = 1.00$	191	141	110	248	155	111	448	227	140

The table reports the loss ratio defined in (16), calculated for varying levels of relative risk aversion γ , the VaR probability $\alpha\%$, the LEL cap relative to the initial endowment $\epsilon/W(0)\%$, and the horizon T . The loss ratio, α , and $\epsilon/W(0)$ are stated in percentage points, and T in annual units. The analysis is for CRRA preferences and log-normal state price density. Panels A, B, and C present the results for γ taking the values 0.5, 1, and 2, respectively. In each panel $\alpha\% \in \{1.0\%, 2.5\%, 5.0\%\}$, $\epsilon/W(0)\% \in \{0.5\%, 1.0\%, 2.0\%\}$, and $T \in \{0.25, 0.50, 1.00\}$. The fixed parameter values are: $\bar{W}/W(0) = 0.9$, $r = 0.05$, $||\kappa|| = 0.4$, $\xi(0) = 1$. For all parameter values the VaR constraint is binding. The LEL constraint is binding for all values, except when $\gamma = 2$, $\epsilon/W(0)\% = 1.0\%$, and $T = 0.25$ and when $\gamma = 2$, $\epsilon/W(0)\% = 2.0\%$, and $T = 0.25$ or $T = 0.50$. The loss ratios between the VaR-RM and the B policy corresponding to the entries in the table range from 101% to 110%.

in the adverse states of the world the losses under VaR-RM are not only larger than under LEL-RM but are considerably so for most parameter values. In many realistic scenarios the loss ratio is found to be in the 200% to 1000% range. For example, for a logarithmic agent, with a 3-month horizon and a VaR probability of 5.0%, the present value of losses is almost double that under LEL-RM with $\epsilon/W(0)$ of 1.0%, and the ratio increases for lower risk aversion, tighter LEL cap, and longer horizon.

4. Equilibrium Implications of VaR-RM

Given that VaR-RM is becoming an industry standard, it is of interest to evaluate the impact of the presence of VaR risk managers on market prices. In this section, to examine price effects of VaR-RM, we develop a pure-exchange general equilibrium model of an economy containing VaR risk managers. Much attention has been directed toward understanding the impact of portfolio insurance on equilibrium prices [Brennan and Schwartz (1989), Donaldson and Uhlig (1993), Basak (1995, 1999), Grossman and Zhou (1996)], and given the relationship between VaR risk managers and portfolio insurers, a comparison of equilibrium effects is warranted.

4.1 The equilibrium setting

A problem with extending the economic setting in Section 1 to a standard pure-exchange general equilibrium model is that the VaR constraint is imposed directly on the agent's terminal wealth, and hence on his terminal consumption. In equilibrium, this imposes restrictions on the exogenous source that supplies the goods for the terminal consumption. Specifically, Proposition 1 (and Figure 1) revealed the VaR agent's wealth to be discontinuous, never taking values between $\underline{W}$ and $\bar{W}$. Therefore, good-market clearing would require a discontinuity in the exogenous terminal consumption source, which seems too contrived a primitive. To circumvent this problem, we instead assume that the VaR horizon, T , is shorter than the agent's lifetime, T' , so that the VaR horizon wealth, $W(T)$ (rather than equating to a lump-sum consumption), represents the value of future consumption. As a result, the VaR constraint is imposed on a quantity, which need not be directly provided by an exogenous consumption supply. A side benefit of this assumption is that it probably renders our model a more realistic description of VaR-RM, because in reality the VaR horizon would rarely coincide with the consumption horizon. To distinguish the setting here from that of Section 1, we refer to the VaR risk manager as the long-lived VaR agent. We will see that the basic optimal (partial equilibrium) behavior presented in Sections 2–3 survives under this modified setting.

We assume that the economy is populated by two types of agents, the normal agent (n) and the long-lived VaR agent (v), who derive utility from intertemporal (continuous) consumption over their lifetime $[0, T']$.¹⁴ As opposed to the normal agent, the long-lived VaR agent is subject to the additional VaR constraint (7) over time- T wealth, where $T < T'$. For simplicity, we specialize to both agents having logarithmic utility of consumption, and assume the (exogenously) given aggregate consumption process $\delta(t) \equiv \sum_{j=1}^N \delta_j(t)$ to follow a geometric Brownian motion process:

$$d\delta(t) = \delta(t) \left[\mu_\delta dt + \sum_{j=1}^N \sigma_{\delta_j} dw_j(t) \right], \quad \in [0, T'],$$

with $\mu_\delta, \sigma_{\delta_j}$ constants, and $\delta(0) > 0$.

We can anticipate [in light of Basak (1995)] that the constraint applied at the VaR horizon T may result in jumps in the equilibrium security and state prices. Hence, we need to modify accordingly our posited price dynamics in (1)–(2). We posit that the price dynamics in $[0, T)$ and $(T, T']$ are still given by (1)–(2), but at time T we allow for an additional jump component,

¹⁴ We find this formulation more appealing than letting the agents consume only at T' . A setting with intertemporal consumption is widely accepted as the more realistic one for dynamic general equilibrium modeling and has the advantage of having a main “work-horse” asset-pricing model, Lucas (1978), as a benchmark. Moreover, under standard preferences and endowment structure, this formulation offers added tractability because it results in a constant r and κ , as in the benchmark.

$\eta dA(t)$, in the changes of security prices. Here, $A(t)$ is a (right-continuous) step function defined by $A(t) \equiv 1_{\{t \geq T\}}$, so that $dA(t)$ is a measure assigning unit mass to time T , and the jump coefficient, η , is an $\mathcal{F}_T$ -measurable random variable related to the price jumps by

$$\begin{aligned}\eta &= \ln(B(T)/B(T-)) = \ln(S_j(T)/S_j(T-)) \\ &= \ln(\xi(T-)/\xi(T)), \quad j = 1, \dots, N,\end{aligned}\quad (17)$$

where $S_j(T-)$ is the left limit of $S_j(\cdot)$ at T . Notice that, because $\mathcal{F}_{T-} = \mathcal{F}_T$, to prevent arbitrage on these jumps, the jump coefficient η in all security prices must be the same so that the deflated prices and wealth, $\xi(t)B(t)$, $\xi(t)S_j(t)$, and $\xi(t)W_i(t)$, remain continuous at all times.

4.2 Optimization of a long-lived VaR agent

The long-lived VaR risk manager solves the following problem:

$$\begin{aligned}& \max_{(c_v, W(T-))} E \left[\int_0^{T'} \ln(c_v(t)) dt \right] \\ \text{subject to } & E \left[\int_0^T \xi(t)c_v(t) dt + \xi(T-)W_v(T-) \right] \leq \xi(0)W_v(0), \quad (18)\end{aligned}$$

$$E \left[\int_T^{T'} \xi(t)c_v(t) dt | \mathcal{F}_T \right] \leq \xi(T-)W_v(T-), \quad (19)$$

$$P(W_v(T-) \geq \underline{W}) \geq 1 - \alpha. \quad (20)$$

The static budget constraint is broken into two components, (18) and (19), to facilitate understanding of the impact of the VaR constraint (18) on the optimization problem. The VaR constraint is imposed on the left limit of time- T wealth to maintain the standard convention of right continuity of wealth processes. The optimal solutions, if they exist, for the long-lived VaR agent and the normal agent are summarized in Proposition 6.

Proposition 6. *The optimal consumption policies and time- T optimal wealth of the two agents are*

$$c_n(t) = \frac{1}{y_n \xi(t)}, \quad t \in [0, T'], \quad (21)$$

$$c_v(t) = \begin{cases} \frac{1}{y_{v1} \xi(t)}, & t \in [0, T), \\ \frac{1}{y_{v2} \xi(t)}, & t \in [T, T'], \end{cases} \quad (22)$$

$$W_n(T-) = \frac{T' - T}{y_n \xi(T-)}, \quad (23)$$

$$W_v(T-) = \begin{cases} \frac{T'-T}{y_{v1}\xi(T-)} & \text{if } \xi(T-) < \frac{T'-T}{y_{v1}\underline{W}}, \\ \underline{W} & \text{if } \frac{T'-T}{y_{v1}\underline{W}} \leq \xi(T-) < \bar{\xi}, \\ \frac{T'-T}{y_{v1}\xi(T-)} & \text{if } \bar{\xi} \leq \xi(T-) , \end{cases} \quad (24)$$

where the constants y_n , y_{v1} , and the $\mathcal{F}_T$ -measurable random variable y_{v2} satisfy

$$\frac{T'}{y_n} = \xi(0)W_n(0), \quad (25)$$

$$\frac{T'}{y_{v1}} + E\left[\left(\xi(T-)\underline{W} - \frac{T'-T}{y_{v1}}\right)1_{\{\frac{T'-T}{y_{v1}\underline{W}} \leq \xi(T-) < \bar{\xi}\}}\right] = \xi(0)W_v(0), \quad (26)$$

$$\frac{T'-T}{y_{v1}} + \left(\xi(T-)\underline{W} - \frac{T'-T}{y_{v1}}\right)1_{\{\frac{T'-T}{y_{v1}\underline{W}} \leq \xi(T-) < \bar{\xi}\}} = \frac{T'-T}{y_{v2}}, \quad (27)$$

and $\bar{\xi}$ is defined by $P(\xi(T-) > \bar{\xi}) \equiv \alpha$.

The solution for the VaR-horizon wealth of the long-lived VaR agent, (24), is analogous to (9), and the intuition for the solution, discussed in Section 2.1, prevails. The only new aspect in which the long-lived VaR agent differs from the normal agent is that he is given differing “weighting” before (y_{v1}) and after (y_{v2}) the VaR horizon. When the VaR constraint is binding, $y_{v1} > y_{v2}$ in states where the agent is insuring himself. This resembles the result in Basak (1995) for the portfolio insurer, the idea being that posthorizon consumption not only provides the VaR risk manager with utility but also contributes toward meeting his VaR constraint.

4.3 Equilibrium state prices

We now define and then characterize the equilibrium in our setting.

Definition 1. An equilibrium is a collection of (r, μ, σ, η) and optimal $(c_n, c_v, \theta_n, \theta_v)$, such that the good, stock, and bond markets clear, that is, $\forall t \in [0, T']$,

$$c_n(t) + c_v(t) = \delta(t), \quad (28)$$

$$\theta_{nj}(t) + \theta_{vj}(t) = S_j(t), \quad j = 1, \dots, N, \quad (29)$$

$$W_n(t) + W_v(t) = \sum_{j=1}^N S_j(t). \quad (30)$$

Proposition 7 solves for the equilibrium state price density and its dynamics.

Proposition 7. The equilibrium state price density is given by

$$\xi(t) = \begin{cases} (y_n^{-1} + y_{v1}^{-1})\delta(t)^{-1}, & t \in [0, T) \\ (y_n^{-1} + y_{v2}^{-1})\delta(t)^{-1}, & t \in [T, T'], \end{cases} \quad (31)$$

where y_n, y_{v1}, y_{v2} satisfy (25)–(27), with (31) substituted in. Moreover, the equilibrium interest rate and market price of risk are constants, at all $t \in [0, T']$, given by $r = \mu_\delta - \|\sigma_\delta\|^2$, and $\kappa_j = \sigma_{\delta,j}$, $j = 1, \dots, N$, and the jump-size parameter is $\eta = \ln((y_n^{-1} + y_{v1}^{-1})/(y_n^{-1} + y_{v2}^{-1})) \leq 0$.

Proposition 7 reveals the anticipated (upward) jump in ξ at time T ; the price of consumption, ξ , jumps up to counteract the upward jump in aggregate consumption demand at the time- T insured states, where the jump in demand is due to the VaR risk manager no longer postponing consumption to meet the VaR constraint.

4.4 Equilibrium market price, volatility, and risk premium

The price of the equity market portfolio, W_{em} , is defined as the aggregate optimally invested wealth in the risky securities. In equilibrium, W_{em} is also equal to both the aggregate optimally invested wealth and the sum of the risky asset prices:

$$W_{em}(t) \equiv \sum_{j=1}^N (\theta_{nj}(t) W_n(t) + \theta_{vj}(t) W_v(t)) = W_n(t) + W_v(t) = \sum_{j=1}^N S_j(t).$$

The equilibrium market dynamics can be represented by

$$dW_{em}(t) + \delta(t) dt = W_{em}(t) \left[\mu_{em}(t) dt + \sum_{j=1}^N \sigma_{em,j}(t) dw_j(t) + \eta dA(t) \right],$$

where μ_{em} is the equity market drift and $\|\sigma_{em}(t)\| = \sqrt{\sum_{j=1}^N \sigma_{em,j}^2(t)}$ is the equity market volatility. Proposition 8 presents these quantities in equilibrium and contrasts them with the benchmark (B) economy with all normal agents.¹⁵

Proposition 8. *The equilibrium market price, volatility, and risk premium in a logarithmic-utility normal-agent benchmark economy are given, $\forall t \in [0, T']$, by*

$$W_{em}^B(t) = (T' - t)\delta(t), \quad \|\sigma_{em}^B(t)\| = \|\sigma_\delta\|, \quad \mu_{em}^B(t) - r = \|\sigma_\delta\|^2.$$

Before the VaR horizon, the corresponding quantities in the economy with one logarithmic-utility long-lived VaR agent and one logarithmic-utility normal

¹⁵ Although not the focus of our discussion, we note that, under appropriate restrictions on exogenous parameters, existence of equilibrium [demonstrated via existence of the y 's in (25)–(27)] can be straightforwardly verified.

agent are

$$\begin{aligned}
 W_{em}^{VaR}(t) = & (T' - t)\delta(t) \\
 & - \left[\frac{T' - T}{y_{v1}} \delta(t) \mathcal{N}(-\hat{d}_1(\bar{\delta})) - \underline{W} e^{-(\mu_{\delta} - \|\sigma_{\delta}\|^2)(T-t)} \mathcal{N}(-\hat{d}_2(\bar{\delta})) \right] \\
 & + \left[\frac{T' - T}{y_{v1}} \delta(t) \mathcal{N}(-\hat{d}_1(\underline{\delta})) \right. \\
 & \quad \left. - \underline{W} e^{-(\mu_{\delta} - \|\sigma_{\delta}\|^2)(T-t)} \mathcal{N}(-\hat{d}_2(\underline{\delta})) \right], \tag{32}
 \end{aligned}$$

$$\begin{aligned}
 \|\sigma_{em}^{VaR}(t)\| &= \hat{q}(t)\|\sigma_{\delta}\|, \\
 \mu_{em}^{VaR}(t) - r &= \hat{q}(t)\|\sigma_{\delta}\|^2,
 \end{aligned}$$

where

$$\begin{aligned}
 \bar{\delta} &\equiv \frac{W y_{v1}}{T' - T}, \\
 \underline{\delta} &\equiv 1/\bar{\xi}, \\
 \hat{d}_1(x) &\equiv \frac{\ln \frac{\delta(t)}{x} + (\mu_{\delta} - \frac{1}{2}\|\sigma_{\delta}\|^2)(T-t)}{\|\sigma_{\delta}\|\sqrt{T-t}}, \\
 \hat{d}_2(x) &\equiv \hat{d}_1(x) - \|\sigma_{\delta}\|\sqrt{T-t}, \\
 \hat{q}(t) &\equiv 1 - \frac{W e^{-(\mu_{\delta} - \|\sigma_{\delta}\|^2)(T-t)} (\mathcal{N}(-\hat{d}_2(\bar{\delta})) - \mathcal{N}(-\hat{d}_2(\underline{\delta})))}{W_{em}^{VaR}(t)} \\
 &\quad + \frac{(\underline{W} - \frac{1}{y_{v1}}(T' - T)\underline{\delta}) e^{-(\mu_{\delta} - \|\sigma_{\delta}\|^2)(T-t)} \phi(\hat{d}_2(\underline{\delta}))}{W_{em}^{VaR}(t)\|\sigma_{\delta}\|\sqrt{T-t}}.
 \end{aligned}$$

After the VaR horizon, market prices, volatility, and risk premia in both economies are identical. Consequently, before the VaR horizon,

- (i) $W_{em}^{VaR}(t) > W_{em}^B(t)$,
- (ii) $\|\sigma_{em}^{VaR}(t)\| > \|\sigma_{em}^B(t)\|$ and $\mu_{em}^{VaR}(t) > \mu_{em}^B(t)$ if, and only if, $\delta(t) < \delta^*(t)$, where $\delta^*(t)$ is deterministic and bounded:

$$\underline{\delta} e^{-(\mu_{\delta} - \|\sigma_{\delta}\|^2/2)(T-t)} \leq \delta^*(t) \leq \sqrt{\underline{\delta} \bar{\delta}} e^{-(\mu_{\delta} - \|\sigma_{\delta}\|^2/2)(T-t)} e^{\|\sigma_{\delta}\|^2(T-t)}.$$

Item (i) reveals the prehorizon market price in the VaR economy to be higher than in the benchmark economy. This result is as in the PI economy and comes about because the long-lived VaR agent values posthorizon dividends more than the prehorizon consumption, because these dividends help

him meet his constraint. The prehorizon value of the equity market is then pushed up because equities are claims against the posthorizon dividends.

When the VaR agent behaves like a portfolio insurer ($\alpha = 0$), it is immediate to verify that $\hat{q}(t) \in [0, 1]$, and equity market volatility is never higher than in the B case, as indeed was shown by Basak (1995). Otherwise, as long as the VaR constraint is binding ($\bar{\delta} > \delta$), item (ii) reveals that there are always states of the world in which the VaR economy stock volatility is higher than in the benchmark. This is a consequence of the risky asset demands of the VaR agent, discussed in Section 2.2. Because the interest rate and the market price of risk are pinned down as constants in equilibrium, favorability of the risky equity market relative to the bond is controlled by its volatility. Whenever the presence of the VaR agent elevates the demand for risky assets, the market volatility will increase to compensate (so to clear markets) and conversely when the VaR agent depresses the demand for risky assets. When the market volatility is increased (decreased), for the market price of risk to remain unchanged, the market risk premium must also increase (decrease) accordingly. Furthermore, item (ii) implies that the increased volatility arises in states of low output, or down stock markets, or more specifically, in the transition from the intermediate states of the world to the bad states. Indeed, the market volatility behavior [as a function of $1/\delta(t)$] inherits the S-shaped form of the demand for risky assets [as a function of $\xi(t)$] seen in Figure 3b.

Note that the equilibrium analysis provides a justification for our identification of low (high) $\xi(t)$ with good (bad) states of the world. Equation (31) reveals $\xi(t)$, the price of consumption, to be decreasing in the consumption supply $\delta(t)$, and (32) reveals the equity market value to be increasing in $\delta(t)$. Hence, what we call “good (bad)” states are those associated with high (low) aggregate output and with high (low) equity prices.

5. Conclusion

We analyze the effects of risk management on optimal wealth and consumption choices and on optimal portfolio policies. We first focus on modeling risk managers as expected utility maximizers, who derive utility from wealth at some horizon and who must comply with a VaR constraint imposed at that horizon, requiring that the wealth may decrease below a given floor only with a prespecified probability. Having embedded VaR into an optimizing framework, we reveal several surprising effects, some of which may be viewed as undesirable by regulators. In particular, VaR risk managers incur larger losses than non-risk managers in the most adverse states of the world. To address that, we next propose an alternative model of risk management, LEL-RM, where expected losses (rather than the probability of losses) are limited. We demonstrate how this alternative model remedies the shortcomings of VaR-RM. In particular, we show that for many empirically relevant scenarios the

expected losses under VaR-RM may range from being two to as high as 10 times larger than under LEL-RM.

Both the partial equilibrium and the general equilibrium analyses of the economy with VaR risk managers yield profoundly different implications compared to the extensively studied case of portfolio insurance: VaR risk managers differ from portfolio insurers both in their endogenously chosen quantities and in their impact on equilibrium prices. In particular, in the worse states of the world, the VaR agents may take on more risk than non-risk managers and consequently increase the stock market volatility, which is exactly the opposite behavior and impact on volatility as compared with portfolio insurers.

In related work, we have also analyzed the case of the VaR constraint applied repeatedly over time. There, the optimal policy retains the structure of the single-constraint case, and many results extend qualitatively (details available upon request). Although in this article we demonstrate how to embed two particular forms of risk management into an optimizing framework, our analysis may also pave the way toward evaluation of further alternative risk-management practices of interest to regulators. In particular, there is room to consider risk-management models that require agents to focus on the higher moments of the distribution of a loss. For example, from an econometric perspective, volatilities can be estimated more efficiently than means; it is therefore of interest to compare the LEL-RM framework with one that binds the second moment of a loss, which may be an easier framework to implement in practice. In studying risk-management practices, within the paradigms underlying our analysis, it is also of interest to address issues pertinent to the presence of credit risk, where the debtor has an option to default. Toward that end, a setting such as that examined by Basak and Shapiro (2000) could be adopted.

Appendix

Proof of Proposition 1. Let $\hat{W}(T) = W^{VaR}(T)$. If $P(\hat{W}(T) < \underline{W}) < \alpha$, then by their definition, $\bar{\xi} < \underline{\xi}$, and $W^{VaR}(T) = I(y\xi(T)) = W^B(T)$, which is optimal following the standard arguments as in the benchmark case. Otherwise, $P(\hat{W}(T) < \underline{W}) = \alpha$, and $\bar{\xi} \geq \underline{\xi}$. The remainder of the proof is for the latter case. We adapt the common convex-duality approach [see, for example, Karatzas and Shreve (1998)] to incorporate the VaR constraint. The expression in Lemma 1 is the convex conjugate of u with an additional term capturing the VaR constraint.

Lemma 1. *Expression (9) solves the following pointwise problem $\forall \xi(T)$:*

$$u(\hat{W}(T)) - y\xi(T)\hat{W}(T) + y_2 1_{\{\hat{W}(T) \geq \underline{W}\}} = \max_W \{u(W) - y\xi(T)W + y_2 1_{\{W \geq \underline{W}\}}\},$$

where $y_2 \equiv u(I(y\bar{\xi})) - y\bar{\xi}I(y\bar{\xi}) - u(\underline{W}) + y\bar{\xi}\underline{W} \geq 0$.

Proof. The function on which $\max\{\cdot\}$ operates is not concave in W , but can only exhibit local maxima at $W = I(y\bar{\xi}(T))$ and/or $W = \underline{W}$. To find the global maximum, we need to

compare the value of these two local maxima. When $\xi(T) < \underline{\xi}$, we have $I(y\xi(T)) > \underline{W}$ and

$$u(I(y\xi(T))) - y\xi(T)I(y\xi(T)) + y_2 > u(\underline{W}) - y\xi(T)\underline{W} + y_2,$$

so $I(y\xi(T))$ is the global maximum. When $\underline{\xi} \leq \xi(T) < \bar{\xi}$, we have $I(y\xi(T)) \leq \underline{W}$ and

$$\begin{aligned} u(\underline{W}) - y\xi(T)\underline{W} + y_2 &= u(I(y\bar{\xi})) - y\bar{\xi}I(y\bar{\xi}) + y\underline{W}(\bar{\xi} - \xi(T)) \\ &> u(I(y\xi(T))) - y\xi(T)I(y\xi(T)), \end{aligned} \quad (A1)$$

where the inequality follows from $\xi(T) < \bar{\xi}$ and $\frac{\partial}{\partial \xi} \{u(I(y\xi)) - y\xi I(y\xi) + y\underline{W}\xi\} = -yI(y\xi) + y\underline{W} \geq 0$ whenever $\xi \geq \underline{\xi}$. So $\underline{W}$ is the global maximum. When $\xi(T) \geq \bar{\xi}$, the inequality in (A1) is reversed and so $I(y\bar{\xi}(T))$ is the global maximum. Finally, to show $y_2 \geq 0$, note that

$$y_2 = [u(I(y\bar{\xi})) - y\bar{\xi}I(y\bar{\xi}) + y\underline{W}\bar{\xi}] - [u(I(y\bar{\xi})) - y\bar{\xi}I(y\bar{\xi}) + y\underline{W}\bar{\xi}] \geq 0,$$

again from $\frac{\partial}{\partial \xi} \{u(I(y\xi)) - y\xi I(y\xi) + y\underline{W}\xi\} \geq 0$ and $\bar{\xi} \geq \underline{\xi}$. ■

Now, let $W(T)$ be any candidate optimal solution, which satisfies the VaR constraint (7) and the static budget constraint (8). We have

$$\begin{aligned} &E[u(\hat{W}(T))] - E[u(W(T))] \\ &= E[u(\hat{W}(T))] - E[u(W(T))] - y\xi(0)W(0) + y\xi(0)W(0) + y_2(1 - \alpha) - y_2(1 - \alpha) \\ &\geq E[u(\hat{W}(T))] - E[u(W(T))] - E[y\xi(T)\hat{W}(T)] + E[y\xi(T)W(T)] \\ &\quad + E[y_2 1_{\{\hat{W}(T) \geq \underline{W}\}}] - E[y_2 1_{\{W(T) \geq \underline{W}\}}] \geq 0, \end{aligned}$$

where the former inequality follows from the static budget constraint and the VaR constraint holding with equality for $\hat{W}(T)$, while holding with inequality for $W(T)$. The latter inequality follows from Lemma 1. Hence $\hat{W}(T)$ is optimal.¹⁶ Finally, because the VaR constraint must hold with equality, we deduce the definition of $\bar{\xi}$. From (9) it is clear that $\partial W^{VaR}(T; y)/\partial \alpha|_y < 0$, and in particular $W^{PI}(T; y) \geq W^{VaR}(T; y)$. Furthermore, except when equal to $\underline{W}$, all wealth policies are decreasing in y . Hence, to allow the static budget constraint hold with equality, we must have y decreasing in α and $y \in [y^B, y^{PI}]$. ■

Proof of Proposition 2.

- (i) It is straightforward to verify that $L_1(W^B) = G_1(a_B, y^B)$, $L_1(W^{VaR}) = G_1(a_V, y)$, where

$$\begin{aligned} G_1(a, x) &= \underline{W}\mathcal{N}(a) - x^{-\frac{1}{\gamma}} e^{\left(\frac{m}{\gamma} + \frac{x^2}{2\gamma^2}\right)} \mathcal{N}\left(a - \frac{s}{\gamma}\right), \\ m &= E[-\ln \xi(T)], \quad s^2 = \text{Var}[-\ln \xi(T)], \\ a_B &= \frac{(\ln \underline{W}^\gamma y^B - m)}{s}, \quad a_V = \frac{(\ln \underline{W}^\gamma y - m)}{s}, \end{aligned}$$

and y solves $E[\xi(T)I(y\xi(T))] = \xi(0)W(0)$. Next, it is also straightforward to show that, for $x > 0$, $\frac{\partial}{\partial a} G_1(a, x) \geq 0$ if, and only if, $a \leq a_V$. Hence, because $a_B \leq a_V$,

¹⁶ The optimization problem is not standard because it is nonconcave; to gain insight into its structure, we found it satisfactory to provide a general proof of sufficiency for optimality. To prove existence, one has to follow the standard path of stating and verifying conditions for integrability of wealth, prices, and portfolio holdings (as mentioned in note 4). In addition, one has to present the appropriate growth conditions on u and moment conditions on ξ , followed by an elaborate analysis to verify that the expectations in the objective function and in the budget constraint are well defined [e.g., as in Cox and Huang (1991)]. To prevent diverting the focus with a series of technical conditions and to not unnecessarily lengthen the article, we chose to solve explicit examples of interest, instead of proving existence in general.

$G_1(a_B, y) \leq G_1(a_V, y)$. Also, as $\frac{\partial}{\partial x} G_1(a, x) \geq 0$ and $y \geq y^B$, $G_1(a, y^B) \leq G_1(a, y)$. Then,

$$\begin{aligned} L_1(W^{VaR}) - L_1(W^B) &= G_1(a_V, y) - G_1(a_B, y^B) \\ &\geq G_1(a_V, y) - G_1(a_B, y) \geq 0. \end{aligned}$$

- (ii) It is straightforward to verify that $L_2(W^B) = G_2(a_B, y^B)$, $L_2(W^{VaR}) = G_2(a_V, y)$, where

$$\begin{aligned} G_2(a, x) &= \left(\underline{W} e^{-m + \frac{\gamma^2}{2}} \mathcal{N}(a + s) - x^{-\frac{1}{\gamma}} e^{\Gamma} \mathcal{N}\left(a - \frac{1 - \gamma}{\gamma} s\right) \right) / \xi(0), \\ \Gamma &= \frac{1 - \gamma}{\gamma} m + \left(\frac{1 - \gamma}{\gamma} \right)^2 \frac{s^2}{2}, \end{aligned}$$

a_B , a_V , as in part (i). Also, for $x > 0$, $\frac{\partial}{\partial a} G_2(a, x) \geq 0$ if, and only if, $a \leq a_V$, and since $\frac{\partial}{\partial x} G_2(a, x) \geq 0$, $G_2(a, y^B) \leq G_2(a, y)$. Therefore,

$$\begin{aligned} L_2(W^{VaR}) - L_2(W^B) &= G_2(a_V, y) - G_2(a_B, y^B) \\ &\geq G_2(a_V, y) - G_2(a_B, y) \geq 0. \end{aligned} \quad \blacksquare$$

Proof of Proposition 3.

- (i) From (3) and (4), Itô's lemma implies that $\xi(t) W^{VaR}(t)$ is a martingale:

$$W^{VaR}(t) = E \left[\frac{\xi(T)}{\xi(t)} W^{VaR}(T) | \mathcal{F}_t \right]. \quad (A2)$$

When r and κ are constant, conditional on $\mathcal{F}_t$, $\ln \xi(T)$ is normally distributed with mean $\ln \xi(t) - (r + \frac{\|\kappa\|^2}{2})(T - t)$ and variance $\|\kappa\|^2(T - t)$. Substituting (9) into (A2), using $I(x) = x^{-\frac{1}{\gamma}}$ and evaluating the conditional expectations over each of the three regions of $\xi(T)$ yields (10).

- (ii) Applying Itô's lemma to (10), using $\kappa = \sigma(t)^{-1}(\mu(t) - r\mathbf{1})$, we get

$$\begin{aligned} \sigma_W^{VaR}(t) &= \frac{1}{\gamma} \frac{e^{\Gamma(t)}}{(y\xi(t))^{\frac{1}{\gamma}}} \left[1 - \mathcal{N}(-d_1(\underline{\xi})) + \mathcal{N}(-d_1(\bar{\xi})) \right. \\ &\quad \left. + \frac{\gamma(\underline{W} - \underline{W})e^{-r(T-t) - \Gamma(t)}\phi(d_2(\bar{\xi}))}{(y\xi(t))^{-\frac{1}{\gamma}} \|\kappa\| \sqrt{T-t}} \right] \kappa. \end{aligned}$$

From (4), $\sigma_W^{VaR}(t)$ must equal $\sigma(t)^\top \theta^{VaR}(t) W^{VaR}(t)$. Using the well-known value of θ^B , we obtain

$$\begin{aligned} q^{VaR}(t) &= \frac{e^{\Gamma(t)}}{W^{VaR}(t)(y\xi(t))^{\frac{1}{\gamma}}} \left[1 - \mathcal{N}(-d_1(\underline{\xi})) + \mathcal{N}(-d_1(\bar{\xi})) \right. \\ &\quad \left. + \frac{\gamma(\underline{W} - \underline{W})e^{-r(T-t) - \Gamma(t)}\phi(d_2(\bar{\xi}))}{(y\xi(t))^{-\frac{1}{\gamma}} \|\kappa\| \sqrt{T-t}} \right] \quad (A3) \end{aligned}$$

Rearranging (A3) yields (11).

- (iii) Inspection of (A3) clearly reveals that it is nonnegative. The limits are immediate to verify.

- (iv) For a given t , to save notation, we suppress the dependence of ξ , q^{VaR} , and W^{VaR} on t . The proof first establishes the existence of ξ^* , for a given t , by explicitly computing a region (in the ξ -space) within which q^{VaR} rises, as a function of ξ , from below to above 1. Then, uniqueness of ξ^* is established. As stated in the proposition, the above region is defined in terms of two sufficient conditions: the first is that $q^{VaR} < 1$ if $\xi < \sqrt{\xi\xi}e^{(r-||\kappa||^2/2)(T-t)}$, the second is that $q^{VaR} > 1$ if $\xi > \bar{\xi}e^{(r-||\kappa||^2/2)(T-t)}e^{||\kappa||^2/\gamma(T-t)}$. For brevity, we only present the proof of the former, as the proof of the latter follows similar steps. For $X \in [\underline{W}, \underline{W}]$, let

$$F(X, \xi) \equiv \frac{\gamma(X - \underline{W})\phi(d_2(\bar{\xi}))}{||\kappa||\sqrt{T-t}} - X(\mathcal{N}(-d_2(X^{-\gamma}/y)) - \mathcal{N}(-d_2(\bar{\xi}))) .$$

Note that $d_2(X^{-\gamma}/y)$ and $d_2(\bar{\xi})$ are functions of ξ , and that $q^{VaR} = 1 + e^{-r(T-t)} F(\underline{W}, \xi)/W^{VaR}$. Hence, for a given t and ξ , $q^{VaR} < 1$ if, and only if, $F(\underline{W}, \xi) < 0$. For analytical tractability, we only derive a sufficient condition for $F(\underline{W}, \xi) < 0$. Noting that $F(\underline{W}, \xi) = 0$, a sufficient condition for $F(\underline{W}, \xi) < 0$ is that $\frac{\partial}{\partial X} F(X, \xi) < 0$, $\forall X \in [\underline{W}, \underline{W}]$. It is straightforward to verify that a sufficient condition for $\frac{\partial}{\partial X} F(X, \xi) < 0$, $\forall X \in [\underline{W}, \underline{W}]$, is that $\xi < \sqrt{\xi\xi}e^{(r-||\kappa||^2/2)(T-t)}$. But, because $\xi = 1/yX^\gamma \leq 1/yX^\gamma$, $\forall X \in [\underline{W}, \underline{W}]$, the latter inequality holds when $\xi < \sqrt{\xi\xi}e^{(r-||\kappa||^2/2)(T-t)}$. To summarize: $\xi < \sqrt{\xi\xi}e^{(r-||\kappa||^2/2)(T-t)} \Rightarrow \xi < \sqrt{\xi/yX^\gamma}e^{(r-||\kappa||^2/2)(T-t)}$, $\forall X \in [\underline{W}, \underline{W}] \Rightarrow \partial/\partial X F(X, \xi) < 0$, $\forall X \in [\underline{W}, \underline{W}] \Rightarrow F(\underline{W}, \xi) < 0 \Rightarrow q^{VaR} < 1$. This, combined with the condition for $q^{VaR} > 1$, using the limits in (iii), and the fact that $F(\underline{W}, \xi)$ is differentiable with respect to ξ imply that there exists a ξ^* for which $F(\underline{W}, \xi^*) = 0$, and there exist ξ_L, ξ_H satisfying $\xi_L < \xi^* < \xi_H$ for which $F(\underline{W}, \xi_L) < 0 < F(\underline{W}, \xi_H)$ and $\frac{\partial}{\partial \xi} F(\underline{W}, \xi_L) = \frac{\partial}{\partial \xi} F(\underline{W}, \xi_H) = 0$. To complete the proof, we need to show that ξ^* is unique. To prove this, it is enough, in our setting, to verify that $\frac{\partial}{\partial \xi} F(\underline{W}, \xi) = 0$ has at most two distinct roots. To verify the latter, note that $\frac{\partial}{\partial \xi} F(\underline{W}, \xi) = 0$ if, and only if, $f(\underline{W}, d_2(\bar{\xi})) = 0$, where $f(\underline{W}, h) = a_1h - a_2e^{a_3h} + \underline{W}$, and $a_1 = \gamma(\underline{W} - \underline{W})/||\kappa||\sqrt{T-t}$, $a_2 = \underline{W}e^{-\frac{a_3^2}{2}}$, $a_3 = \gamma \ln(\underline{W}/\underline{W})/||\kappa||\sqrt{T-t}$, are all positive and independent of ξ . But $\frac{\partial^2}{\partial h^2} f(\underline{W}, h) = -a_2a_3^2e^{a_3h} < 0$, for all h , and hence $f(\underline{W}, h) = 0$ has no more than two distinct roots: h_1, h_2 . Assume, without loss of generality, $h_1 > h_2$. Because there is a one-to-one mapping between $d_2(\bar{\xi})$ and ξ , $\xi_L = \bar{\xi}e^{(r-||\kappa||^2/2)(T-t)-h_1||\kappa||\sqrt{T-t}}$ and $\xi_H = \bar{\xi}e^{(r-||\kappa||^2/2)(T-t)-h_2||\kappa||\sqrt{T-t}}$ are the unique, global minimizer and maximizer, respectively, of F . ■

Proof of Proposition 4. This is the direct analog of Proposition 1. Let $\hat{W}(T) = W^{LEL}(T)$. If $E[\xi(T)(\underline{W} - \hat{W}(T))1_{\{\hat{W}(T) \leq \underline{W}\}}] < \epsilon$, then $z_2 = 0$ and $\bar{\xi}_\epsilon = \xi_\epsilon$, and $W^{LEL}(T) = I(z_1\xi(T)) = W^B(T)$, which is optimal following the standard arguments. Otherwise, $E[\xi(T)(\underline{W} - \hat{W}(T))1_{\{\hat{W}(T) \leq \underline{W}\}}] = \epsilon$, and $\bar{\xi}_\epsilon \geq \xi_\epsilon$. The remainder of the proof is for the latter case.

Lemma 2. Expression (14) solves the following pointwise problem $\forall \xi(T)$:

$$\begin{aligned} & u(\hat{W}(T)) - z_1\xi(T)\hat{W}(T) - z_2\xi(T)(\underline{W} - \hat{W}(T))1_{\{\hat{W}(T) < \underline{W}\}} \\ & = \max_W \{u(W) - z_1\xi(T)W - z_2\xi(T)(\underline{W} - W)1_{\{W < \underline{W}\}}\}. \end{aligned}$$

Proof. The function on which $\max\{\cdot\}$ operates is not concave in W but can only exhibit local maxima at $W = I(z_1\xi(T))$ if $I(z_1\xi(T)) \geq \underline{W}$, or $W = I((z_1 - z_2)\xi(T))$ if $I((z_1 - z_2)\xi(T)) < \underline{W}$, or $W = \underline{W}$. When $\xi(T) < \xi_\epsilon$, $I(z_1\xi(T)) > \underline{W}$, $I((z_1 - z_2)\xi(T)) > \underline{W}$, and

$$u(I(z_1\xi(T))) - z_1\xi(T)I(z_1\xi(T)) > u(\underline{W}) - z_1\xi(T)\underline{W},$$

so $I(z_1\xi(T))$ is the global maximum. When $\xi(T) \geq \bar{\xi}_\epsilon$, $I((z_1 - z_2)\xi(T)) < \underline{W}$, $I(z_1\xi(T)) < \underline{W}$ and

$$u(I((z_1 - z_2)\xi(T))) - (z_1 - z_2)\xi(T)I((z_1 - z_2)\xi(T)) \geq u(\underline{W}) - (z_1 - z_2)\xi(T)\underline{W},$$

so $I((z_1 - z_2)\xi(T))$ is the global maximum. When $\xi_\epsilon \leq \xi(T) < \bar{\xi}_\epsilon$, $I(z_1\xi(T)) \leq \underline{W}$, $I((z_1 - z_2)\xi(T)) > \underline{W}$, so $W = \underline{W}$ is the only local maximum and hence the solution. ■

Now, let $W(T)$ be any candidate optimal solution, which satisfies the static budget constraint and the LEL constraint in (12). We have

$$\begin{aligned} & E[u(\hat{W}(T))] - E[u(W(T))] \\ &= E[u(\hat{W}(T))] - E[u(W(T))] - z_1\xi(0)W(0) + z_1\xi(0)W(0) - z_2\epsilon + z_2\epsilon \\ &\geq E[u(\hat{W}(T))] - E[u(W(T))] - E[z_1\xi(T)\hat{W}(T)] + E[z_1\xi(T)W(T)] \\ &\quad - E[z_2\xi(T)(\underline{W} - \hat{W}(T))1_{\{\hat{W}(T) \leq \underline{W}\}}] + E[z_2\xi(T)(\underline{W} - W(T))1_{\{W(T) \leq \underline{W}\}}] \geq 0, \end{aligned}$$

where the former inequality follows from the static budget constraint and the LEL constraint holding with equality for $\hat{W}(T)$, while holding with inequality for $W(T)$. The latter inequality follows from Lemma 1. Hence, $\hat{W}(T)$ is optimal. Suppose $z_1 > z_1^{PI}$. Then $W^{PI}(T) > W^{LEL}(T)$ in all states, contradicting the budget constraint holding with equality for both. Hence by contradiction, $z_1 \leq z_1^{PI}$. Suppose $z_1 - z_2 \leq z_1 < z_1^B$. Then $W^{LEL}(T) > W^B(T)$ in all states. Similarly, if $z_1^B < z_1 - z_2 < z_1$, then $W^{LEL}(T) < W^B(T)$ in all states. Either case contradicts the budget constraint holding with equality for both, so we must have $z_1 - z_2 \leq z_1^B \leq z_1$. ■

Proof of Proposition 5. The proof is as of Proposition 3, except with ξ and $\bar{\xi}$ replaced appropriately by ξ_ϵ and $\bar{\xi}_\epsilon$. ■

Proof of Proposition 6. Equations (21), (23), and (25) are well known to solve the unconstrained optimization. To show that (22), (24), (26), and (27) are the optimal solution to the optimization problem of the long-lived VaR agent is a straightforward extension of the proof of Proposition 1 and is therefore omitted. ■

Proof of Proposition 7. Equation (31) follows from the clearing of the consumption good market. Then, r and κ are determined by applying Itô's lemma to (31) and equating terms with (3), and η follows by substituting (31) into (17). ■

Proof of Proposition 8. In Equations (25)–(27), the y 's are only determined up to a multiplicative constant, and we therefore, without loss of generality, set $y_n^{-1} + y_v^{-1} = 1$.¹⁷ The expression for $W_{em}^{VaR}(t)$ follows by substituting $(T' - T)\delta(t)/y_{v1}$, $\|\sigma_\delta\|$, $\mu_\delta - \|\sigma_\delta\|^2$, for $1/y\xi(t)$, $\|\kappa\|$, r , respectively, in the time- t wealth equation (10) of Proposition 3, and adding the $(T' - t)\delta(t)$ term to account for intermediate consumption. Applying Itô's lemma to $W_{em}^{VaR}(t)$ yields the expressions for $\|\sigma_{em}^{VaR}(t)\|$, $\mu_{em}^{VaR}(t)$. To show property (i), use (23)–(24) and (31) to note that when $\hat{W}_v(T-) \neq \underline{W}$ then $\hat{W}_n(T-) + \hat{W}_v(T-) = (T' - T)\delta(T-)$, and when $\hat{W}_v(T-) = \underline{W}$, then $\hat{W}_n(T-) + \hat{W}_v(T-) > (T' - T)\delta(T-)$. Hence, $W_{em}^{VaR}(T-) > W_{em}^B(T-)$, which implies (i). Property (ii) follows by substituting the appropriate equilibrium quantities in part (iv) of Proposition 3. ■

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¹⁷ This normalization is purely for expositional convenience; we could alternatively have adopted the normalization $y_n^{-1} + y_v^{-1} = \delta(0)^{-1}$, so that $\xi(0) = 1$, without affecting any of our conclusions.

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