316-466 Monetary Economics — Midsemester Exam

1. **Welfare costs of inflation**. (30 marks). Consider the following money in the utility function model. Time is discrete and there is no uncertainty. There is a single representative consumer with time-separable preferences over consumption and real balances

$$\sum_{t=0}^{\infty} \beta^t U(c_t, m_t), \qquad 0 < \beta < 1$$

where $m_t \equiv M_{t+1}/P_t$ denotes end-of-period real money balances. The flow budget constraint of the consumer is

$$P_t c_t + M_{t+1} + q_t B_{t+1} \leq P_t y_t + M_t + B_t - T_t$$

where P_t denotes the price level, $q_t \equiv 1/(1+i_t)$ denotes the price of a one-period nominal bond, y_t is a given endowment, and T_t denotes lump-sum taxes paid to the government. The given initial conditions are $M_0 > 0$ and $B_0 = 0$.

   (a) (5 marks). Show that the consumer's first order condition characterizing money demand can be written

$$\frac{U_m(c_t, m_t)}{U_c(c_t, m_t)} = \frac{i_t}{1+i_t}$$

   What is the economic intuition for this condition? What does the term $i_t/(1+i_t)$ measure?

   (b) (5 marks). Suppose that the period utility function is

$$U(c,m) = \frac{1}{1-\sigma}\left\{\left[c\varphi\left(\frac{m}{c}\right)\right]^{1-\sigma} - 1\right\}, \qquad \sigma > 0$$

   where σ denotes the inverse of the intertemporal elasticity of substitution and where the function φ is strictly increasing and concave. Show that this utility function implies that money demand is characterized by

$$\frac{\varphi'(x_t)}{\varphi(x_t) - x_t\varphi'(x_t)} = \frac{i_t}{1+i_t}$$

   where $x_t \equiv m_t/c_t$.

   (c) (5 marks). Suppose that this economy is in a steady state with output normalized to $y = 1$. This implies steady state welfare is maximized when $U(1,m)$ is maximized. What nominal interest rate maximizes steady state welfare. Denote by $m^\star$ the real balances at this welfare maximizing level. Characterize $m^\star$ in terms of the function φ. [Hint: $U(1,m)$ is strictly concave in m].

   (d) (5 marks). Let

$$\varphi(x) \equiv \frac{1}{1+(B/x)}, \qquad B > 0$$

Find steady state money demand as a function of the nominal interest rate i. Denote this solution $m(i)$. Given this function, what is $m^\star$? What is steady state welfare at this point (i.e., what is $U(1, m^\star)$)? [Hint: You might find it helpful to sketch the function $\varphi(x)$].

(e) (5 marks). Let $w(i)$ denote the percentage rise or fall in consumption that the consumer must receive to keep them indifferent between a steady state with nominal interest rate i and the optimum. That is, $w(i)$ solves

$$U[c(1 + w(i)), m(i)] = U(1, m^\star)$$

Using your answers from earlier parts, solve for $w(i)$.

(f) (5 marks). Let $B = 0.0018$ (as discussed by Robert Lucas in a 2000 *Econometrica* paper). What is $w(i)$ for a nominal interest rate of 10% (i.e., for $i = 0.1$)? Interpret your answer in terms of the welfare costs of inflation. Do you think this is a large number? Why or why not?

2. **Asset pricing — Hansen-Jagannathan bounds**. (30 marks). Consider the following consumption Euler equation from a representative consumer asset pricing model

$$1 = E_t \left\{ \beta \frac{U'(c_{t+1})}{U'(c_t)} R^x_{t+1} \right\}$$

for some asset return R^x_{t+1}. Consider two assets, safe (risk-free) bonds with sure return R^f_{t+1} and risky equity with random return R^e_{t+1}.

(a) (2 marks). Show that the following condition must hold

$$0 = E \left\{ \beta \frac{U'(c_{t+1})}{U'(c_t)} [R^e_{t+1} - R^f_{t+1}] \right\}$$

(b) (8 marks). Denoting the excess return (or "equity premium") by $R_{t+1} \equiv R^e_{t+1} - R^f_{t+1}$ and the stochastic discount factor (SDF) by m_{t+1}, show that the following condition must hold

$$\frac{E\{R_{t+1}\}}{\sigma\{R_{t+1}\}} = -\text{corr}\{m_{t+1}, R_{t+1}\} \frac{\sigma\{m_{t+1}\}}{E\{m_{t+1}\}}$$

where $\sigma\{X\}$ and $\text{corr}\{X, Y\}$ denote the standard deviation of the random variable X and the correlation coefficient between the random variables X and Y, respectively.

(c) (5 marks) In US data, the "Sharpe ratio" of the equity premium, $E\{R_{t+1}\}/\sigma\{R_{t+1}\}$ is approximately

$$\frac{E\{R_{t+1}\}}{\sigma\{R_{t+1}\}} = \frac{0.08}{0.16} = 0.5$$

Using this fact and the properties of the correlation coefficient, put bounds on the coefficient of variation of the SDF. That is. Find numbers $\underline{m}$ and $\overline{m}$ such that

$$\frac{\sigma\{m_{t+1}\}}{E\{m_{t+1}\}} \leq \overline{m}$$
$$\frac{\sigma\{m_{t+1}\}}{E\{m_{t+1}\}} \geq \underline{m}$$

(d) (3 marks). Using the consumption Euler equation, find a formula for the mean of the SDF, $E\{m_{t+1}\}$. If the safe return on bonds is about 1.01, what must the mean of the SDF be?

(e) (5 marks). Suppose that the representative consumer has constant relative risk aversion preferences

$$U(c) = \frac{1}{1-\gamma}(c^{1-\gamma} - 1), \qquad \gamma > 0$$

Express the standard deviation of the SDF, $\sigma\{m_{t+1}\}$, in terms of the time discount factor β, the coefficient of relative risk aversion γ, and the growth rate of consumption.

(f) (7 marks). If the standard deviation of consumption growth is approximately 1%, $\sigma\{c_{t+1}/c_t\} = 0.01$, and the discount factor is approximately $\beta = 1$, what order of magnitude does the coefficient of relative risk aversion have to be in order for the model to be consistent with the Sharpe ratio for the equity premium? That is, what value of γ do you need to make the model consistent with the bounds obtained in part (c)? [Hint: How well does the model do if $\gamma = 1$?]

3. **Monetary economics with linear production**. Consider a stochastic money in the utility function model. Time is discrete and denoted $t = 0, 1, ...$ There is a representative consumer who faces a random sequence of money growth shocks and productivity shocks. The consumer supplies labor to a competitive spot market in return for a nominal wage. Money growth shocks are denoted μ_t and productivity shocks are denoted A_t and both are Markov processes. The state at t is $s_t = (\mu_t, A_t)$ and a finite history is $s^t = (s_t, s_{t-1}, \cdots s_0)$ with s_0 known at date zero. As of date zero, the probability of s^t is $f(s^t \mid s_0)$. The consumer has expected utility preferences over consumption, end of period real balances $m_t(s^t) \equiv M_{t+1}(s^t)/P_t(s^t)$, and leisure $\ell_t(s^t)$.

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t U[c_t(s^t), m_t(s^t), \ell_t(s^t)] f(s^t \mid s_0), \qquad 0 < \beta < 1$$

The utility function U is assumed to be strictly increasing and strictly concave. The representative consumer trades in money and in a complete set of nominal, one-period, state contingent bonds. She has the flow budget constraint

$$\begin{aligned} & P_t(s^t)c_t(s^t) + M_{t+1}(s^t) + \sum_{s'} q_t(s^t, s')B_{t+1}(s^t, s') \\ \leq \ & W_t(s^t)n_t(s^t) + M_t(s^{t-1}) + B_t(s^{t-1}, s_t) - T_t(s^t) \end{aligned}$$

where $W_t(s^t)$ denotes the nominal wage, $T_t(s^t)$ denotes lump-sum taxes and where $M_0 > 0$ and $B_0 = 0$ are given initial conditions. She also has the following constraint on her time endowment

$$\ell_t(s^t) + n_t(s^t) \leq 1$$

There is also a representative firm with the production function $y_t(s^t) = A_t(s^t)n_t(s^t)$ that seeks to maximize profits by hiring labor. That is, taking as given prices and wages, the firm chooses $n_t(s^t)$ so as to maximize

$$P_t(s^t)A_t(s^t)n_t(s^t) - W_t(s^t)n_t(s^t)$$

The government's flow budget constraint is

$$\mathcal{M}_{t+1}(s^t) + T_t(s^t) = \mathcal{M}_t(s^{t-1})$$

where $\mathcal{M}_t(s^{t-1})$ denotes the exogenous money supply. The government's policy is

$$\frac{\mathcal{M}_{t+1}(s^t)}{\mathcal{M}_t(s^{t-1})} = 1 + \mu_t(s^t)$$

where $\mu_t(s^t)$ denotes the exogenously given stochastic money growth rate.

(a) (2 marks). Solve the firm's decision problem. What does this imply about the *real* wage? How does the real wage relate to productivity?

(b) (5 marks). Form a Lagrangian for the consumer's decision problem. Derive first order conditions characterizing the choice of $c_t(s^t)$, $M_{t+1}(s^t)$, $n_t(s^t)$, for each t and s^t and for $B_{t+1}(s^t, s')$ for each t and s^t and $s' \in \mathcal{S}$. Interpret these conditions. Use these first order conditions to derive a formula for the price $q_t(s^t, s')$ of a state contingent nominal bond.

(c) (5 marks). Describe how a one-period bond that is safe in nominal terms can be synthesized from the state-contingent bonds. Let $i_t(s^t)$ denote the one-period nominal interest rate on this bond. Derive a formula characterizing the nominal interest rate. [Hint: Relate the nominal interest rate to the set of state contingent bond prices]. Is this bond also safe in real terms? Why or why not?

(d) (8 marks). Suppose that the period utility function is

$$U(c, m, \ell) = \log(c) + \gamma \log(m) + \eta \log(\ell), \qquad \gamma > 0, \eta > 0$$

Solve for the non-stochastic steady state of this model (where $\mu_t(s^t) = \bar{\mu}$ and $A_t(s^t) = \bar{A}$ all t and s^t). Make sure you find solutions for consumption, labor, real balances, the real interest rate, the real wage rate, the nominal interest rate, the inflation rate, the money supply, and the price level in terms of the primitives of the model. [Hint: Be clear about the model's equilibrium conditions].

(e) (10 marks). For this question you can simplify the notation by suppressing the dependence on s^t and by using the conditional expectations notation where appropriate. Suppose that in log-deviations, the money growth rate and the productivity shock follow mean zero AR(1) processes with independent, mean zero homoskedastic innovations

$$\begin{aligned}
\hat{\mu}_{t+1} &= \rho_\mu \hat{\mu}_t + \epsilon^\mu_{t+1}, \qquad 0 \leq \rho_\mu \leq 1 \\
\hat{A}_{t+1} &= \rho_A \hat{A}_t + \epsilon^A_{t+1}, \qquad 0 \leq \rho_A \leq 1
\end{aligned}$$

Log linearize the model around its non-stochastic steady state. Show that the equations of the model can be reduced to a one-dimensional stochastic difference equation in real balances. Solve this difference equation by iterating forward. Then use the shock processes and the solution to your difference equation to find equilibrium processes for nominal interest rates and inflation. Discuss the sensitivity of your answers to the serial correlation coefficients ρ_μ and ρ_A. Relate your answers to a similar model where consumers do not value leisure and where they instead have an exogenous endowment that follows a stochastic process similar to the one assumed here for technology.