316-466 Monetary Economics — Note 2

I now turn to multiperiod uncertainty and competitive equilibrium with a complete set of securities. Uncertainty is modeled as a Markov chain with *transition* probabilities $\pi(s' \mid s)$ and given initial distribution $\pi_0(s)$ on a discrete state space $\mathcal{S}$ with typical element s. That is

$$\begin{aligned} \pi(s' \mid s) &\equiv \Pr(s_{t+1} = s' \mid s_t = s) \\ \pi_0(s) &\equiv \Pr(s_0 = s) \end{aligned}$$

The Markov chain induces *unconditional* probabilities via recursion. A finite history

$$s^t \equiv (s_0, s_1, \cdots, s_t) = (s^{t-1}, s_t)$$

has conditional probabilities obtained by chaining together the transitions and unconditional probabilities obtained by taking this chain all the way back

$$\begin{aligned} \pi(s^t \mid s^\tau) &= \pi(s_t \mid s_{t-1}) \times \pi(s_{t-1} \mid s_{t-2}) \times \cdots \times \pi(s_{\tau+1} \mid s_\tau), \qquad t \geq \tau \\ \pi(s^t) &= \pi(s_t \mid s_{t-1}) \times \pi(s_{t-1} \mid s_{t-2}) \times \cdots \times \pi(s_1 \mid s_0) \times \pi_0(s_0) \end{aligned}$$

We will typically assume that the state is known in the initial period (s_0 has been realized) so that the relevant probability distributions are $\pi(s^t \mid s_0)$.

The market structure is Arrow-Debreu. There is a complete set of markets for claims to consumption goods indexed by date and state. These claims are traded in a completely frictionless environment at time $t = 0$. At later dates, these claims are executed as agreed to at date $t = 0$. There are no enforcement problems.

An individual i has the endowment vector

$$y^i = \left\{ y_t^i(s_t) \right\}_{t=0}^{\infty}$$

with endowment realizations depending on the state. This individual takes as given prices

$$q^0 = \left\{ q_t^0(s^t) \right\}_{t=0}^{\infty}$$

which may depend on the entire history. (Prices are endogenous variable; whether they depend on the entire history or not is something that the model has to answer — it's not something that we can dictate *a priori*). The superscript nought is to indicate that these are prices of consumption claims traded at date $t = 0$. The budget constraint of this individual is then

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)[y_t^i(s_t) - c_t^i(s^t)] \geq 0$$

Resource feasibility in this economy requires that

$$\sum_i [y_t^i(s_t) - c_t^i(s^t)] \geq 0$$

for each date and state.

An individual i has a time- and state-separable expected utility function. Primarily for expositional convenience, I will assume that everyone has the same preferences and thus individuals differ only in their income streams. Specifically

$$\begin{aligned} U(c^i) &= \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t^i(s^t)] \pi(s^t \mid s_0), \qquad 0 < \beta < 1 \\ &= E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t^i) \right\} \end{aligned}$$

The Lagrangian of an individual i is, then,

$$L^i = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t^i(s^t)] \pi(s^t \mid s_0) + \lambda^i \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)[y_t^i(s_t) - c_t^i(s^t)]$$

The first order conditions of a consumer are given by

$$\frac{\partial L^i}{\partial c_t^i(s^t)} = 0 \qquad \Longleftrightarrow \qquad \beta^t u'[c_t^i(s^t)] \pi(s^t \mid s_0) = \lambda^i q_t^0(s^t)$$

for any date and state.

Example 1. Risk sharing

The first order conditions imply that for any two consumers i and j,

$$\frac{u'[c_t^i(s^t)]}{u'[c_t^j(s^t)]} = \frac{\lambda^i}{\lambda^j}$$

Hence at any date and state, the consumption of individual i is simply proportional to that of individual j with the constant of proportionality depending only on their Lagrange multipliers — i.e., on the value of their intertemporal wealth. Suppose that the period utility function has the constant relative risk aversion form

$$u(c) = \frac{1}{1-\sigma}(c^{1-\sigma} - 1), \qquad \sigma > 0$$

Then the ratio of marginal utilities satisfies

$$\left(\frac{c_t^i(s^t)}{c_t^j(s^t)} \right)^{-\sigma} = \frac{\lambda^i}{\lambda^j}$$

or

$$c_t^i(s^t) = \kappa_{ij} c_t^j(s^t), \qquad \kappa_{ij} \equiv \left(\frac{\lambda^i}{\lambda^j} \right)^{-1/\sigma}$$

The constant of proportionality κ_{ij} does not depend on the date or the state. It depends only on who these individuals *are* — and depends on who these individuals are only through the value of their wealths.

With complete markets, the Arrow-Debreu model implies extensive risk sharing over time and across states of nature.

Example 2. No aggregate risk

Suppose that there is no aggregate risk in the economy. That is, suppose

$$\sum_i y_t^i(s_t) = y$$

for any date and state. For illustrative purposes, suppose also that there are only two people; $i = 1, 2$. Then

$$y_t^1(s_t) + y_t^2(s_t) = y$$

Thus whenever $i = 1$ is lucky and has a relatively high income, it must be the case that $i = 2$ is correspondingly unlucky and has a relatively low income.

Guess that in this economy, each individual has constant consumption

$$c_0^i = c_t^i(s^t)$$

for any date and state. Then the first order conditions can be written

$$\beta^t \frac{u'(c_0^i)}{\lambda^i} \pi(s^t \mid s_0) = q_t^0(s^t)$$

Plug this into i's intertemporal budget constraint to get

$$\frac{u'(c_0^i)}{\lambda^i} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t \mid s_0)[y_t^i(s_t) - c_0^i] = 0$$

Which is easily solved for the constant level of consumption

$$c_0^i \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t \mid s_0) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t \mid s_0) y_t^i(s_t)$$

or

$$c_0^i = (1-\beta) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t \mid s_0) y_t^i(s_t)$$

Summing up these endowments, we see that

$$\begin{aligned} c_0^1 + c_0^2 &= (1-\beta) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t \mid s_0)[y_t^1(s_t) + y_t^2(s_t)] \\ &= (1-\beta) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t \mid s_0) y \\ &= y \end{aligned}$$

So markets are indeed clearing with these allocations and the implied prices.

Example 3. Pricing one-period returns

The date $t = 0$ Arrow-Debreu prices provide a powerful intellectual framework. In applications, however, it is common to work with various transformations of these prices. In particular, denote by $q_t^k(s^t)$ the price at date k state s^k of one unit of consumption delivered at date $t \geq k$ state s^t. This is just the ratio of two date zero prices

$$q_t^k(s^t) = \frac{q_t^0(s^t)}{q_k^0(s^k)}$$

Hence, using the first order conditions,

$$q_t^k(s^t) = \frac{q_t^0(s^t)}{q_k^0(s^k)} = \frac{\beta^t u'[c_t^i(s^t)]\pi(s^t \mid s_0)}{\beta^k u'[c_k^i(s^k)]\pi(s^k \mid s_0)} = \beta^{t-k}\frac{u'[c_t^i(s^t)]}{u'[c_k^i(s^k)]}\pi(s^t \mid s^k)$$

In particular, the one-period-ahead price is

$$q_{t+1}^t(s^{t+1}) = \beta\frac{u'[c_{t+1}^i(s^{t+1})]}{u'[c_t^i(s^t)]}\pi(s^{t+1} \mid s^t)$$

Now consider an asset that has a random payoff in the next period, $t+1$, if s_{t+1} is realized. Denote this payoff $x_{t+1}(s_{t+1})$. The price $p_t^x(s^t)$ of this asset at date t state s^t is

$$p_t^x(s^t) = \sum_{s_{t+1}} q_{t+1}^t(s^{t+1})x_{t+1}(s_{t+1})$$

Notice that the sum is taken over s_{t+1}. (Since $s^{t+1} \equiv (s^t, s_{t+1})$, many people find it more attractive to write an expression of this kind as

$$p_t^x(s^t) = \sum_{s'} q_{t+1}^t(s^t, s')x_{t+1}(s')$$

where the dummy variable s' indexes the possible histories that are continuations of s^t).

Using the formula for one-period-ahead contingent claims, this is just

$$p_t^x(s^t) = \sum_{s_{t+1}} \beta\frac{u'[c_{t+1}^i(s^{t+1})]}{u'[c_t^i(s^t)]}\pi(s^{t+1} \mid s^t)x_{t+1}(s_{t+1})$$

or

$$p_t^x = E_t\left\{\beta\frac{u'(c_{t+1}^i)}{u'(c_t^i)}x_{t+1}\right\}$$

The price of an asset with random payoff x is the conditional expectation of x times a "stochastic discount factor" (i.e., the intertemporal marginal rate of substitution).

The one period gross return R_{t+1}^x on an asset with non-zero price p_t^x and payoff x_{t+1} is $R_{t+1}^x = x_{t+1}/p_t^x$. In this case, the asset pricing equation can be rewritten

$$1 = E_t\left\{\beta\frac{u'(c_{t+1}^i)}{u'(c_t^i)}R_{t+1}^x\right\}$$

And indeed, for two assets with payoffs x_{t+1} and y_{t+1}, the gross returns R^x_{t+1} and R^y_{t+1} must line up so that

$$1 = E_t\left\{\beta\frac{u'(c^i_{t+1})}{u'(c^i_t)}R^x_{t+1}\right\} = E_t\left\{\beta\frac{u'(c^i_{t+1})}{u'(c^i_t)}R^y_{t+1}\right\}$$

or

$$0 = E_t\left\{\beta\frac{u'(c^i_{t+1})}{u'(c^i_t)}[R^x_{t+1} - R^y_{t+1}]\right\}$$

A formula like this places strong testable restrictions on patterns of asset returns.

Example 4. Consumption-based asset pricing

In what follows, I will assume a representative agent and so drop the is from the marginal rate of substitution formulas. In this model, assets are claims to consumption streams and are priced according to the valuation that the consumer ascribes to such streams.

The random variable

$$m_{t+1}(s^{t+1}) = \beta\frac{u'[c_{t+1}(s^{t+1})]}{u'[c_t(s^t)]}$$

gives the valuation the representative consumer ascribes to a unit of consumption goods in s^{t+1} if the state is s^t. This valuation is affected by both time preference (as measured by β) and risk aversion (as measured by the curvature of the utility function u). Indeed, if the consumer is *risk neutral* so that marginal utility is a constant, the stochastic discount factor reduces to just β since a risk neutral consumer does not care about the volatility of consumption when valuing assets that deliver claims to consumption streams.

Many asset pricing theories reduce to an expression of the form

$$p^x_t = E_t\{m_{t+1}x_{t+1}\} \qquad \text{or} \qquad 1 = E_t\{m_{t+1}R^x_{t+1}\}$$

The particular formula $m_{t+1} = \beta u'(c_{t+1})/u'(c_t)$ for the stochastic discount factor (SDF) is an artifact of the time- and state-separable expected utility function I assumed on behalf of the representative consumer. More generally, the SDF may also depend on the utility of leisure, consumption in other dates/states or indeed the returns on other assets.

Here are some examples of assets that we might want to price:

- A one-period riskless bond pays $x_{t+1} = 1$ for sure next period and has the price $p^1_t = E_t\{m_{t+1} \cdot 1\} = E_t\{m_{t+1}\}$. The one-period risk-free return is then $R^f_t = 1/p^1_t = 1/E_t\{m_{t+1}\}$. We observe (something close to) the return on riskless bonds in the data. This data puts strong restrictions on the admissible means of the stochastic discount factor that theorists can propose.

- A j-period riskless bond pays $x_{t+j} = 1$ for sure in $j \geq 1$ periods time and has the price $p^j_t = E_t\{m_{t+1} \cdot m_{t+2} \cdots m_{t+j} \cdot 1\}$. Combining the returns on bonds of different maturities, we can construct theories of *term premia.*

- Risky equity. A share in a long-lived asset entitles you to dividends d_{t+1} and the ability to resell the asset. Thus $x_{t+1} = d_{t+1} + p^{eq}_{t+1}$. The gross return is

$$R^{eq}_{t+1} = \frac{d_{t+1} + p^{eq}_{t+1}}{p^{eq}_t} = \frac{p^{eq}_{t+1}}{p^{eq}_t}\left(1 + \frac{d_{t+1}}{p^{eq}_{t+1}}\right)$$

The gross return is increasing in the capital gain p^{eq}_{t+1}/p^{eq}_t and decreasing in next period's price/earnings ratio p^{eq}_{t+1}/d_{t+1}. We can solve for the price of equity by recursively substituting out future prices in the stochastic difference equation

$$p^{eq}_t = E_t\{m_{t+1}(d_{t+1} + p^{eq}_{t+1})\}$$

Using the formulas

$$\begin{aligned} m_{t+1} &= \beta\frac{u'(c_{t+1})}{u'(c_t)} \\ m_{t+j} &= \beta^j\frac{u'(c_{t+j})}{u'(c_t)} \end{aligned}$$

and the fact that

$$\begin{aligned} m_{t+1} \times m_{t+2} \times \cdots \times m_{t+j} &= \beta\frac{u'(c_{t+1})}{u'(c_t)} \times \beta\frac{u'(c_{t+2})}{u'(c_{t+1})} \times \cdots \times \beta\frac{u'(c_{t+j})}{u'(c_{t+j-1})} \\ &= \beta^j\frac{u'(c_{t+j})}{u'(c_t)} \end{aligned}$$

we get

$$p^{eq}_t = \lim_{T\longrightarrow\infty} E_t\left\{\sum_{j=1}^{T}\beta^j\frac{u'(c_{t+j})}{u'(c_t)}d_{t+j}\right\} + \lim_{T\longrightarrow\infty} E_t\left\{\beta^T\frac{u'(c_{t+T})}{u'(c_t)}p^{eq}_{t+T}\right\}$$

If we assume that the limiting capital gain term goes to zero as $T \longrightarrow \infty$ we have that

$$p^{eq}_t = E_t\left\{\sum_{j=1}^{\infty}\beta^j\frac{u'(c_{t+j})}{u'(c_t)}d_{t+j}\right\}$$

The price of equity is the expected discounted value of the dividend payouts it provides with the dividends weighted by the SDF.

- The consumption capital asset pricing model (C-CAPM). Consider two assets, a one-period risk free asset with gross return R^f_t and a risky asset with gross return R^x_{t+1}. These must satisfy

$$\begin{aligned} 1 &= E_t\{m_{t+1}R^f_t\} = E_t\{m_{t+1}\}R^f_t \\ 1 &= E_t\{m_{t+1}R^x_{t+1}\} = E_t\{m_{t+1}\}E_t\{R^x_{t+1}\} + \text{Cov}_t\{m_{t+1}, R^x_{t+1}\} \end{aligned}$$

But $E_t\{m_{t+1}\} = 1/R_t^f$, so

$$1 = E_t\left\{\frac{R_{t+1}^x}{R_t^f}\right\} + \mathrm{Cov}_t\{m_{t+1}, R_{t+1}^x\}$$

or

$$E_t\left\{\frac{R_{t+1}^x}{R_t^f} - 1\right\} = -\mathrm{Cov}_t\{m_{t+1}, R_{t+1}^x\}$$

The term on the left hand side is the expected excess return on the asset x over the risk free return (the *risk premium*). To see this, notice that $R_{t+1}^x/R_t^f \cong 1 + r_{t+1}^x - r_t^f$ where the little rs denote net returns. Hence

$$E_t\left\{\frac{R_{t+1}^x}{R_t^f} - 1\right\} \cong E_t\{r_{t+1}^x - r_t^f\} = E_t\{r_{t+1}^x\} - r_t^f$$

The expected excess return is proportional to the *covariance* of the return of that asset with the SDF. The intuition for this is easiest to see if we use our particular formula for m_{t+1} to illustrate

$$E_t\left\{\frac{R_{t+1}^x}{R_t^f} - 1\right\} = -\mathrm{Cov}_t\left\{\beta\frac{u'(c_{t+1})}{u'(c_t)}, R_{t+1}^x\right\}$$

A high risk premium is demanded of an asset that is poor from an insurance perspective, i.e., an asset that has a high return only when the marginal utility of consumption is low (and consumption itself is relatively high) so that the covariance is a large negative number. Put differently, an asset is more risky the more its return covaries negatively with the SDF. Notice that in general the risk premium is time-varying.

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