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The small open economy under uncertainty

We now extend our small open endowment economy to allow for shocks.

A. Two-period example

Let there be two dates, $t = 0, 1$ and let there be S possible states of nature that may be realized at date $t = 1$. Index the **states** by $s \in \mathcal{S} = \{1, 2, ..., S\}$ and let the probability of state s be $\pi(s)$.

- *Preferences*: The representative consumer has a time and state separable utility function over consumption c_0 and $c_1(s)$. The consumer maximizes **expected utility**

$$u(c) = U(c_0) + \beta \sum_s U[c_1(s)]\pi(s)$$

  with constant time discount factor $0 < \beta < 1$. The period utility function $U(c)$ is assumed to be strictly increasing and concave. Consumption at date $t = 1$ is indexed by which of the S possible states realizes.

- *Endowments*: There is no production. Instead, there is simply an exogenously given supply of the consumption good at each date and state, y_0 and $\{y_1(s)\}_{s=1}^S$. As of date $t = 0$, the second period endowment is random because the consumer does not know which state will realize.

- *Market structure*: We will assume that the consumer can freely borrow or lend in a complete set of asset markets. Specifically, we assume the existence of S securities $\{B_1(s)\}_{s=1}^S$ that pay one unit of consumption if and only if state s is realized at date $t = 1$. These are sometimes known as **Arrow securities**. Let the price of an Arrow security be $q_1(s)$.

- *Flow budget constraints*: In the first period the consumer has initial endowment y_0 and can consume or buy securities so that

$$c_0 + \sum_s q_1(s)B_1(s) = y_0$$

  In the second period, if state s is realized, the consumer will have $B_1(s)$ from the Arrow securities that pay off, random endowment $y_1(s)$ and can uses this to consume. Hence

$$c_1(s) = B_1(s) + y_1(s)$$

- *Intertemporal budget constraint*: Plugging the set of $t = 1$ state s budget constraints into the date $t = 0$ budget constraint gives the single intertemporal budget constraint

$$c_0 + \sum_s q_1(s)c_1(s) = y_0 + \sum_s q_1(s)y_1(s)$$

  Note that no probabilities enter this constraint: budget constraints have to hold at every date and state.

- *Optimization*: The consumer chooses a complete contingent plan of consumption, c_0 and $c_1(s)$ for each s. There is a single Lagrange multiplier $\lambda \geq 0$ to go along with the single intertemporal budget constraint so that

$$\mathcal{L} = U(c_0) + \beta \sum_s U[c_1(s)]\pi(s) + \lambda \left[(y_0 - c_0) + \sum_s q_1(s)[y_1(s) - c_1(s)] \right]$$

  The key first order condition for consumption c_0 is

$$\frac{\partial \mathcal{L}}{\partial c_0} = 0 \iff U'(c_0) = \lambda$$

  And similarly for each state that may occur at date $t = 1$,

$$\frac{\partial \mathcal{L}}{\partial c_1(s)} = 0 \iff \beta U'[c_1(s)]\pi(s) = \lambda q_1(s)$$

  Hence the marginal rate of substitution between consumption today and consumption in any state tomorrow is

$$\beta \frac{U'[c_1(s)]}{U'(c_0)} \pi(s) = q_1(s) \tag{1}$$

  and similarly for the marginal rate of substitution between consumption in any two states s and z,

$$\frac{U'[c_1(s)]\pi(s)}{U'[c_1(z)]\pi(z)} = \frac{q_1(s)}{q_1(z)}$$

  Notice that this implies $c_1(s) = c_1(z)$ if and only if $\frac{\pi(s)}{\pi(z)} = \frac{q_1(s)}{q_1(z)}$, otherwise, if the relative probabilities do not line up with the relative prices, consumption tilts towards one state or the other.

- *Safe real returns*: A portfolio of Arrow securities that pays one unit of consumption for sure (irrespective of s) can be constructed by buying one of each separate security. Since it involves buying one of each security, this portfolio — **a Bond** — has a price

$$p_1 = \sum_s q_1(s)$$

and we can define the real interest rate r_1 associated with this bond by

$$p_1 = \frac{1}{1 + r_1}$$

(the subscript 1 used here is in anticipation of multi-period uncertainty, which we'll turn to shortly). The stochastic consumption Euler of a consumer is found by summing the formula (1) over all the states

$$\sum_s \beta \frac{U'[c_1(s)]}{U'(c_0)} \pi(s) = \sum_s q_1(s) = p_1$$

or

$$U'(c_0) = \beta(1 + r_1) \sum_s U'[c_1(s)]\pi(s)$$

You will sometimes see this as written

$$U'(c_0) = \beta(1 + r_1)\mathsf{E}_0\{U'(c_1)\}$$

where $\mathsf{E}_0\{\}$ denotes expectations conditional on date $t = 0$ information and the notation indicates that c_1 (and hence $U'(c_1)$) is a random variable.

- Notation: Let's introduce some dummy variables. Index all date $t = 0$ variables by s_0 and all date $t = 1$ variables by s_1. Since there is no uncertainty at date $t = 0$, let $\pi_0(s_0) = 1$ for some s_0 and let $\pi_1(s_1)$ denote the probabilities of the states in the second period. Also, let $q_0 = 1$. Then we can write the consumer's utility function and budget constraint in terms of the double-summation formulas

$$u(c) = \sum_{t=0}^{1} \sum_{s_t} U[c_t(s_t)]\pi_t(s_t) \tag{2}$$

and

$$\sum_{t=0}^{1}\sum_{s_t} q_t(s_t)c_t(s_t) = \sum_{t=0}^{1}\sum_{s_t} q_t(s_t)y_t(s_t) \tag{3}$$

These have direct multi-period analogues.

B. Dynamic stochastic models

The many period generalization of this model is quite straightforward. We keep a fixed state space $\mathcal{S} = \{1, 2, ..., S\}$ and let $t = 0, 1, 2, ...$ A **history** of states, denoted s^t, is a vector

$$s^t = (s_0, s_1, ..., s_t) = (s^{t-1}, s_t)$$

The unconditional probability of a history s^t being realized as of date zero is denoted $\pi_t(s^t)$. The *conditional* probability of a state s_{t+1} given s^t is $\pi_{t+1}(s^t, s_{t+1})/\pi_t(s^t)$. We will always assume that the initial state s_0 is known. The endowments of individuals are given by $y = \{y_t(s^t)\}_{t=0}^{\infty}$ while preferences are over consumption plans $c = \{c_t(s^t)\}_{t=0}^{\infty}$.

The time and state separable utility function is now

$$u(c) = \sum_{t=0}^{\infty}\sum_{s^t} \beta^t U[c_t(s^t)]\pi_t(s^t)$$

(compare this to equation (2)). Sometimes this is written more simply as

$$u(c) = \mathsf{E}_0\left\{\sum_{t=0}^{\infty} \beta^t U(c_t)\right\}$$

where E_0 denotes expectations conditional on date $t = 0$ information.

We again assume a complete set of Arrow securities with $q_t(s^t)$ denoting the price as of date zero of a unit of consumption delivered in date t state s^t. The intertemporal budget constraint is then

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t(s^t)c_t(s^t) = \sum_{t=0}^{\infty}\sum_{s^t} q_t(s^t)y_t(s^t)$$

(compare this to equation (3)).

The Lagrangian of a consumer is, then,

$$\mathcal{L} = \sum_{t=0}^{\infty}\sum_{s^t} \beta^t U[c_t(s^t)]\pi_t(s^t) + \lambda \sum_{t=0}^{\infty}\sum_{s^t} q_t(s^t)[y_t(s^t) - c_t(s^t)]$$

The key first order conditions are given by the choice of $c_t(s^t)$ for each date and state

$$\frac{\partial \mathcal{L}}{\partial c_t(s^t)} = 0 \Longleftrightarrow \beta^t U'[c_t(s^t)]\pi_t(s^t) = \lambda q_t(s^t)$$

Now consider consumption choices at different dates and states

$$\begin{aligned}
\beta^t U'[c_t(s^t)]\pi_t(s^t) &= \lambda q_t(s^t) \\
\beta^{t+1} U'[c_{t+1}(s^{t+1})]\pi_{t+1}(s^{t+1}) &= \lambda q_{t+1}(s^{t+1})
\end{aligned}$$

Recalling that $s^{t+1} = (s^t, s_{t+1})$ and rearranging

$$\beta \frac{U'[c_{t+1}(s^t, s_{t+1})]}{U'[c_t(s^t)]} \frac{\pi_{t+1}(s^t, s_{t+1})}{\pi_t(s^t)} = \frac{q_{t+1}(s^t, s_{t+1})}{q_t(s^t)}$$

At date t, the history s^t is known, but which event s_{t+1} will realize next is unknown. A unit of consumption for sure can be obtained by buying one of each Arrow security that pays off at $t+1$. As of date zero these securities have price $q_{t+1}(s^t, s_{t+1})$ but **as of date** t they have price

$$Q_t(s^t, s_{t+1}) \equiv \frac{q_{t+1}(s^t, s_{t+1})}{q_t(s^t)} \tag{4}$$

Hence the price at date t of a bond that delivers one unit of consumption for sure in date $t+1$ is

$$p_t(s^t) = \sum_{s_{t+1}} Q_t(s^t, s_{t+1}) = \sum_{s_{t+1}} \frac{q_{t+1}(s^t, s_{t+1})}{q_t(s^t)}$$

The one-period real interest rate $r_t(s^t)$ on this bond is given by

$$p_t(s^t) = \frac{1}{1 + r_t(s^t)}$$

The stochastic Euler equation for the consumer is therefore

$$U'[c_t(s^t)] = \beta[1 + r_t(s^t)] \sum_{s_{t+1}} U'[c_{t+1}(s^t, s_{t+1})] \frac{\pi_{t+1}(s^t, s_{t+1})}{\pi_t(s^t)}$$

which is sometimes simplified to

$$U'(c_t) = \beta(1 + r_t)\mathsf{E}_t\left\{U'(c_{t+1})\right\}$$

where $\mathsf{E}_t\{\}$ denotes expectations conditional on date t information and it is understood that c_t is known at date t but c_{t+1} is random.

C. Streamlined notation

Having introduced the cumbersome state-contingent notation, we'll often suspend it's use and do things more simply. For example, we might write the utility function as

$$u(c) = \mathsf{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t U(c_t) \right\}$$

with flow constraints

$$c_t + \sum_{s'} Q_t(s') B_{t+1}(s') = y_t + B_t$$

where the dependence of consumption, bond holdings, etc, on the underlying history s^t is suppressed.

D. Linear-quadratic permanent income model

As a simple example, let's consider a small open economy perfectly integrated into world capital markets facing constant world real interest rate $r > 0$ with an exogenous random supply of goods y_t (the stochastic process for y_t will be discussed below). Also, suppose that the representative consumer can only trade in one-period riskless bonds — bonds that pay a unit of consumption no matter which state realizes. Then the flow budget constraint can be written

$$c_t + B_{t+1} = y_t + (1+r)B_t$$

Substituting this into the consumer's utility function gives

$$\mathcal{L} = \mathsf{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t U[y_t + (1+r)B_t - B_{t+1}] \right\}$$

and the key first order condition for bond holdings B_{t+1} can be written

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \Longleftrightarrow \mathsf{E}_0 \left\{ -\beta^t U'(c_t) + \beta^{t+1} U'(c_{t+1})(1+r) \right\} = 0$$

or

$$U'(c_t) = \beta(1+r)\mathsf{E}_t\{U'(c_{t+1})\}$$

From the point of view of date t, the only thing that is random is the marginal utility of consumption tomorrow.

Now suppose that period utility is quadratic

$$U(c) = c - \frac{a}{2}c^2, \qquad a > 0$$

Then marginal utility is linear in consumption, $U'(c) = 1 - ac$ and we can write the consumption Euler equation as

$$1 - ac_t = \beta(1+r) - \beta(1+r)a\mathsf{E}_t\{c_{t+1}\}$$

or

$$\mathsf{E}_t\{c_{t+1}\} = \frac{\beta(1+r)-1}{a\beta(1+r)} + \frac{1}{\beta(1+r)}c_t$$

In the special case that $\beta(1+r) = 1$, consumption is a **random walk** (cf. Hall 1978). In this special case,

$$\mathsf{E}_t\{c_{t+1}\} = c_t \Longleftrightarrow c_{t+1} = c_t + \text{noise}$$

and then by the **law of iterated expectations**,

$$\mathsf{E}_t\{c_{t+k}\} = c_t, \qquad t, k \geq 0$$

To see that this implies about the level of consumption, recall the intertemporal budget constraint and take date $t = 0$ conditional expectations on both sides

$$\mathsf{E}_0\left\{\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t c_t\right\} = (1+r)B_0 + \mathsf{E}_0\left\{\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t y_t\right\}$$

And on simplifying

$$c_0 = rB_0 + \frac{r}{1+r}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \mathsf{E}_0\{y_t\}$$

The linear-quadratic setup gives rise to a "**certainty-equivalence**" result.

We can go a bit further if we specify a particular stochastic process for y_t. Suppose y_t is an autoregression with mean $\bar{y} > 0$, specifically

$$y_{t+1} = (1-\phi)\bar{y} + \phi y_t + \epsilon_{t+1}, \qquad 0 < \phi < 1$$

where ϵ_{t+1} is white noise. Then the conditional expectations satisfy

$$\mathsf{E}_t\{y_{t+k} - \bar{y}\} = \phi^k (y_t - \bar{y}), \qquad t, k \geq 0$$

So the initial level of consumption is therefore

$$\begin{aligned} c_0 &= rB_0 + \frac{r}{1+r} \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \mathsf{E}_0\{y_t - \bar{y} + \bar{y}\} \\ &= rB_0 + \bar{y} + \frac{r}{1+r} \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \phi^t (y_0 - \bar{y}) \\ &= rB_0 + \bar{y} + \frac{r}{1+r-\phi}(y_0 - \bar{y}) \end{aligned}$$

or at any date $t \geq 0$ (there is nothing special about $t = 0$)

$$c_t = rB_t + \bar{y} + \frac{r}{1+r-\phi}(y_t - \bar{y})$$

If output were a constant $\bar{y}$ in all dates and states, consumption would be $rB_0 + \bar{y}$. With random y_t, there is a correction term that raises or lowers consumption depending on whether y_t is bigger or smaller than its mean.

The impact of a transitory shock to output is

$$0 < \frac{\partial c_t}{\partial y_t} = \frac{r}{1+r-\phi} < 1$$

The more persistent the shock process (i.e., $\phi \to 1$) the closer the impact effect is to one. If the shock process is close to white noise ($\phi \to 0$), the impact effect is only $\frac{r}{1+r}$ which is a number like 0.05 or so. Hence transitory shocks to output have a small effect on consumption, permanent shocks have a big (nearly one-for-one) effect on consumption. This leads to an important implication of the small open economy model: the trade balance $y_t - c_t$ is **pro-cylical**. When output is unusually high, consumption increases by less than the movement in output so the trade balance increases. This is counter-factual for most countries. In the data a boom tends to be associated with a trade deficit, not a trade surplus.

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