316-632 International Monetary Economics

Problem Set #3 *Selected answers*

Question 1. The planner's problem is to maximize

$$\sum_i \sum_{t=0}^{\infty} \sum_{z^t} \omega_i \beta^t U[c_t^i(z^t)] \varphi_t(z^t)$$

by choice of $\{c_t^i(z^t)\}$ for $i = 1, 2$ subject to the resource constraints

$$\sum_i c_t^i(z^t) \leq x_t(z^t)$$

The Lagrangian for this problem can be written

$$L = \sum_i \sum_{t=0}^{\infty} \sum_{z^t} \omega_i \beta^t U[c_t^i(z^t)] \varphi_t(z^t) + \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \varphi_t(z^t) q_t(z^t) \left[x_t(z^t) - \sum_i c_t^i(z^t) \right]$$

where $\{q_t(z^t)\}$ are non-negative multipliers. The key first order conditions are

$$\omega_i U'[c_t^i(z^t)] = q_t(z^t), \qquad i = 1, 2$$

Making use of the functional form $U'(c) = c^{-\sigma}$ we can re-write the first order conditions as

$$c_t^i(z^t) = \left(\frac{\omega_i}{q_t(z^t)} \right)^{1/\sigma}, \qquad i = 1, 2$$

and now summing over $i = 1, 2$ we get

$$x_t(z^t) q_t(z^t)^{1/\sigma} = (\omega_1^{1/\sigma} + \omega_2^{1/\sigma})$$

Hence the Lagrange multipliers are given by

$$q_t(z^t) = (\omega_1^{1/\sigma} + \omega_2^{1/\sigma})^{\sigma} x_t(z^t)^{-\sigma}$$

and the solutions for consumption are

$$\begin{aligned} c_t^1(z^t) &= \frac{\omega_1^{1/\sigma}}{\omega_1^{1/\sigma} + \omega_2^{1/\sigma}} x_t(z^t) \\ c_t^2(z^t) &= \frac{\omega_2^{1/\sigma}}{\omega_1^{1/\sigma} + \omega_2^{1/\sigma}} x_t(z^t) \end{aligned}$$

Question 2. The Lagrangian for the home ($i = 1$) country will be

$$\begin{aligned} L^1 &= \sum_{t=0}^{\infty} \sum_{z^t} \beta^t U[c_t^1(z^t)] \varphi_t(z^t) \\ &+ \sum_{t=0}^{\infty} \sum_{z^t} \lambda_t^1(z^t) \left[\begin{array}{c} P_{t-1}^1(z^{t-1}) y_{t-1}^1(z^{t-1}) + M_{t-1}^1(z^{t-1}) - P_{t-1}^1(z^{t-1}) c_{t-1}^1(z^{t-1}) \\ + B_t^1(z^{t-1}, z_t) + \mathcal{E}_t(z^t) B_t^{12}(z^{t-1}, z_t) \end{array} \right] \\ &- \sum_{t=0}^{\infty} \sum_{z^t} \lambda_t^1(z^t) \left[\begin{array}{c} M_t^1(z^t) + P_t^1(z^t) \tau_t^1(z^t) \\ + \sum_{z'} q_t^1(z^t, z') B_{t+1}^1(z^t, z') + \mathcal{E}_t(z^t) \sum_{z'} q_t^2(z^t, z') B_{t+1}^{12}(z^t, z') \end{array} \right] \\ &+ \sum_{t=0}^{\infty} \sum_{z^t} \eta_t^1(z^t) [M_t^1(z^t) - P_t^1(z^t) c_t^1(z^t)] \end{aligned}$$

where $\{\lambda_t^1(z^t)\}$ denote multipliers on the budget constraint in the asset market and $\{\eta_t^1(z^t)\}$ denote the multipliers on the cash-in-advance constraint. The key first order conditions for this problem include

$$\begin{aligned} c_t^1(z^t) &: \quad \beta^t U'[c_t^1(z^t)]\varphi_t(z^t) = P_t^1(z^t)\left[\eta_t^1(z^t) + \sum_{z'} \lambda_{t+1}^1(z^t, z')\right] \\ M_t^1(z^t) &: \quad \lambda_t^1(z^t) = \eta_t^1(z^t) + \sum_{z'} \lambda_{t+1}^1(z^t, z') \end{aligned}$$

which can be combined to give a familiar expression for the marginal utility of a dollar, namely

$$\beta^t \frac{U'[c_t^1(z^t)]}{P_t^1(z^t)}\varphi_t(z^t) = \lambda_t^1(z^t)$$

For bond holdings we have the conditions

$$\begin{aligned} B_{t+1}^1(z^t, z') &: \quad \lambda_t^1(z^t) q_t^1(z^t, z') = \lambda_{t+1}^1(z^t, z') \\ B_{t+1}^{12}(z^t, z') &: \quad \lambda_t^1(z^t)\mathcal{E}_t(z^t) q_t^2(z^t, z') = \lambda_{t+1}^1(z^t, z')\mathcal{E}_{t+1}(z^t, z') \end{aligned}$$

which imply that the nominal pricing kernel for dollar assets is

$$q_t^1(z^t, z') = \frac{\lambda_{t+1}^1(z^t, z')}{\lambda_t^1(z^t)} = \beta \frac{U'[c_{t+1}^1(z^t, z')]}{U'[c_t^1(z^t)]} \frac{P_t^1(z^t)}{P_{t+1}^1(z^t, z')} \frac{\varphi_{t+1}(z^t, z')}{\varphi_t(z^t)}$$

and the relationship between dollar and euro pricing kernels is

$$q_t^1(z^t, z') = q_t^2(z^t, z') \frac{\mathcal{E}_t(z^t)}{\mathcal{E}_{t+1}(z^t, z')}$$

We can use similar methods to characterize the solution to the household's problem for country $i = 2$. As in the lecture notes, it's easiest if we express the household's budget constraints in dollars. If so, the appropriate Lagrangian is

$$\begin{aligned} L^2 = & \sum_{t=0}^{\infty} \sum_{z^t} \beta^t U[c_t^2(z^t)]\varphi_t(z^t) \\ & + \sum_{t=0}^{\infty} \sum_{z^t} \lambda_t^2(z^t)\mathcal{E}_t(z^t) \left[\begin{array}{c} P_{t-1}^2(z^{t-1}) y_{t-1}^2(z^{t-1}) + M_{t-1}^2(z^{t-1}) - P_{t-1}^2(z^{t-1}) c_{t-1}^2(z^{t-1}) \\ + \frac{1}{\mathcal{E}_t(z^t)} B_t^{21}(z^{t-1}, z_t) + B_t^2(z^{t-1}, z_t) \end{array} \right] \\ & - \sum_{t=0}^{\infty} \sum_{z^t} \lambda_t^2(z^t)\mathcal{E}_t(z^t) \left[\begin{array}{c} M_t^2(z^t) + P_t^2(z^t)\tau_t^2(z^t) \\ + \frac{1}{\mathcal{E}_t(z^t)} \sum_{z'} q_t^1(z^t, z') B_{t+1}^{21}(z^t, z') + \sum_{z'} q_t^2(z^t, z') B_{t+1}^2(z^t, z') \end{array} \right] \\ & + \sum_{t=0}^{\infty} \sum_{z^t} \eta_t^2(z^t)[M_t^2(z^t) - P_t^2(z^t) c_t^2(z^t)] \end{aligned}$$

and this leads to familiar conditions

$$\begin{aligned} c_t^2(z^t) &: \quad \beta^t U'[c_t^2(z^t)]\varphi_t(z^t) = \mathcal{E}_t(z^t) P_t^2(z^t)\left[\eta_t^2(z^t) + \sum_{z'} \lambda_{t+1}^2(z^t, z')\right] \\ M_t^2(z^t) &: \quad \lambda_t^2(z^t) = \eta_t^2(z^t) + \sum_{z'} \lambda_{t+1}^2(z^t, z') \end{aligned}$$

which gives the marginal utility of a euro

$$\beta^t \frac{U'[c_t^2(z^t)]}{P_t^2(z^t)} \varphi_t(z^t) = \mathcal{E}_t(z^t)\lambda_t^2(z^t)$$

We also have the conditions for bond holdings

$$\begin{aligned} B_{t+1}^{21}(z^t, z') &: \qquad \lambda_t^2(z^t) q_t(z^t, z') = \lambda_{t+1}^2(z^t, z') \\ B_{t+1}^{2}(z^t, z') &: \qquad \lambda_t^2(z^t)\mathcal{E}_t(z^t) q_t^2(z^t, z') = \lambda_{t+1}^2(z^t, z')\mathcal{E}_{t+1}(z^t, z') \end{aligned}$$

(Compare these to their equivalents from the home country).

The relationship of the Lagrange multipliers $\lambda_t^i(z^t)$ to the planning weights ω_i is somewhat subtle. First, notice that the $\lambda_t^i(z^t)$ are the multiplier on the **period** budget constraints of the households, and are not the single multipliers that we would have on the **intertemporal** or Arrow-Debreu budget constraints. If we wrote down an Arrow-Debreu problem with a single Lagrange multiplier λ_i for each country's intertemporal budget constraint we would get first order conditions for consumption of the form

$$\beta^t U'[c_t^i(z^t)]\varphi_t(z^t) = \lambda_i Q_t^i(z^t)$$

where $\{Q_t^i(z^t)\}$ are the **nominal** "date-zero" prices (which are not necessarily common to both countries). However, because each country faces the same Arrow-Debreu international goods market, the **real** "date-zero" prices will be the same for each country, so we can write

$$Q_t(z^t) = \frac{Q_t^i(z^t)}{P_t^i(z^t)} \qquad \text{independent of } i$$

or alternatively

$$\beta^t \frac{U'[c_t^i(z^t)]}{P_t^i(z^t)} \varphi_t(z^t) = \lambda_i Q_t(z^t)$$

If we then introduce the harmless normalization

$$Q_t(z^t) = \beta^t \varphi_t(z^t) q_t(z^t)$$

We see that the relationship between the planning and decentralized problems is that **if**

$$\lambda_i = \frac{1}{\omega_i}$$

then the solution to the planner's problem and the equilibrium allocations will coincide (see Lecture Note 4a for more on this).

Question 3. The market clearing conditions are for goods,

$$c_t^1(z^t) + c_t^2(z^t) = x_t(z^t) = y_t^1(z^t) + y_t^2(z^t)$$

for money,

$$\begin{aligned} M_t^1(z^t) &= \mathcal{M}_t^1(z^t) \\ M_t^2(z^t) &= \mathcal{M}_t^2(z^t) \end{aligned}$$

and for bonds,

$$
\begin{aligned}
B^1_{t+1}(z^t, z') + B^{21}_{t+1}(z^t, z') &= \mathcal{B}^1_{t+1}(z^t, z') \\
B^{12}_{t+1}(z^t, z') + B^2_{t+1}(z^t, z') &= \mathcal{B}^2_{t+1}(z^t, z')
\end{aligned}
$$

Question 4. To solve the model, it's easiest to move between the planner's solution and the decentralized problem as needed. We assume for convenience that the planner's weights are indeed

$$
\begin{aligned}
\lambda_1 &= \frac{1}{\omega_1} \\
\lambda_2 &= \frac{1}{\omega_2}
\end{aligned}
$$

If so, we can write the equilibrium consumption allocations as

$$
\begin{aligned}
c^1_t(z^t) &= \Lambda_1 x_t(z^t), \qquad \Lambda_1 \equiv \frac{(1/\lambda_1)^{1/\sigma}}{(1/\lambda_1)^{1/\sigma} + (1/\lambda_2)^{1/\sigma}} \\
c^2_t(z^t) &= \Lambda_2 x_t(z^t), \qquad \Lambda_2 \equiv \frac{(1/\lambda_2)^{1/\sigma}}{(1/\lambda_1)^{1/\sigma} + (1/\lambda_2)^{1/\sigma}}
\end{aligned}
$$

Assuming that the cash-in-advance constraints bind,

$$
\begin{aligned}
M^1_t(z^t) &= P^1_t(z^t) c^1_t(z^t) \\
M^2_t(z^t) &= P^2_t(z^t) c^2_t(z^t)
\end{aligned}
$$

Now using the money-market clearing conditions and the solutions for consumption, we can solve for the dollar and euro price levels

$$
\begin{aligned}
P^1_t(z^t) &= \frac{M^1_t(z^t)}{c^1_t(z^t)} = \frac{\mathcal{M}^1_t(z^t)}{\Lambda_1 x_t(z^t)} \\
P^2_t(z^t) &= \frac{M^2_t(z^t)}{c^2_t(z^t)} = \frac{\mathcal{M}^2_t(z^t)}{\Lambda_2 x_t(z^t)}
\end{aligned}
$$

Notice that these conditions imply that inflation in each country has a common and a country-specific component

$$
\begin{aligned}
\log\left(\frac{P^1_{t+1}}{P^1_t}\right) &= \log\left(\frac{\mathcal{M}^1_{t+1}}{\mathcal{M}^1_t}\right) - \log\left(\frac{x_{t+1}}{x_t}\right) \\
\log\left(\frac{P^2_{t+1}}{P^2_t}\right) &= \log\left(\frac{\mathcal{M}^2_{t+1}}{\mathcal{M}^2_t}\right) - \log\left(\frac{x_{t+1}}{x_t}\right)
\end{aligned}
$$

Whenever money growth in a country is faster than the growth rate of world output, inflation in that country is positive. Inflation differentials are driven purely by money-growth differentials, namely

$$
\log\left(\frac{P^1_{t+1}}{P^1_t}\right) - \log\left(\frac{P^2_{t+1}}{P^2_t}\right) = \log\left(\frac{\mathcal{M}^1_{t+1}}{\mathcal{M}^1_t}\right) - \log\left(\frac{\mathcal{M}^2_{t+1}}{\mathcal{M}^2_t}\right)
$$

We can characterize real and nominal exchange rates using the conditions that determine the marginal utility of a dollar and the marginal utility of a euro. These are

$$\begin{aligned}\frac{U'[c_t^1(z^t)]}{P_t^1(z^t)} &= \lambda_1 \\ \frac{U'[c_t^2(z^t)]}{P_t^2(z^t)} &= \lambda_2 \mathcal{E}_t(z^t)\end{aligned}$$

and using the assumed functional form

$$\begin{aligned}U'[\Lambda_1 x_t(z^t)] &= [\Lambda_1 x_t(z^t)]^{-\sigma} = \lambda_1 P_t^1(z^t) \\ U'[\Lambda_2 x_t(z^t)] &= [\Lambda_2 x_t(z^t)]^{-\sigma} = \lambda_2 \mathcal{E}_t(z^t) P_t^2(z^t)\end{aligned}$$

Hence the real exchange rate is

$$\mathcal{Q}_t\left(z^t\right) \equiv \mathcal{E}_t(z^t)\frac{P_t^2(z^t)}{P_t^1(z^t)} = \frac{[\Lambda_2 x_t(z^t)]^{-\sigma}}{\lambda_2}\frac{\lambda_1}{[\Lambda_1 x_t(z^t)]^{-\sigma}} = \frac{\Lambda_2^{-\sigma}}{\lambda_2}\frac{\lambda_1}{\Lambda_1^{-\sigma}}$$

which is constant.

The nominal exchange rate is therefore

$$\mathcal{E}_t(z^t) = \frac{\Lambda_2^{-\sigma}}{\lambda_2}\frac{\lambda_1}{\Lambda_1^{-\sigma}}\frac{P_t^1(z^t)}{P_t^2(z^t)} = \frac{\Lambda_2^{-\sigma}}{\lambda_2}\frac{\lambda_1}{\Lambda_1^{-\sigma}}\frac{\mathcal{M}_t^1(z^t)}{\Lambda_1 x_t(z^t)}\frac{\Lambda_2 x_t(z^t)}{\mathcal{M}_t^2(z^t)} = \frac{\Lambda_2^{1-\sigma}}{\lambda_2}\frac{\lambda_1}{\Lambda_1^{1-\sigma}}\frac{\mathcal{M}_t^1(z^t)}{\mathcal{M}_t^2(z^t)}$$

Upto a constant of proportionality, the nominal exchange rate is just the ratio of the supply of dollars to euros. Hence the depreciation of the nominal exchange rate is given by the money growth differential (which is itself equal to the inflation differential)

$$\log\left(\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right) = \log\left(\frac{\mathcal{M}_{t+1}^1}{\mathcal{M}_t^1}\right) - \log\left(\frac{\mathcal{M}_{t+1}^2}{\mathcal{M}_t^2}\right) = \log\left(\frac{P_{t+1}^1}{P_t^1}\right) - \log\left(\frac{P_{t+1}^2}{P_t^2}\right)$$

If money growth is faster in the home country (domestic inflation is higher), then the nominal exchange rate depreciates.

The pricing kernel for dollar denominated assets is given by

$$\begin{aligned}q_t^1(z^t, z') &= \frac{\lambda_{t+1}^1(z^t, z')}{\lambda_t^1(z^t)} \\ &= \beta\frac{U'[c_{t+1}^1(z^t, z')]}{U'[c_t^1(z^t)]}\frac{P_t^1(z^t)}{P_{t+1}^1(z^t, z')}\frac{\varphi_{t+1}(z^t, z')}{\varphi_t(z^t)} \\ &= \beta\left(\frac{x_{t+1}(z^t, z')}{x_t(z^t)}\right)^{-\sigma}\frac{\mathcal{M}_t^1(z^t)}{\Lambda_1 x_t(z^t)}\frac{\Lambda_1 x_{t+1}(z^t, z')}{\mathcal{M}_{t+1}^1(z^t, z')}\frac{\varphi_{t+1}(z^t, z')}{\varphi_t(z^t)} \\ &= \beta\left(\frac{x_{t+1}(z^t, z')}{x_t(z^t)}\right)^{1-\sigma}\frac{\mathcal{M}_t^1(z^t)}{\mathcal{M}_{t+1}^1(z^t, z')}\frac{\varphi_{t+1}(z^t, z')}{\varphi_t(z^t)}\end{aligned}$$

and the pricing kernel for euro assets is similarly

$$\begin{aligned}q_t^2(z^t, z') &= q_t^1(z^t, z')\frac{\mathcal{E}_{t+1}(z^t, z')}{\mathcal{E}_t(z^t)} \\ &= \beta\left(\frac{x_{t+1}(z^t, z')}{x_t(z^t)}\right)^{1-\sigma}\frac{\mathcal{M}_t^1(z^t)}{\mathcal{M}_{t+1}^1(z^t, z')}\frac{\mathcal{E}_{t+1}(z^t, z')}{\mathcal{E}_t(z^t)}\frac{\varphi_{t+1}(z^t, z')}{\varphi_t(z^t)} \\ &= \beta\left(\frac{x_{t+1}(z^t, z')}{x_t(z^t)}\right)^{1-\sigma}\frac{\mathcal{M}_t^2(z^t)}{\mathcal{M}_{t+1}^2(z^t, z')}\frac{\varphi_{t+1}(z^t, z')}{\varphi_t(z^t)}\end{aligned}$$

The price of a riskless dollar bond is therefore

$$\begin{aligned}
\frac{1}{1+i_t^1(z^t)} &= \sum_{z'} q_t^1(z^t, z') \\
&= \sum_{z'} \beta \left(\frac{x_{t+1}(z^t, z')}{x_t(z^t)}\right)^{1-\sigma} \frac{\mathcal{M}_t^1(z^t)}{\mathcal{M}_{t+1}^1(z^t, z')} \frac{\varphi_{t+1}(z^t, z')}{\varphi_t(z^t)} \\
&= \mathsf{E}_t \left\{ \beta \left(\frac{x_{t+1}}{x_t}\right)^{1-\sigma} \frac{\mathcal{M}_t^1}{\mathcal{M}_{t+1}^1} \right\}
\end{aligned}$$

(and similarly for the price of a riskless euro bond). Nominal interest differentials are driven by differences in the money growth too.

Real interest rates are given by

$$\begin{aligned}
\frac{1}{1+r_t^1(z^t)} &= \sum_{z'} q_t^1(z^t, z') \frac{P_{t+1}^1(z^t, z')}{P_t^1(z^t)} = \mathsf{E}_t \left\{ \beta \left(\frac{x_{t+1}}{x_t}\right)^{1-\sigma} \right\} \\
\frac{1}{1+r_t^2(z^t)} &= \sum_{z'} q_t^2(z^t, z') \frac{P_{t+1}^2(z^t, z')}{P_t^2(z^t)} = \mathsf{E}_t \left\{ \beta \left(\frac{x_{t+1}}{x_t}\right)^{1-\sigma} \right\}
\end{aligned}$$

Hence real interest rates are common (real interest differentials are zero).

Chris Edmond
7 October 2004