NYU STERN

Global Economy
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Problem Set #1

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Before attempting this problem set, you will probably need to read over the lecture notes on the '**Aggregate Production Function**' and the '**Solow Model**'. It might also be helpful if you revise the material on maximizing a function in the '**Mathematics Review**'.

A. Wages and returns to capital in the Solow model

The main point of this exercise is to help you understand the Solow model's implications for payments like wages and returns to capital. Along the way, we'll find out why a good rule of thumb is $\alpha = 1/3$ in a production function like

$$Y = AK^{\alpha}L^{1-\alpha}$$

In this problem we will explain some of the reasoning behind the $\alpha = 1/3$ number.

To begin with, let's suppose that we can think of the production function as being operated by a single firm that pays wage rate w for labor input and rental rate r for capital input. The profits of this firm are

$$\text{Profits} = Y - wL - rK$$

Profits are sales of output less total labor costs (wages w times labor L hired) and less total capital costs (rental rate r times capital hired K). Output Y is given by the production function, so we can also write this as

$$\text{Profits} = AK^{\alpha}L^{1-\alpha} - wL - rK$$

Now:

1. Suppose that the firm chooses to hire capital input K and labor input L to maximize its profits. Show that this leads to the following profit-maximizing conditions

$$\begin{aligned}
\frac{\partial \text{Profits}}{\partial K} &= \alpha AK^{\alpha-1}L^{1-\alpha} - r = 0 \\
\frac{\partial \text{Profits}}{\partial L} &= (1-\alpha)AK^{\alpha}L^{-\alpha} - w = 0
\end{aligned}$$

   Manipulate these equations to show that they imply i) that the firm's demand for capital is decreasing in r, and similarly that the firm's demand for labor is decreasing in w, ii) that the firm's demand for capital and labor is increasing in productivity A, and iii) that the firm's demand for labor is increasing in the amount K of capital it uses and similarly the firm's demand for capital is increasing in the amount L of labor it uses. Explain whether you think these results are 'reasonable'. Which ones do you think are realistic? Which ones are unrealistic? Why? (20 points).

2. Show that the profit maximizing conditions can be manipulated so that we can also write output as

$$Y = AK^{\alpha}L^{1-\alpha} = wL + rK$$

What does this imply for the firm's profits? (10 points).

3. For many countries, economic data suggests that labor's share of national income, wL/Y has an average value of something like

$$\frac{wL}{Y} \approx 0.67$$

Explain how you can use your previous calculations and this piece of data to calculate a suitable value for the parameter α in the production function. (10 points).

4. For the US in 2000, capital per worker was about $K/L = 138,910$ dollars and real output per worker was about $Y/L = 64,537$ dollars. Use this data and the value for α you obtained in above to calculate an implied rental rate for capital, r. The economic *return* on physical capital is often measured as the rental rate r minus the rate of depreciation of the capital stock. If depreciation is $\delta = 0.08$, what is the return on capital implied by this data? Does this seem like a reasonable number to you? Why or why not? (10 points).

Answer.

1. Recall that to maximize a function, it is often sufficient to find what value makes the first derivative of the function zero. Because the profit function that we want to maximize has two arguments, we need to find the values that simultaneously make its two *partial* derivatives equal to zero. The economics can be seen by noting that both conditions can be written in the form 'marginal benefit = marginal cost'. For example, the marginal benefit of hiring a bit more capital is the extra output that the firm can produce using that capital — in other words, the marginal product of capital: the (partial) derivative of the production function with respect to capital. The marginal cost of hiring a bit more capital is the rental rate r. For the firm to be maximizing profits, the marginal benefit and marginal cost need to be equal, otherwise the firm could increase profits by either hiring more capital (if the marginal benefit is too high) or hiring less capital (if the marginal benefit is too low). The same argument goes for labor, where the marginal cost is the wage rate w.

To see how the firm's optimal choice of capital and labor varies, we begin by solving the profit maximizing condition for capital to get an expression for K. The algebra is as follows

$$\begin{aligned} \alpha A K^{\alpha-1} L^{1-\alpha} = r, \qquad \Rightarrow \left(\frac{K}{L}\right)^{\alpha-1} &= \frac{r}{\alpha A} \\ \Rightarrow \left(\frac{K}{L}\right)^{1-\alpha} &= \frac{\alpha A}{r} \\ \Rightarrow K = \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}} L \end{aligned}$$

Notice that $1/(1-\alpha) > 0$. With this formula, we can differentiate K with respect to r, A, or L to get the results we're after, namely that the firm's demand for capital is decreasing in the price it pays r, increasing in its productivity A and increasing in the amount of labor it uses, L.

We can do the same calculations for labor

$$\begin{aligned} (1-\alpha) A K^{\alpha} L^{-\alpha} = w, \qquad \Rightarrow \left(\frac{L}{K}\right)^{-\alpha} &= \frac{w}{(1-\alpha)A} \\ \Rightarrow \left(\frac{L}{K}\right)^{\alpha} &= \frac{(1-\alpha)A}{w} \\ \Rightarrow L = \left(\frac{(1-\alpha)A}{w}\right)^{\frac{1}{\alpha}} K \end{aligned}$$

With this formula, we can differentiate L with respect to w, A, or K to get that the firm's demand for labor is decreasing in the price it pays w, increasing in its productivity A and increasing in the amount of capital it uses, K.

I think all of the implications from this exercise are reasonable. In particular, I would expect when any input is relatively expensive (either r or w is high) then firms will try to use less of it. I expect that when a firm is more productive, it will be more willing to pay given prices for capital and labor and so at those prices will tend to use more of both. Perhaps the most controversial result is that with this Cobb-Douglas production function capital and labor are *complements*. If the firm uses more labor, it also wants to use more capital (and vice-versa). Basically, by using more capital and more labor at the same time the firm helps to stave off the effects of diminishing returns. (Remember with the Cobb-Douglas production function if we double all inputs we double output, but if we double capital while holding labor constant, then we get less than double output).

2. We can re-write the equations governing the firm's optimal choices as

$$\begin{aligned} \alpha A K^{\alpha} L^{1-\alpha} &= rK \\ (1-\alpha) A K^{\alpha} L^{1-\alpha} &= wL \end{aligned}$$

But the production function is just $Y = AK^{\alpha}L^{1-\alpha}$. So we can also write

$$\begin{aligned} \alpha Y &= rK \\ (1-\alpha)Y &= wL \end{aligned}$$

Therefore

$$wL + rK = (1-\alpha)Y + \alpha Y = Y$$

as required. Clearly, this implies that profits are zero. Is this silly? We hear talk of positive profits all the time. To some extent, this is just a matter of differences in terminology. To an economist, the accounting concept known as 'corporate profits' is really just a payment to the owners of capital and so in this example should be treated as part of rK. As you might recall from Firms and Markets, in economics the term 'profit' is often reserved for discussing (say) the ability of a monopolist to exploit its market power. Since this is a perfectly competitive market, it's not surprising that economic profits are zero.

3. As we've just seen in question 2

$$(1-\alpha)Y = wL$$

So

$$1 - \alpha = \frac{wL}{Y} = 0.67$$

Therefore

$$\alpha = 0.33$$

The point here is that the concept known as 'labor's share of GDP,' wL/Y, is something we can measure in the data. On average, this number is about 0.67. In principle, α could be any number between zero and one. But for our theory to be consistent with the data, we aren't free to choose any α we like, we ought to choose an α value of 0.33 or thereabouts.

4. Again, from question 2 we have

$$\alpha Y = rK$$

And this is just the same as

$$\alpha = \frac{rK}{Y} = r\frac{K/L}{Y/L}$$

Since we have numbers for α, K/L and Y/L we can solve this equation for r. The solution is

$$r = \alpha\frac{Y/L}{K/L} = 0.33 \times \frac{64,537}{138,910} = 0.15$$

The annual rental rate of capital is about 15% (the units are annual because the GDP number is annual). This might seem high but remember that the owners of capital have to be compensated for physical depreciation. So an estimate of the annual 'return to capital' should be a number like $r - \delta$ where δ is the physical depreciation rate. A good estimate of the depreciation rate for US data is $\delta = 0.08$ or $\delta = 0.10$ per year so the return to capital seems to be around 0.05 to 0.07 (5% to 7%) per year. This seems pretty reasonable to me.

B. Wages, returns to capital, and productivity growth

Recall that in the Solow model with Cobb-Douglas production function, steady state capital per worker is given by the formula

$$\frac{K}{L} = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$$

where s is the constant national savings rate. Now:

1. Using your calculations from Part A, show that the steady state wage rate is

$$w = (1-\alpha)A^{\frac{1}{1-\alpha}}\left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

   and that the steady state rental rate of capital is

$$r = \frac{\alpha\delta}{s}$$

   (20 points).

2. Suppose that $\alpha = 0.33$, $\delta = 0.08$, $s = 0.20$ and $A = 1$. Compute the steady state wage and rental rates. Now suppose that national savings increases from $s = 0.20$ to $s = 0.25$. Compute the new values for the wage and rental rates. Given economic intuition for your answers. Hint: in the Solow model what happens to the long run (i.e., steady state) supply of capital as the savings rate increases? How does this affect the rental rate r? What happens to the demand for capital as r changes in this way? How do these effects balance out in the long run? Do you think these results are reasonable? Why or why not? (15 points).

3. Suppose that we have the same parameters, $\alpha = 0.33$, $\delta = 0.08$ and $s = 0.20$ as before, but TFP doubles from $A = 1$ to $A = 2$. Explain the effects that this has on wages and rental rates. Which input (capital or labor) benefits most from productivity growth? Explain why. Give economic intuition in terms of the demand and supply for labor and capital. Are you surprised by this finding? Why or why not? (15 points).

Answer.

1. It's useful to notice that the two conditions governing the firm's optimal choices can be re-written in terms of the K/L ratio using the fact that $K^{\alpha-1}L^{1-\alpha} = (K/L)^{\alpha-1}$ and similarly $K^{\alpha}L^{-\alpha} = (K/L)^{\alpha}$. So we can write the firm's profit maximizing conditions as

$$\begin{aligned}
\alpha A\left(\frac{K}{L}\right)^{\alpha-1} &= r \\
(1-\alpha)A\left(\frac{K}{L}\right)^{\alpha} &= w
\end{aligned}$$

But we have a formula for the steady state K/L ratio. Plugging the formula for steady state K/L into the previous two equations gives, for the rental rate of capital

$$\begin{aligned} r &= \alpha A \left(\frac{K}{L}\right)^{\alpha-1} \\ &= \alpha A \left[\left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}\right]^{\alpha-1} \\ &= \alpha A \left(\frac{sA}{\delta}\right)^{-1} \\ &= \frac{\alpha\delta}{s} \end{aligned}$$

as required. Similarly, for wages

$$\begin{aligned} w &= (1-\alpha)A \left(\frac{K}{L}\right)^{\alpha} \\ &= (1-\alpha)A \left[\left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}\right]^{\alpha} \\ &= (1-\alpha)A \left(\frac{sA}{\delta}\right)^{\frac{\alpha}{1-\alpha}} \\ &= (1-\alpha)A^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

and we're done.

2. This is just a matter of plugging in the right numbers. To begin with, we have

$$r = \frac{0.33 \times 0.08}{0.20} = 0.132$$

and

$$w = 0.67 \times 1 \times 2.5^{\frac{1}{2}} = 1.059$$

If the savings rate increases to $s = 0.25$, we now have

$$r = \frac{0.33 \times 0.08}{0.25} = 0.106$$

and

$$w = 0.67 \times 1 \times 3.125^{\frac{1}{2}} = 1.1844$$

Notice that an increase in the savings rate reduces the steady state rental rate (from 13.2% to 10.6%) and increases the wage rate. This is because a higher savings rate increases the steady state capital/worker ratio which makes capital relatively more abundant and so pushes down the rental price of capital. However, more capital makes labor more productive, so wages increase.

3. Same kind of idea. Plugging in the numbers

$$r = \frac{0.33 \times 0.08}{0.20} = 0.132$$

Notice that this does not depend on A, so when TFP increases the steady state rental rate is *unchanged*! The wage rate with $A = 1$ is, as before,

$$w = 0.67 \times 1 \times 2.5^{\frac{1}{2}} = 1.059$$

but with higher TFP of $A = 2$ it increases to

$$w = 0.67 \times 2^{\frac{3}{2}} \times 2.5^{\frac{1}{2}} = 2.995$$

What's going on here? Why does an increase in the level of TFP increase the wage rate but not increase the return to capital? Basically, there are two effects: (i) an increase in A increases the demand for capital and labor, which tends to increase both r and w. But (ii) over time, an increase in A also increases the steady state K/L ratio and an increase in the K/L ratio reduces r while increasing w. So both forces tend to increase w while the two forces counteract each other when it comes to r. It turns out that the net effect on r is exactly zero: the two offsetting forces just exactly cancel out! This is very relevant to real-world economies. Over time, the return to capital has no particular trend, it bounces around 5-6%. But over time, real wages do tend to keep increasing. Put another way, over time most of the benefits from higher productivity show up as higher wages not as higher returns to capital. Temporary increases in the return to capital just encourage more saving and investing which then wipe out any temporary increase in r while at the same time making labor more productive thereby increasing wages.