

Solow's Model of Economic Growth

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Robert Solow received the 1987 Nobel Prize in economics for developing the leading model of economic growth. The model is based on the premise that cross-country differences in income per person are the result (primarily) of differences in national savings rates (savings finances increases in the capital stock). We illustrate its properties and show how it can be used as a tool for exploring the implications of changes in saving rates, the labor force, and productivity.

The model

Solow's model has four relatively simple components. The first is our friend the production function:

$$Y = AF(K, L) = AK^\alpha L^{1-\alpha}.$$

Changes in output therefore come from changes in productivity, capital, and/or labor. The second component is a link between investment and saving. You'll recall that the flow identity $S = I + (G - T) + NX$ linked saving to investment, the government deficit, and net exports. Solow ruled out the last two (we can put them back in later if we like), giving us

$$S = I.$$

Lurking behind the scenes here is a national income accounting expenditure identity: $Y = C + I$ in this case. The third component is a description of saving behavior: people save a constant fraction s of their income,

$$S = sY,$$

where the saving rate s is a number between zero and one. This is a little simplistic — you might expect saving to depend on the rate of return or expectations of future income — but simplicity has a lot to be said for it. Finally, the capital stock depreciates at a constant rate δ , so that

$$K_{t+1} = (1 - \delta)K_t + I_t,$$

where the constant depreciation rate δ is a number between zero and one.

The model consists of these four equations. This seems kind of simple for a Nobel Prize, but they really are good equations. Now let's see where they lead.

Capital dynamics

Let's think about how the model behaves if the labor force L and productivity A are constant. Analysis of the model in this case consists of describing how the capital stock evolves through time. Other variables follow from their relations to the capital stock: we can find output from the production function, saving (= investment) from output, and consumption from the expenditure identity ($Y = C + I$).

The key step is to describe how the capital stock changes from one period to the next. With a little work, we see that the capital stock behaves like this:

$$\begin{aligned}
 K_{t+1} &= (1 - \delta)K_t + I_t \\
 &= (1 - \delta)K_t + S_t \\
 &= (1 - \delta)K_t + sY_t \\
 &= (1 - \delta)K_t + sAK_t^\alpha L^{1-\alpha}.
 \end{aligned} \tag{1}$$

Note that each step follows from one of the components of the model. If we have values for the parameters (A, α, s, δ) , we can program this up and see how K moves through time.

Example. A numerical example will show you how this works. Let $L = 100$, $A = 1$, $s = 0.2$, $\delta = 0.1$, and $\alpha = 1/3$. (We'll use the same parameters throughout.) If the initial capital stock is 250, we can compute future values of the capital stock by applying equation (1) repeatedly. (You could do this in a spreadsheet.) We then compute output from the capital stock using the production function. The results for this case are summarized in Table 1. [Suggestion: try to

Date	Capital Stock	Output
0	250.0	135.7
1	252.1	136.1
2	254.2	136.5
3	256.0	136.8
4	257.8	137.1
5	259.4	137.4
6	261.0	137.7
7	262.4	137.9
8	263.8	138.2
9	265.0	138.4
10	266.2	138.6

Table 1: Simulation of the Solow model.

reproduce a couple periods of the table to make sure you understand how it works]. You can see in the table that capital and output both increase over time. Will they increase forever? The answer

is no, but it takes a little work to show. (Alternatively, you could extend the simulation and see what happens).

The dynamics of the capital stock reflect a balance of two factors: saving tends to increase the capital stock by financing new investment and depreciation tends to reduce it. A modest change to equation (1) makes this clear:

$$\Delta K_{t+1} \equiv K_{t+1} - K_t = sAK_t^\alpha L^{1-\alpha} - \delta K_t. \quad (2)$$

(The equal sign with three lines means that the equation defines the expression that comes before it, in this case ΔK_{t+1} .) You can see that the change is zero (the capital stock doesn't change) when

$$\bar{K} = \left(\frac{sA}{\delta} \right)^{1/(1-\alpha)} L,$$

where \bar{K} is the 'steady state' capital stock. This is a little complicated, but remember: it's just a formula. In our example, $\bar{K} = 282.8$, so we have a ways to go before the model reaches its steady state. What happens if we are above or below \bar{K} ? You can get a sense of the dynamics from Figure 1. The top line is output, which is related to the capital stock through the production

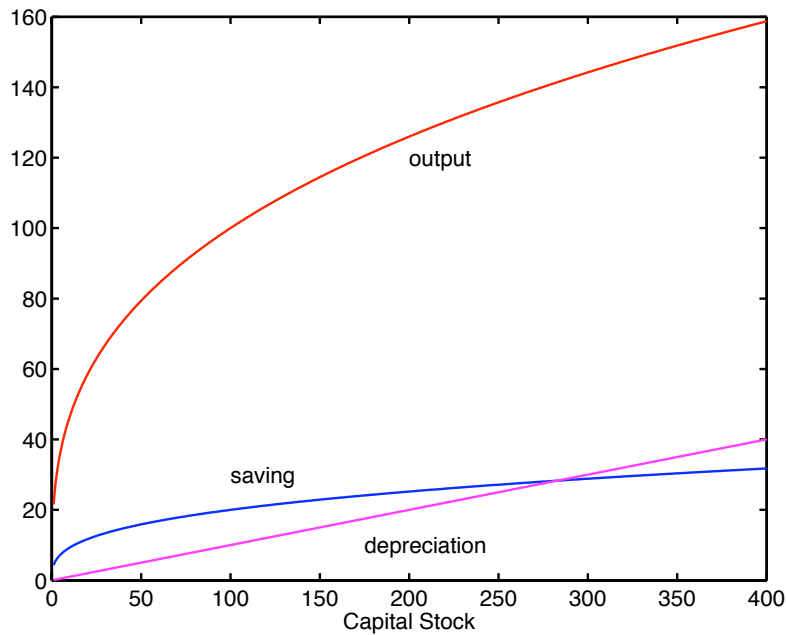


Figure 1: The Solow model.

function. The next line is saving, a constant fraction of output and the first expression on the right side of equation (2): $sAK^\alpha L^{1-\alpha}$. The third line is depreciation, a constant fraction δ of the capital stock and the second object on the right side of equation (2): δK . Diminishing returns to capital gives the saving line its curvature. It leads to higher saving than depreciation at low values of the capital stock, so the capital stock is increasing. Similarly, saving is lower than depreciation

at high values of the capital stock, so the capital stock falls. The crossing point is \bar{K} , where saving is just enough to make up for depreciation, leaving the capital stock unchanged.

Convergence

The central feature of the model is what we call the convergence property: if countries have the same parameters, they will eventually converge to the same level of output per worker. We haven't quite shown this yet, but the only thing missing is the 'per worker' qualification.

Let us consider, then, a version of the model in per worker terms. The first step is to divide both sides of (1) by L . If $k \equiv K/L$ is capital per worker (or the capital-labor ratio), the equation becomes

$$k_{t+1} = (1 - \delta)k_t + sAk_t^\alpha$$

or

$$\Delta k_{t+1} \equiv k_{t+1} - k_t = sAk_t^\alpha - \delta k_t. \quad (3)$$

You'll note a resemblance to equation (2).

Figure 2 illustrates the model's dynamics. It's based on the same parameter values as our earlier example: $A = 1$, $s = 0.2$, $\delta = 0.1$, and $\alpha = 1/3$. The line marked 'saving per worker' is the first

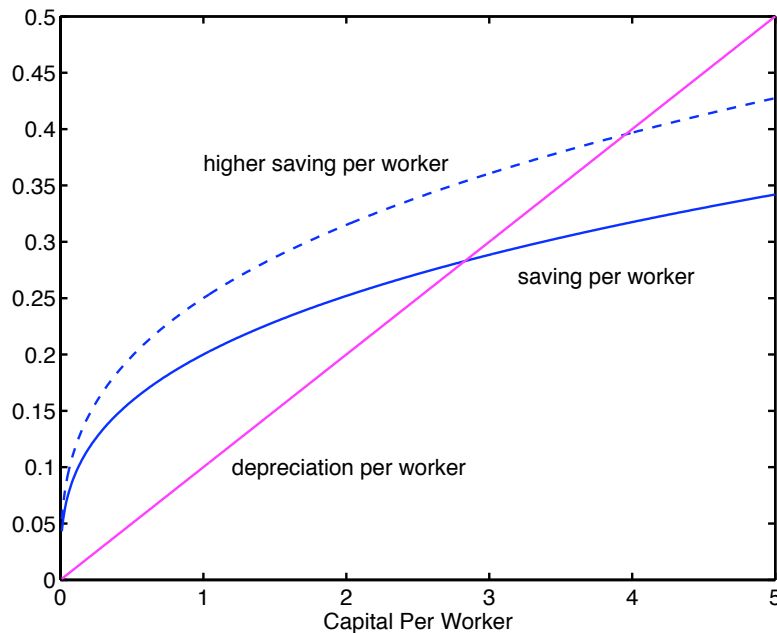


Figure 2: The per worker Solow model and the impact of higher saving.

expression on the right side of equation (3): sAk_t^α . The line marked 'depreciation per worker' is

the second expression on the right side of equation (3): δk . For small values of k , saving per worker is greater than depreciation per worker, so k increases. For large values of k , saving per worker is less than depreciation per worker, so k decreases. The two lines cross at the steady state, where the capital-labor ratio is constant. We can find this steady state value of k from equation (3) by setting $\Delta k_{t+1} = 0$. This leads to

$$\bar{k} = \left(\frac{sA}{\delta} \right)^{1/(1-\alpha)},$$

a minor variant of our earlier expression for the steady state capital stock.

We have shown that the capital-labor ratio eventually converges to its steady state value. What about output per worker? The production function in per worker form is $Y/L = Ak^\alpha$, so steady state output per worker depends on steady state capital per worker:

$$\overline{(Y/L)} = A\bar{k}^\alpha = A \left(\frac{sA}{\delta} \right)^{\alpha/(1-\alpha)} = A^{1/(1-\alpha)} \left(\frac{s}{\delta} \right)^{\alpha/(1-\alpha)}.$$

The algebra isn't pretty, but it tells us how the steady state depends on the various parameters. More important, the existence of a steady state tells us that if two countries have the same parameter values, they will converge to the same output per worker. We refer to this as the convergence property. In this model, any long-term differences between countries must come from differences in their parameters.

Saving rates

We've seen that steady state output per worker depends on the parameters, including the saving rate. This is apparent from the formula for steady state output per worker above, but the logic is more transparent in Figure 2. The line marked 'saving per worker' is based on a saving rate of $s = 0.2$ or 20%. Suppose the saving rate were 25%; what would the impact be on output per worker? In the figure, the saving line shifts up, as shown by the dashed line marked 'higher saving per worker.' Why? Because sAk^α is higher at every value of k . With this new line, the steady state value of capital per worker (where the saving line crosses the depreciation line) is higher, as shown.

Example (continued). We can get a sense of the magnitudes involved by returning to our numerical example. Recall that with $s = 0.2$ the steady state capital-labor ratio is

$$\bar{k} = \left(\frac{sA}{\delta} \right)^{1/(1-\alpha)} = \left(\frac{0.20 \times 1}{0.1} \right)^{3/2} = 2.828.$$

If s rises to 0.25, k_{ss} rises to 3.953 (replace 0.2 with 0.25 in the formula above). Similarly, steady state output per worker is

$$\overline{(Y/L)} = A\bar{k}^\alpha = 1 \times (2.828)^{1/3} = 1.414.$$

When $s = 0.25$, this rises to 1.581.

In short: if two countries have different saving rates, the one with the higher rate has higher steady state capital-labor ratio and income per worker.

Labor force growth

Countries differ dramatically in their rate of labor force growth, with poor countries typically growing faster in this respect than rich countries. What impact does that have on steady state output per worker? This is a more complex experiment and requires some modification of the model.

Let's see how the model changes if we introduce labor force growth. If it grows at a constant rate n , then

$$L_{t+1}/L_t = 1 + n.$$

With this modification, equation (1) becomes

$$K_{t+1} = (1 - \delta)K_t + sAK_t^\alpha L_t^{1-\alpha}.$$

If we divide both sides by L_t , we get

$$K_{t+1}/L_t = (K_{t+1}/L_{t+1})(L_{t+1}/L_t) = k_{t+1}(1 + n) = (1 - \delta)k_t + sAk_t^\alpha.$$

The change in the capital-labor ratio is therefore

$$\Delta k_{t+1} = \left(\frac{s}{1 + n} \right) Ak^\alpha - \left(\frac{\delta + n}{1 + n} \right) k.$$

This is a little different from before, but we can study it the same way. We can show, for example, that the capital-labor ratio converges. The steady state follows from setting $\Delta k_{t+1} = 0$:

$$\bar{k} = \left(\frac{sA}{\delta + n} \right)^{1/(1-\alpha)}.$$

If $n = 0$, this reduces to our earlier expression.

Note that the country with the higher rate of labor force growth has lower steady state capital per worker. Why? Because the addition of labor each period is a second force (depreciation being the other) driving the capital-labor ratio down. If the saving rate doesn't change, the result is lower steady state capital per worker. What about output per worker? Its steady state value is

$$\overline{(Y/L)} = A\bar{k}^\alpha = A \left(\frac{sA}{\delta + n} \right)^{\alpha/(1-\alpha)} = A^{1/(1-\alpha)} \left(\frac{s}{\delta + n} \right)^{\alpha/(1-\alpha)}.$$

A country with higher labor force growth thus has lower steady state output per worker.

Example (continued). Consider our original parameter values: $A = 1$, $s = 0.2$, $\delta = 0.1$, $\alpha = 1/3$, and (implicitly) $n = 0$. What happens to steady state capital per worker and output per worker if n rises to 0.01? Recall that for the original parameter values, steady state k is 2.828 and steady state Y/L is 1.414. If n is 0.01, these numbers fall to 2.452 and 1.348, respectively.

Executive summary

1. Solow's model bases growth on saving and investment.
2. Convergence property: countries with similar parameters will (eventually) have similar output per worker. Conversely: longstanding differences in output per worker must stem from differences in (primarily) productivity and saving rates.
3. If two countries have different saving rates, the one with the higher rate has higher steady state income per worker.
4. If two countries have different labor force growth rates, the one with the higher rate has lower steady state income per worker.

Review questions

1. For the numerical example in the text:
 - (a) Suppose the economy starts with the steady state capital stock. What are the steady state levels of output, investment, and consumption?
 - (b) If 25% of the capital stock is destroyed in a war, how long does it take the economy to eliminate half the fall in output?

Answer.

- (a) The steady state capital stock is (as we've seen) $\bar{K} = 282.8$. Using this value, the production function tells us that output is 141.4. Investment equals the depreciation of the capital stock, 28.3. We can find consumption two ways. The first is through the expenditure identity: $Y = C + I$. We know Y and I , so C is 113.1. The second is through the flow identity. Saving is fraction s of output, so consumption is fraction $1 - s$, $0.8 \times 141.4 = 113.1$.

(b) This requires a simulation. Let the capital stock fall to 212.1, 75% of its steady state value. Then output is 128.5, 90.9% of its steady state value. We recover half the fall if output rises to 135.0. If we simulate the model, we see that it reaches 135.1 in 10 periods (years).

2. Consider the model with population growth, and suppose $\delta = 0.1$, $s = 0.2$, $A = 1$, and $n = 0.01$. How much does steady state output per worker fall if n rises to 0.02?

Answer. With $n = 0.01$, steady state output per worker is 1.348. When $n = 0.02$, it falls to 1.291, a drop of about 4%.

Further reading

Goldman Sachs has used the Solow model (and some heroic assumptions about fundamentals) to forecast the importance of the BRICs (Brazil, Russia, India, and China) to the world economy in 50 years. See '[Dreaming with BRICs](#).' It illustrates how assumptions about productivity, population growth, and education can be used to generate plausible scenarios for the relative sizes of economies in the distant future.