**B30.3332**
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**Risk-sharing with endogenous segmentation**
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This note introduces a simple version of the endogenous segmentation setup used by Alvarez, Atkeson and Kehoe (2002). The setting is a static endowment economy without money.

# 1 Review of complete risk sharing

**Preferences and endowments.** Consider a static endowment economy populated by a continuum of households. The households have identical preferences expressed by a strictly increasing strictly concave utility function $U(c)$ over a single consumption good. The households have heterogeneous endowments $y \in [0, \infty)$ of this single consumption good distributed with density $f(y) > 0$ in the population. Let $Y := \int_0^\infty y f(y) dy$ denote the aggregate endowment.

Now consider a social planner who weights these households equally and seeks to choose a consumption allocation $c(y) \geq 0$ for each household type to maximize social welfare

$$\int_0^\infty U[c(y)] f(y) dy \tag{1}$$

subject to the resource constraint

$$\int_0^\infty c(y) f(y) dy \leq Y \tag{2}$$

**Optimization.** Let $\lambda \geq 0$ denote the multiplier on the planner's resource constraint. The Lagrangian for this problem can be written

$$L = \int_0^\infty \{U[c(y)] + \lambda[y - c(y)]\} f(y) dy \tag{3}$$

and the key first order condition is

$$U'[c(y)] = \lambda, \qquad \text{for all } y \in [0, \infty) \tag{4}$$

Therefore consumption is a constant, $\bar{c}$ say, for all $y$. The idiosyncratic endowment risk is perfectly shared in the sense that no matter what the household's endowment $y$, all households have the same consumption allocation. We solve for $\bar{c}$ using the resource constraint

$$Y = \int_0^\infty c(y) f(y) dy = \int_0^\infty \bar{c} f(y) dy = \bar{c} \tag{5}$$

The particular level of the constant consumption allocated to each household is, of course, that each household receive an allocation equal to the aggregate endowment. Notice that this implies the solution for the multiplier is $\lambda = U'(Y)$. In a decentralized version of this model, the real asset pricing kernel would be proportional to this multiplier. Loosely speaking, asset prices would be determined by the marginal utility of the 'representative agent'.

# 2 Endogenous segmentation and incomplete risk sharing

**Fixed costs.** Now consider the same environment with the added assumption that a household can only be insured if the planner pays a real resource cost $\gamma > 0$ denoted in units of the consumption good. Let $z(y) \in \{0, 1\}$ denote an indicator function $z(y) = 1$ if the planner pays the fixed cost $\gamma$ for household $y$ and $z(y) = 0$ if the planner does not pay the fixed cost $\gamma$ for household $y$.

With this extra restriction, the resource constraint facing the planner is instead

$$\int_0^\infty [c(y) + \gamma z(y)] f(y) dy \leq Y \tag{6}$$

To solve this problem it's convenient to define $x(y)$ as a household's net transfer from the pool of aggregate resources if the planner pays the fixed cost $\gamma$. With this notation we can write $c(y) = y + x(y)z(y)$ so that $c(y) = y + x(y)$ if $z(y) = 1$ and $c(y) = y$ if $z(y) = 0$. Using this notation we can say that the planner's problem is to choose $x(y)$ and $z(y) \in \{0, 1\}$ to maximize social welfare

$$\int_0^\infty U[y + x(y)z(y)] f(y) dy \tag{7}$$

subject to the resource constraint

$$\int_0^\infty [y + x(y)z(y) + \gamma z(y)] f(y) dy \leq Y \tag{8}$$

Equivalently the resource constraint is

$$\int_0^\infty [x(y) + \gamma] z(y) f(y) dy \leq 0 \tag{9}$$

**Optimization.** Let $\lambda \geq 0$ denote the multiplier on the planner's resource constraint. The Lagrangian for this problem can be written

$$L = \int_0^\infty \{U[y + x(y)z(y)] - \lambda[x(y) + \gamma]z(y)\} f(y) dy \tag{10}$$

and the key first order condition associated with the choice of $x(y)$ is

$$U'[c(y)]z(y) = \lambda z(y), \qquad \text{for all } y \in [0, \infty) \tag{11}$$

If $z(y) = 0$, when the fixed cost is not paid, this first order condition has no content. In this case we know that $c(y) = y$ and $x(y) = 0$. If $z(y) = 1$, when the fixed cost is paid, then we have

$$U'[c(y)] = \lambda, \qquad \text{for all } y \text{ such that } z(y) = 1 \tag{12}$$

All households that have $z(y) = 1$ have constant consumption $c(y) = \bar{c}$ for some constant $\bar{c}$ to be determined. Moreover, $\lambda = U'(\bar{c})$. This implies that all household that have $z(y) = 1$ have transfers $x(y) = \bar{c} - y$. We now need to determine when in fact $z(y) = 0$ and when $z(y) = 1$.

**Solving for $z(y)$ given $\bar{c}$.** Fix a $y$ and consider the integrand of the Lagrangian

$$V(z, y) := U[y + x(y)z] - \lambda[x(y) + \gamma]z \tag{13}$$

The contribution to the Lagrangian when $z = 0$ is

$$V(0, y) = U(y)$$

meaning that if the fixed cost is not paid, the net contribution to social welfare is just the autarkic utility $U(y)$.

The contribution to the Lagrangian when $z = 1$ is

$$V(1, y) = U[y + x(y)] - \lambda[x(y) + \gamma]$$

Now using $x(y) = \bar{c} - y$ and $\lambda = U'(\bar{c})$ for any $y$ such that $z = 1$ we can write this as

$$V(1, y) = U(\bar{c}) - U'(\bar{c})(\bar{c} + \gamma - y)$$

meaning that if the fixed cost is paid, the contribution to social welfare is the insured utility $U(\bar{c})$ less the utility lost by paying the fixed cost $\gamma$ and switching to consumption $\bar{c}$ from $y$ evaluated using the shadow price $\lambda = U'(\bar{c})$ of resources available to the planner.

It is optimal for $y$ to have $z = 1$ if and only if

$$V(1, y) - V(0, y) \geq 0 \tag{14}$$

equivalently if and only if

$$U(\bar{c}) - U'(\bar{c})(\bar{c} + \gamma - y) - U(y) \geq 0 \tag{15}$$

Now for given $\bar{c}$ we can define a function

$$h(y, \bar{c}) := U(\bar{c}) - U(y) - U'(\bar{c})(\bar{c} + \gamma - y) \tag{16}$$

that measures the net contribution to social welfare obtained from switching a household with $y$ from non-participation to participation in the risk sharing arrangement. So for given $\bar{c}$, the planner will choose $z(y) = 1$ for all households with $y$ such that $h(y, \bar{c}) \geq 0$ and $z(y) = 0$ for all households with $y$ such that $h(y, \bar{c}) < 0$.

Notice that

$$\frac{\partial h}{\partial y} = -U'(y) + U'(\bar{c}) \geq 0 \Leftrightarrow \bar{c} \leq y \tag{17}$$

and

$$\frac{\partial^2 h}{\partial y^2} = -U''(y) > 0 \tag{18}$$

So $h(y, \bar{c})$ is strictly convex in $y$, strictly decreasing for all $y < \bar{c}$, reaching a minimum at $y = \bar{c}$ and then increasing for all $y > \bar{c}$. Moreover $h(\bar{c}, \bar{c}) = -U'(\bar{c})\gamma < 0$ and $h(\bar{c}+\gamma, \bar{c}) = U(\bar{c}) - U(\bar{c}+\gamma) < 0$. Therefore $h(y, \bar{c}) < 0$ for all $y \in (\bar{c}, \bar{c}+\gamma)$ and so certainly for these $y$ we know $z(y) = 0$.

Given $\bar{c}$ the function $h(y, \bar{c}) = 0$ typically has two solutions, call these $y_L$ and $y_H$ with $y_L < \bar{c} < \bar{c} + \gamma < y_H$ such that

$$h(y_L, \bar{c}) = 0 = h(y_H, \bar{c})$$

To summarize we have an *inaction region*, an interval $(y_L, y_H)$ on which $h(y, \bar{c}) < 0$ such that $z(y) = 0$, and *active regions* $y \leq y_L$ and $y \geq y_H$ on which $h(y, \bar{c}) \geq 0$ such that $z(y) = 1$. The fixed cost $\gamma$ is only paid for households with relatively high or low endowments who are pooled together to share their idiosyncratic risk. Households with moderate endowments are left to their own and do not receive any risk sharing.

Notice that the cutoffs $y_L$ and $y_H$ are a function of $\bar{c}$. To acknowledge this, write the cutoffs $y_L(\bar{c})$ and $y_H(\bar{c})$. Moreover

$$\frac{\partial h}{\partial \bar{c}} = -U''(\bar{c})(\bar{c} - y + \gamma) \geq 0 \Leftrightarrow \bar{c} + \gamma \geq y$$

Using this and the implicit function theorem gives that $y_L'(\bar{c}) > 0$ and $y_H'(\bar{c}) > 0$ so that an increase in the amount of consumption allocated to households participating in the risk sharing arrangement increases shifts the inaction region $(y_L, y_H)$ to the right.

**Solving for $\bar{c}$.** To complete the solution of the model we have to determine the value of $\bar{c}$. As usual, this is done using the resource constraint. Writing

$$\begin{aligned} Y &= \int_0^\infty [y + x(y)z(y) + \gamma z(y)] f(y) dy \\ &= \int_0^{y_L} (\bar{c}+\gamma) f(y) dy + \int_{y_L}^{y_H} y f(y) dy + \int_{y_H}^\infty (\bar{c}+\gamma) f(y) dy \end{aligned}$$

we need to find the $\bar{c}$ which solves

$$G(\bar{c}, y_L, y_H) = 0$$

where

$$G(\bar{c}, y_L, y_H) := (\bar{c}+\gamma)[F(y_L) + 1 - F(y_H)] + \int_{y_L}^{y_H} y f(y) dy - Y$$

and where $y_L, y_H$ are both implicitly functions of $\bar{c}$ with $F(y) := \int_0^y f(t)dt$ the cumulative distribution implied by $f$.

The function $G$ has derivatives

$$\begin{aligned} \frac{\partial G}{\partial \bar{c}} &= F(y_L) + 1 - F(y_H) > 0 \\ \frac{\partial G}{\partial y_L} &= (\bar{c} + \gamma - y_L) f(y_L) > 0 \\ \frac{\partial G}{\partial y_H} &= (y_H - \bar{c} - \gamma) f(y_H) > 0 \end{aligned}$$

where the second inequality follows from $y_L < \bar{c} < \bar{c} + \gamma$ and the third inequality follows from $y_H > \bar{c} + \gamma$. Now let

$$g(\bar{c}) := G[\bar{c}, y_L(\bar{c}), y_H(\bar{c})]$$

Since $y_L'(\bar{c}) > 0$ and $y_H'(\bar{c}) > 0$ we know that $g(\bar{c})$ is continuous and strictly increasing so by the intermediate value theorem $g(\bar{c}) = 0$ has a unique solution (since there exist $c_L, c_H$ such that $g(\bar{c}) < 0$ for all $\bar{c} < c_L$ and $g(\bar{c}) > 0$ for all $\bar{c} > c_H$.)

To wrap up, if we fix a $\gamma$ and an endowment distribution $f(y)$ we solve $h(y_L, \bar{c}) = 0 = h(y_H, \bar{c})$ for the implicit functions $y_L(\bar{c})$ and $y_H(\bar{c})$. Given those functions we construct the composite function $g(\bar{c})$ and solve $g(\bar{c}) = 0$ to determine the amount of consumption allocated to households who participate in the risk sharing arrangement (for whom the planner pays the fixed cost $\gamma$). This in turn determines the exact boundaries of the inaction region, an interval $(y_L, y_H)$ such that $z(y) = 0$ with $c(y) = y$ for all $y \in (y_L, y_H)$ and otherwise $z(y) = 1$ with $c(y) = \bar{c}$. This $\bar{c}$ also determines the multiplier $\lambda = U'(\bar{c})$. In a decentralized version of this model, the real asset pricing kernel would be proportional to this multiplier. Loosely speaking, asset prices would therefore be determined by the marginal utility of the active or participating agents, $U'(\bar{c})$, not the marginal utility of the representative agent, $U'(Y)$.

Notice finally that if we make fixed costs small, $\gamma \to 0$, we have $h(y, \bar{c})$ reaching its minimum at $h(\bar{c}, \bar{c}) = 0$ so that $h(y, \bar{c}) \geq 0$ for all $y$, all households have $z(y) = 1$ and $\bar{c} = Y$ as in the complete risk sharing example.

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