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Endogenous asset market segmentation: interest rate dynamics

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This note outlines the response of nominal and real interest rates to money growth shocks in a model with segmented asset markets. The basic objective is to decompose exogenous changes in money growth μ_t into changes in real interest rates r_t and expected inflation $\mathbb{E}_t\{\pi_{t+1}\}$.

Log-linearization of marginal utility

Recall from Note 3 that in the Alvarez, Atkeson, Kehoe (2002) model the level of consumption of an active household can be written as a function of the current money growth realization $\bar{c}(\mu_t)$.

With CRRA utility the marginal utility of an active household is

$$U'(\bar{c}(\mu_t)) = \bar{c}(\mu_t)^{-\sigma}, \qquad \sigma > 0 \tag{1}$$

Let μ_t follow a stationary stochastic process with long-run mean $\bar{\mu}$ and let $\hat{\mu}_t := \log(\mu_t) - \log(\bar{\mu})$ denote the log-deviation of money growth from its long-run mean. Then log-linearization of the marginal utility of active households gives

$$\log U'(\bar{c}(\mu_t)) - \log U'(\bar{c}(\bar{\mu})) \approx -\phi(\bar{\mu})\hat{\mu}_t \tag{2}$$

with elasticity

$$\phi(\bar{\mu}) := \sigma \left.\frac{\partial \log(\bar{c}(\mu))}{\partial \log(\mu)}\right|_{\mu=\bar{\mu}} \tag{3}$$

Alvarez, Atkeson and Kehoe (2002) assume that this elasticity is positive, i.e., that increases in money growth increase the consumption of active households (and reduce the consumption of inactive households). This is not true in general but does seem to be true for reasonable parameterizations of the model.

Bond prices and interest rates

Nominal and real interest rates in this model satisfy the usual asset pricing formulas with one twist: the marginal utility entering the stochastic discount factor is the marginal utility of active households, not the marginal utility of the representative household. That is, $U'(\bar{c}(\mu_t))$ instead of $U'(Y)$.

The one-period nominal interest rate is therefore

$$\frac{1}{1+i_t} = \beta \mathbb{E}_t \left\{ \frac{U'(\bar{c}(\mu_{t+1}))}{U'(\bar{c}(\mu_t))} \frac{P_t}{P_{t+1}} \right\}, \qquad \frac{P_{t+1}}{P_t} = \mu_{t+1} \tag{4}$$

(using the fact that velocity is constant so inflation is equal to money growth). Log-linearizing this we have

$$-\hat{\imath}_t \approx \mathbb{E}_t \left\{ -\phi(\bar{\mu})(\hat{\mu}_{t+1} - \hat{\mu}_t) - \hat{\mu}_{t+1} \right\} \tag{5}$$

or

$$\hat{\imath}_t = \phi(\bar{\mu}) \mathbb{E}_t \left\{ \hat{\mu}_{t+1} - \hat{\mu}_t \right\} + \mathbb{E}_t \left\{ \hat{\mu}_{t+1} \right\} \tag{6}$$

This is a log-linearized Fisher equation. The first term on the right hand side is the log-deviation of the real interest rate, $\hat{r}_t = \phi(\bar{\mu}) \mathbb{E}_t \left\{ \hat{\mu}_{t+1} - \hat{\mu}_t \right\}$ while the second term is expected inflation $\mathbb{E}_t \left\{ \hat{\pi}_{t+1} \right\} = \mathbb{E}_t \left\{ \hat{\mu}_{t+1} \right\}$.

In a basic cash-in-advance economy with a constant aggregate endowment, the real interest rate would be constant and an increase in money growth would only have an expected inflation effect. Here, an increase in money growth can also change the real interest rate.

Yield curve. Similarly the price of bonds of any maturity $n \geq 1$ is given by

$$q_t^n = \beta^n \mathbb{E}_t \left\{ \frac{U'(\bar{c}(\mu_{t+n}))}{U'(\bar{c}(\mu_t))} \frac{P_t}{P_{t+n}} \right\} \tag{7}$$

Log-linearizing this gives

$$\hat{q}_t^n \approx \mathbb{E}_t \left\{ -\phi(\bar{\mu})(\hat{\mu}_{t+n} - \hat{\mu}_t) - \sum_{j=1}^{n} \hat{\mu}_{t+j} \right\} \tag{8}$$

So that the yield curve is given by

$$\hat{\imath}_t^n \approx \frac{1}{n} \phi(\bar{\mu}) \mathbb{E}_t \left\{ \hat{\mu}_{t+n} - \hat{\mu}_t \right\} + \frac{1}{n} \mathbb{E}_t \left\{ \sum_{j=1}^{n} \hat{\mu}_{t+j} \right\} \tag{9}$$

(using $\hat{\imath}_t^n = -\hat{q}_t^n / n$ to turn bond prices into yields, both written in log deviations). Given a stochastic process for $\hat{\mu}_t$, it's straightforward to calculate bond prices and yields at all maturities $n \geq 1$. We'll now do these calculations for a couple of different stochastic processes for $\hat{\mu}_t$.

AR(1) example. Suppose that money growth follows an AR(1) in log-deviations with Gaussian errors that have constant conditional variance

$$\hat{\mu}_{t+1} = \rho \hat{\mu}_t + \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim \text{ IID and } N(0, \sigma_\varepsilon^2), \qquad 0 \leq \rho < 1 \tag{10}$$

Then expected inflation is $\mathbb{E}_t\{\hat{\mu}_{t+1}\} = \rho\hat{\mu}_t$ and real and nominal interest rates are

$$\begin{aligned}\hat{r}_t &= \phi(\bar{\mu})(\rho-1)\hat{\mu}_t \\ \hat{\imath}_t &= [\phi(\bar{\mu})(\rho-1)+\rho]\hat{\mu}_t\end{aligned}$$

Since $\rho < 1$ and $\phi(\bar{\mu}) > 0$ the coefficient $\phi(\bar{\mu})(\rho-1) < 0$ so that an increase in money growth reduces real interest rates on impact. Since expected inflation is $\rho\hat{\mu}_t$, this also means that the covariance between real interest rates and expected inflation is negative, so this is at least qualitatively consistent with the evidence in Barr and Campbell (1997).

Does an increase in money growth also reduce nominal interest rates (i.e., produce a liquidity effect)? It depends on how big the asset market segmentation effect is

$$\frac{d\hat{\imath}_t}{d\hat{\mu}_t} = [\phi(\bar{\mu})(\rho-1)+\rho] < 0 \Leftrightarrow \phi(\bar{\mu}) > \frac{\rho}{1-\rho}$$

If $\rho = 0$ so that money growth is IID, then any segmentation effect $\phi(\bar{\mu}) > 0$ is sufficient to deliver a liquidity effect because in this case an increase in money growth does not increase expected inflation. If $\rho = 1$, then the expected inflation effect will always dominate. Typical univariate estimates of the autocorrelation ρ are in the range 0.5 or 0.6 for quarterly data which implies we need something like $\phi(\bar{\mu}) > 1$ to get a liquidity effect. Recall that $\phi(\bar{\mu})$ is the product of two terms, σ, the coefficient of relative risk aversion, and the elasticity of active household's consumption to money growth. So high risk aversion helps generate a liquidity effect as does a large elasticity of active household's consumption to money growth. Also $\phi(\bar{\mu})$ depends on the long run money growth rate $\bar{\mu}$ and for large enough $\bar{\mu}$ it will be the case that $\phi(\bar{\mu}) \to 0$. This is because high inflation increases the incentive to pay the fixed cost and be active, but if every household is active the consumption of active households is just $\bar{c}(\mu_t) = Y$ all μ_t and so money growth ceases to have any redistributive effect. In short, in economies with high inflation/money growth we ought to find that the segmentation effect is small and dominated by the Fisher expected inflation effect.

This rationalizes the 'stylized fact' that the non-neutrality of money is 'non-linear' in the average level of money growth. That is, in countries with higher average inflation rates, the Fisher effect tends to be dominant so that an increase in the money supply merely raises expected inflation and nominal interest rates without changing real rates, but in countries with low average inflation rates, the Fisher effect is more modest and increases in the money supply can, at least in the the short run, decrease real and nominal interest rates.

How do interest rates on bonds of longer maturities respond to a money growth shock? Recall that in log deviations yields are given by

$$\hat{\imath}_t^n = \frac{1}{n}\phi(\bar{\mu})\mathbb{E}_t\{\hat{\mu}_{t+n} - \hat{\mu}_t\} + \frac{1}{n}\mathbb{E}_t\left\{\sum_{j=1}^{n}\hat{\mu}_{t+j}\right\} \tag{11}$$

Since money growth is an AR(1) we have $\mathbb{E}_t\{\hat{\mu}_{t+n}\} = \rho^n \hat{\mu}_t$ and also

$$\mathbb{E}_t\left\{\sum_{j=1}^{n} \hat{\mu}_{t+j}\right\} = \sum_{j=1}^{n} \mathbb{E}_t\{\hat{\mu}_{t+j}\} = \sum_{j=1}^{n} \rho^j \hat{\mu}_t = \frac{\rho}{1-\rho}(1-\rho^n)\hat{\mu}_t$$

Therefore the yields in log deviations are given by

$$\hat{\imath}_t^n = \frac{1}{n}\phi(\bar{\mu})(\rho^n - 1)\hat{\mu}_t + \frac{1}{n}\frac{\rho}{1-\rho}(1-\rho^n)\hat{\mu}_t = \frac{\eta(\bar{\mu})}{n}\frac{1-\rho^n}{1-\rho}\hat{\mu}_t \tag{12}$$

where

$$\eta(\bar{\mu}) := \phi(\bar{\mu})(\rho - 1) + \rho$$

is the response of the one period nominal interest rate $\hat{\imath}_t := \hat{\imath}_t^1$ to a money growth shock. As discussed above, there is a liquidity effect if this coefficient $\eta(\bar{\mu}) < 0$. In either case, the whole yield curve moves together in the sense that the *sign* of the response to $\hat{\mu}_t$ is the same for every n. The *size* of the response to $\hat{\mu}_t$ is not the same, however. Since $(1-\rho^n)/n$ is strictly decreasing in n, the biggest response to $\hat{\mu}_t$ is for $n = 1$. For large n the response is small. This means that a money growth shock *steepens* the yield curve, with (say) short rates falling more than long rates in response to an increase in money growth (if $\eta(\bar{\mu}) < 0$). Over time, the money growth impulse fades away and all yields revert to their long run levels. At what speed do they revert? Since every yield is a linear function of a single state variable $\hat{\mu}_t$ and $\hat{\mu}_t$ has persistence ρ, all yields also have persistence ρ.

General MA(∞) money growth. Suppose money growth has the general MA representation

$$\hat{\mu}_t = \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j}, \qquad \varepsilon_{t-j} \sim \text{ IID and } N(0, \sigma_\varepsilon^2),$$

(the AR(1) example being the special case $\theta_j = \rho^j$ for $j = 0, 1, ...$). The one-step-ahead forecast of an MA(∞) is simple

$$\hat{\mu}_{t+1} = \theta_0 \varepsilon_{t+1} + \theta_1 \varepsilon_{t+1-1} + \theta_2 \varepsilon_{t+1-2} + \dots$$

so

$$\mathbb{E}_t\{\hat{\mu}_{t+1}\} = 0 + \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-1} + \cdots = \sum_{j=1}^{\infty} \theta_j \varepsilon_{t+1-j} = \sum_{j=0}^{\infty} \theta_{j+1} \varepsilon_{t-j}$$

The short nominal rate is therefore

$$\begin{aligned}
\hat{\imath}_t &= \phi(\bar{\mu})\mathbb{E}_t\left\{\hat{\mu}_{t+1} - \hat{\mu}_t\right\} + \mathbb{E}_t\left\{\hat{\mu}_{t+1}\right\} \\
&= \phi(\bar{\mu})\left[\sum_{j=0}^{\infty} \theta_{j+1}\varepsilon_{t-j} - \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j}\right] + \sum_{j=0}^{\infty} \theta_{j+1}\varepsilon_{t-j} \\
&= \sum_{j=0}^{\infty} [\phi(\bar{\mu})(\theta_{j+1} - \theta_j) + \theta_{j+1}]\varepsilon_{t-j}
\end{aligned}$$

so the short nominal rate is also an MA(∞) which we can write $\hat{\imath}_t = \sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}$ with coefficients

$$\alpha_j := \phi(\bar{\mu})(\theta_{j+1} - \theta_j) + \theta_{j+1}$$

We can generalize this for all yields by noting that the n-step-ahead forecast of an MA(∞) is

$$\mathbb{E}_t\{\hat{\mu}_{t+n}\} = \sum_{j=n}^{\infty} \theta_j \varepsilon_{t+n-j} = \sum_{j=0}^{\infty} \theta_{j+n} \varepsilon_{t-j}$$

and also

$$\mathbb{E}_t\left\{\sum_{j=1}^{n} \hat{\mu}_{t+j}\right\} = \sum_{j=1}^{n} \mathbb{E}_t\{\hat{\mu}_{t+j}\} = \sum_{j=1}^{n}\sum_{k=0}^{\infty} \theta_{k+j}\varepsilon_{t-k} = \sum_{j=0}^{\infty}\sum_{k=1}^{n} \theta_{j+k}\varepsilon_{t-j}$$

Therefore the yields in log deviations are given by

$$\begin{aligned} \hat{\imath}_t^n &= \frac{1}{n}\phi(\bar{\mu})\mathbb{E}_t\left\{\hat{\mu}_{t+n} - \hat{\mu}_t\right\} + \frac{1}{n}\mathbb{E}_t\left\{\sum_{j=1}^{n}\hat{\mu}_{t+j}\right\} \\ &= \frac{1}{n}\phi(\bar{\mu})\left[\sum_{j=0}^{\infty}\theta_{j+n}\varepsilon_{t-j} - \sum_{j=0}^{\infty}\theta_j\varepsilon_{t-j}\right] + \frac{1}{n}\sum_{j=0}^{\infty}\sum_{k=1}^{n}\theta_{j+k}\varepsilon_{t-j} \end{aligned}$$

so that the yield curve is also an MA(∞) which we can write $\hat{\imath}_t^n = \sum_{j=0}^{\infty} \alpha_j^n \varepsilon_{t-j}$ with coefficients

$$\alpha_j^n := \frac{1}{n}\left[\phi(\bar{\mu})(\theta_{j+n} - \theta_j) + \sum_{k=1}^{n}\theta_{j+k}\right]$$

Notice that this recovers the MA coefficients for the short nominal rate given above when $n = 1$.

We'll now use these representations to study the effects of money growth on interest rates when money growth follows a so-called 'long memory' stochastic process.

Aside on long memory/fractionally integrated processes. An integrated process x_t has the lag operator representation

$$(1 - L)^d x_t = \varepsilon_t$$

where L is the lag operator $Lx_t = x_{t-1}$ and where ε_t is white noise. The process x_t is said to be $I(d)$ or 'integrated of order d'. If d is an integer, then this defines ordinary differences of the form $\Delta = 1 - L$, for $d = 1$ etc and the process x_t is said to be *integrated*. If $d = 0$ then x_t is white noise, if $d = 1$ then x_t is a pure random walk, etc. If d is not an integer the process x_t is said to be *fractionally integrated*. Such processes are often used in applied work in macro and finance because they imply autocorrelation functions which decay at hyperbolic rates, slower than the geometric rates implied by AR processes. To see this note that we can write

$$x_t = (1 - L)^{-d}\varepsilon_t$$

Using the binomial expansion

$$(1-L)^{-d} = 1 + dL + \frac{d(d+1)}{2!}L^2 + \frac{d(d+1)(d+2)}{3!}L^3 + \dots$$

we have that x_t has a valid MA(∞) representation given by

$$x_t = (1-L)^{-d}\varepsilon_t = \left(1 + dL + \frac{d(d+1)}{2!}L^2 + \frac{d(d+1)(d+2)}{3!}L^3 + \dots\right)\varepsilon_t = \sum_{j=0}^{\infty}\theta_j\varepsilon_{t-j}$$

where the coefficients are given by

$$\theta_j = \frac{1}{\Gamma(d)}\frac{\Gamma(j+d)}{\Gamma(j+1)}, \qquad j = 0, 1, 2, ...$$

where $\Gamma(z)$ is Euler's gamma function $\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt$ which generalizes the factorial function $z!$ to non-natural z and has the same recursive property $\Gamma(z+1) = z\Gamma(z)$. This implies the MA coefficients satisfy the difference equation

$$\frac{\theta_j}{\theta_{j-1}} = \frac{\Gamma(j+d)}{\Gamma(j+d-1)}\frac{\Gamma(j)}{\Gamma(j+1)} = \frac{j+d-1}{j}$$

or

$$\theta_j = \left(1 - \frac{1-d}{j}\right)\theta_{j-1} > 0$$

(for $j \geq 1$ with $\theta_0 := 1$).

It turns out that $d < 1/2$ is required for the process defined this way to be stationary and $d > -1/2$ is required for the process to be invertible. Thus in applications we usually restrict $d \in (-1/2, +1/2)$. The order of fractional integration d controls the rate of decay of the MA coefficients. In particular, the coefficients decay at a rate $(1-d)/j$ which slows down as j increases. This is the source of the long memory.

Another way to see this is to use the asymptotic approximation

$$\frac{1}{\Gamma(d)}\frac{\Gamma(j+d)}{\Gamma(j+1)} \to \frac{1}{\Gamma(d)}j^{d-1}, \qquad \text{as } j \to \infty$$

This implies that the MA coefficients exhibit slow hyperbolic decay for large j.

Now back to economics.

Implications of long memory processes for the yield curve. Suppose money growth is fractionally integrated so that the MA coefficients of the money growth process satisfy the difference equation

$$\theta_j = \left(1 - \frac{1-d}{j}\right)\theta_{j-1} > 0$$

for $j \geq 1$ with $\theta_0 := 1$. Then the MA coefficients for nominal interest rates are

$$\alpha_j := \left[-\phi(\bar{\mu})\frac{1-d}{j} + \left(1 - \frac{1-d}{j}\right)\right]\theta_j$$

Notice that for large enough j we have $\alpha_j = \theta_j > 0$ so that the expected inflation effect dominates for high j. This implies that an increase in money growth will tend to raise the yields on nominal bonds of long maturity. Since $\theta_j > 0$ for all j, we have that α_j becomes negative if

$$j < [\phi(\bar{\mu}) + 1](1-d)$$

So if the segmentation effect is sufficiently large, $\alpha_j < 0$ for small j. This implies that an increase in money growth will tend to lower the yields on nominal bonds of short maturity. Putting these responses together we have the implication that an increase in money growth will tend to raise long rates and lower short rates. This is a particularly strong form of yield curve steepening. With the AR process for money growth we have all yields move in the same direction, albeit with bigger responses for short rates. With the long memory process we have long yields and short yields move in different directions. Alvarez, Atkeson and Kehoe (2002) call this 'twisting' the yield curve.

In all cases, a large segmentation effect $\phi(\bar{\mu})$ is crucial for generating liquidity effects. With the AR process for money growth, the relative strength of the segmentation effect is the same at all horizons and there is a liquidity effect if and only if $\phi(\bar{\mu})$ is sufficiently large. With the long memory process for money growth, the relative strength of the segmentation effect is larger at short horizons and smaller at long horizons. In this case, $\phi(\bar{\mu})$ determines for how long the segmentation effect can dominate the Fisher expected inflation effect.

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