THE ARIMA(0,d,0) MODEL

The Fractional $ARIMA(0,d,0)$ process $\{x_t\}$ is defined for any real $d > -1$ by the implicit relationship

$$\Delta^d x_t = \varepsilon_t \quad ,$$

where $\{\varepsilon_t\}$ is zero-mean white noise, and $\Delta^d = (1-B)^d$. If d is an integer, this says that the d'th difference of $\{x_t\}$ is white noise. But what does it mean to take the d'th difference if d is not an integer? A reasonable answer is to define Δ^d as a power series in B,

$$\Delta^d = (1-B)^d = \sum_{j=0}^{\infty} \pi_j B^j \quad ,$$

where π_j is the coefficient of B^j in the binomial series expansion

$$(1-B)^d = \sum_{j=0}^{\infty} (-1)^j \binom{d}{j} B^j \quad .$$

So $\pi_j = (-1)^j \binom{d}{j}$, where the binomial coefficient $\binom{d}{j}$ is defined for any real d by

$$\binom{d}{j} = \frac{d(d-1) \cdot \cdot \cdot (d-j+1)}{j\,!} \quad .$$

If d is a positive integer, then Δ^d reduces to the ordinary d'th difference,

$$\Delta^d = \sum_{j=0}^{d} (-1)^j \binom{d}{j} B^j \quad ,$$

and the π_j are zero for $j > d$. If d is not an integer, however, then all of the π_j will be nonzero, and $\Delta^d x_t = \sum_{j=0}^{\infty} \pi_j x_{t-j}$ will involve infinitely many past observations. We will show that in this case the π_j decay to zero as j^{-d-1}, which is much slower than the exponential rate found for stationary $ARMA$ processes.

To analyze the $\{\pi_j\}$, we now define the **Gamma function** $\Gamma(p)$ for any real p by $\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x}\, dx$ if $p > 0$, $\Gamma(0) = \infty$, and $\Gamma(p) = \frac{1}{p}\Gamma(1+p)$ if $p < 0$. If p is a nonnegative integer, then

$\Gamma(p+1)=p\,!$ The π_j can now be written in terms of the Gamma function as

$$\pi_j = \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)} = \prod_{0<k\le j} \frac{k-1-d}{k} \qquad j=1,2,\cdots \quad .$$

To show this, note that

$$\frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)} = \frac{\Gamma(j-d)}{\Gamma(j+1)\cdot \frac{1}{-d}\cdot\frac{1}{-d+1}\cdots\frac{1}{-d+j-1}\cdot\Gamma(-d+j)} = \frac{(-1)^j\, d\,(d-1)\cdots(d-j+1)}{j\,!} = \pi_j \quad .$$

To analyze the behavior of π_j for large j, we use **Stirling's Formula:** $\Gamma(p)\sim\sqrt{2\pi}\,e^{-p+1}(p-1)^{p-1/2}$ as $p\to\infty$, where " $\sim$ " means that the ratio of the lefthand side to the righthand side approaches 1. Thus, for large j,

$$\log\left\{\frac{\Gamma(j-d)}{\Gamma(j+1)}\right\}\sim(d+1)+(j-d-1/2)\log(j-d-1)-(j+1/2)\log j$$

$$\sim(d+1)+(j-d-1/2)\,(\log j-\frac{d+1}{j})-(j+1/2)\log j$$

$$\sim(d+1)+(j-d-1/2)\log j-(d+1)-(j+1/2)\log j=-(d+1)\log j \quad .$$

Therefore, $\pi_j\sim j^{-d-1}/\Gamma(-d)$ as $j\to\infty$.

If $d\in(-.5\,,.5)$, then $\{x_t\}$ is weakly stationary, and has the one-sided $MA(\infty)$ representation

$$x_t=\Delta^{-d}\,\varepsilon_t=\sum_{j=0}^{\infty}\psi_j\,\varepsilon_{t-j} \quad ,$$

where

$$\psi_j=(-1)^j\begin{pmatrix}-d\\ j\end{pmatrix}=\frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}=\prod_{0<k\le j}\frac{k-1+d}{k} \quad ,$$

and $\psi_j\sim j^{d-1}/\Gamma(d)$ as $j\to\infty$. Because $x_t=\Delta^{-d}\,\varepsilon_t$, the process $\{x_t\}$ is referred to as **fractionally integrated noise.** Weak stationarity follows from the fact that if $d\in(-.5\,,.5)$, then $2d-2<-1$, so that $\sum_j j^{2d-2}<\infty$ and $\sum_j\psi_j^2<\infty$. In this case, $\{x_t\}$ has the spectral representation

$$x_t=\int_{-\pi}^{\pi}e^{i\lambda t}\,(1-e^{-i\lambda})^{-d}\,dZ_\varepsilon(\lambda) \quad .$$

To construct an $ARIMA(0,d,0)$ process for d outside the range $(-.5,.5)$, we can proceed as follows. Start with an $ARIMA(0,d^*,0)$ using the $d^* \in (-.5,.5)$ such that $d-d^*$ is an integer. If $d>.5$, integrate $d-d^*$ times. If $d<-.5$, difference d^*-d times. The result will be an $ARIMA(0,d,0)$. It is important to note, however, that the $ARIMA(0,d,0)$ process will not be stationary if $d \geq .5$.

An important feature of any weakly stationary long memory process is the behavior of its spectral density near zero frequency. Using the spectral representation, we find that $\{x_t\}$ has spectral density

$$f(\lambda) = \frac{\sigma^2}{2\pi} \, |1-e^{-i\lambda}|^{-2d}$$

$$= \frac{\sigma^2}{2\pi} \, |2\sin(\lambda/2)|^{-2d} \quad ,$$

where $\sigma^2 = var\ \varepsilon_t$. For λ close to zero, $\sin\lambda \sim \lambda$, so that

$$f(\lambda) \sim \frac{\sigma^2}{2\pi} \, |\lambda|^{-2d} \quad , \quad \lambda \to 0 \quad .$$

It follows that if $d>0$ then $\lim_{\lambda \to 0} f(\lambda) = \infty$, so $f(\cdot)$ has a pole at zero frequency. On the other hand, if $d<0$ then $\lim_{\lambda \to 0} f(\lambda) = 0$, so $f(\cdot)$ has a zero at zero frequency. This is an important distinction between the cases of positive and negative d. In either case, however, the spectral density has infinite dynamic range, that is, $\max_{\lambda} f(\lambda) / \min_{\lambda} f(\lambda) = \infty$. By contrast, stationary invertible ARMA processes always have finite dynamic range.

It can be shown that if $d \neq 0$ the autocovariances $\{c_r\}$ satisfy $c_r \sim k\, r^{2d-1}$ as $r \to \infty$, where k is a constant. Consequently, if $d<0$ then $\sum |c_r| < \infty$ (the autocovariances are summable), but if $d>0$ then the autocovariances decay to zero so slowly that they are not summable, i.e., $\sum |c_r| = \infty$. This is another important distinction between the cases of positive and negative d. In either case, however, the rate of decay of the autocovariances is much slower than the exponential rate found for stationary invertible ARMA processes.

Negative d can arise in practice if we take the first difference of a series that was originally nonstationary, but not so strongly nonstationary as a unit root process. For example, if the original process had $d=.7$, then the first difference has $d=-.3$.