11: MORE ON LINEAR FILTERS

Basic Properties

Koopmans (p. 81) gives a general definition of a linear filter as a transformation L which takes an input time series and produces an output time series, such that:

(1) $L(\alpha\{x(t)\})=\alpha L(\{x(t)\})$ (Scale Preservation),

(2) $L(\{x(t)\}+\{y(t)\})=L(\{x(t)\})+L(\{y(t)\})$ (Superposability),

(3) If $L(\{x(t)\})=\{y(t)\}$ then $L(\{x(t+h)\})=\{y(t+h)\}$ (Time Invariance).

If α_λ do not depend on t, then (1) and (2) imply

$$L(\sum_\lambda \alpha_\lambda e^{i\lambda t}) = \sum_\lambda \alpha_\lambda L(e^{i\lambda t}) \quad .$$

Thus for any time series which can be represented in the form $x(t)=\sum_\lambda \alpha_\lambda e^{i\lambda t}$, the action of a linear filter on $x(t)$ is completely determined by what it does to the complex exponential functions $e^{i\lambda t}$ for all λ. Furthermore, virtually all time series studied in this course have such a representation.

To find the form of $L(e^{i\lambda t})$, let $\phi_\lambda(t)$ denote the output at time t:

$$L(e^{i\lambda t}) = \phi_\lambda(t) \quad .$$

By property (3), for any h, $L(e^{i\lambda(t+h)})=\phi_\lambda(t+h)$. But

$$L(e^{i\lambda(t+h)}) = L(e^{i\lambda t}\, e^{i\lambda h}) = e^{i\lambda h}\, L(e^{i\lambda t}) = e^{i\lambda h}\,\phi_\lambda(t) \quad .$$

Thus, $\phi_\lambda(t+h)=e^{i\lambda h}\,\phi_\lambda(t)$. Setting $t=0$ yields $\phi_\lambda(h)=e^{i\lambda h}\,\phi_\lambda(0)$. Defining $B(\lambda)=\phi_\lambda(0)$ and replacing h by t (this is just a renaming), we get

$$L(e^{i\lambda t}) = B(\lambda) e^{i\lambda t} \quad .$$

Thus, any linear filter L transforms $e^{i\lambda t}$ back into $e^{i\lambda t}$ multiplied by the **transfer function** $B(\lambda)$.

From the above discussion, it follows that if the input to the linear filter is

$$x(t) = \sum_\lambda \alpha_\lambda e^{i\lambda t} \quad ,$$

then the output is

$$L(x(t)) = \sum_{\lambda} \alpha_{\lambda} B(\lambda) e^{i\lambda t} \quad .$$

The action of L on $x(t)$ is completely determined by the transfer function $B(\lambda)$. In general, $B(\lambda)$ will be a complex number. Writing $B(\lambda)$ in polar form, $B(\lambda) = |B(\lambda)|\, e^{i\theta(\lambda)}$, we obtain

$$L(x(t)) = \sum_{\lambda} \alpha_{\lambda} |B(\lambda)|\, e^{i(\lambda t + \theta(\lambda))} \quad .$$

Thus, the amplitude at each frequency λ is multiplied by the **gain function** $|B(\lambda)|$, and the phase is shifted by the **phase function** $\theta(\lambda)$.

Now let $X(t)$ be a weakly stationary (and therefore finite-variance) stochastic process. The spectral representation is

$$X(t) = \int e^{i\lambda t} Z_X(d\lambda) \quad .$$

The limits of integration are $-\infty, \infty$ and $-\pi, \pi$ in the continuous and discrete time cases, respectively. The condition that $X(t)$ have finite power (i.e., finite variance) is $\int F_X(d\lambda) = C(0) < \infty$. Now,

$$Y(t) = L(X(t)) = \int e^{i\lambda t} B(\lambda) Z_X(d\lambda)$$

represents the output of L as a weakly stationary process with spectral measure

$$Z_Y(d\lambda) = B(\lambda)\, Z_X(d\lambda) \quad .$$

Thus,

$$F_Y(d\lambda) = E\,|Z_Y(d\lambda)|^2 = |B(\lambda)|^2 F_X(d\lambda) \quad .$$

It follows that the spectral functions and spectral density functions of input and output are related by the simple expressions

$$p_Y(\lambda) = |B(\lambda)|^2\, p_X(\lambda) \quad ,$$

$$f_Y(\lambda) = |B(\lambda)|^2\, f_X(\lambda) \quad .$$

If the output $Y(t)$ is to be a weakly stationary process, it must have finite power: $\int F_Y(d\lambda) < \infty$. The condition that $Y(t)$ has finite power is called the **matching condition,** and will be satisfied if

$$\int |B(\lambda)|^2 F_X(d\lambda) < \infty \quad .$$

Inverting Linear Filters

Suppose a linear filter L with transfer function $B(\lambda)$ is applied to the weakly stationary process $X(t)$ yielding the weakly stationary output $Y(t) = L(X(t))$. **Inverting the linear filter** is the task of recovering $X(t)$ from $Y(t)$ by applying another linear filter L^* to $Y(t)$. Thus, it is desired to find a linear filter L^* such that $L^*(Y(t)) = X(t)$. This is easily achieved if $B(\lambda) \neq 0$ (all λ). In this case, we can take L^* to be the linear filter with transfer function $B^*(\lambda) = 1/B(\lambda)$. Then the filter with transfer function $B^*(\lambda) B(\lambda) \equiv 1$ is the do-nothing filter, and is matched to $X(t)$ since

$$\int |B^*(\lambda) B(\lambda)|^2 F_X(d\lambda) = \int F_X(d\lambda) < \infty \quad .$$

Thus,

$$L^*(Y(t)) = L^*(L(X(t))) = L^* L(X(t)) = X(t) \quad ,$$

as required. Rephrasing the above result in terms of the spectral representation: If $X(t) = \int e^{i\lambda t} Z_X(d\lambda)$, then we can write

$$L(X(t)) = Y(t) = \int e^{i\lambda t} B(\lambda) Z_X(d\lambda) = \int e^{i\lambda t} Z_Y(d\lambda) \quad ,$$

where

$$Z_Y(d\lambda) = B(\lambda) Z_X(d\lambda) \quad .$$

Now, given $Y(t) = \int e^{i\lambda t} Z_Y(d\lambda)$, we obtain

$$L^*(Y(t)) = \int e^{i\lambda t} \frac{1}{B(\lambda)} Z_Y(d\lambda) = \int e^{i\lambda t} \frac{1}{B(\lambda)} B(\lambda) Z_X(d\lambda) = \int e^{i\lambda t} Z_X(d\lambda) = X(t) \quad ,$$

as required.

A problem related to (but not the same as) the one described above is: Given a weakly stationary process $X(t)$ and a linear filter L with transfer function $B(\lambda)$, find a weakly stationary process $Y(t)$ such that

$$X(t) = L(Y(t)) \quad .$$

This is a stochastic functional equation which is to be solved for $Y(t)$. The desired solution is of form $Y(t) = L^*(X(t))$ for some linear filter L^*. A weakly stationary solution $Y(t)$ may not exist, but if it does, we must have

$$Y(t) = L^*(X(t)) = L^*L(Y(t)) \quad .$$

Thus, L^*L must be the do-nothing filter, and it follows that the transfer function of L^* must be $B^*(\lambda) = \frac{1}{B(\lambda)}$. In order for a weakly stationary solution to exist, however, $L^*(X(t))$ must have finite variance. We therefore need

$$\int \frac{1}{|B(\lambda)|^2} F_X(d\lambda) < \infty \quad .$$

If this matching condition is satisfied, the solution is

$$Y(t) = \int e^{i\lambda t} \frac{1}{B(\lambda)} Z_X(d\lambda) \quad .$$