

3: LEAKAGE, AND ITS REDUCTION BY DATA WINDOWS

Consider the complex exponential series $x_t = \exp(i\omega t)$, which oscillates at frequency ω . (The complex exponential is easier to work with than the real series $\cos\omega t$ and $\sin\omega t$, and the results for the real series are similar to those given below for the complex exponential.) Suppose we perform a harmonic analysis of $\{x_t\}$ by taking its discrete Fourier transform. $\{x_t\}$ has DFT

$$J_j = \frac{1}{n} \sum_{t=0}^{n-1} x_t \exp(-i\omega_j t) = \frac{1}{n} \sum_{t=0}^{n-1} \exp(i(\omega - \omega_j)t) = \exp(i(n-1)(\omega - \omega_j)/2) D_n(\omega - \omega_j) \quad .$$

If $\omega = \omega_k$ is a Fourier frequency, then $J_k = 1$ and $J_j = 0$ ($j \neq k$). Thus, in this case the harmonic analysis of x_t shows clearly that ω_k is the only frequency present in the series x_t . Hence the harmonic analysis clearly separates the effects of different frequencies. If ω is not a Fourier frequency, however, then all harmonic components J_j will be nonzero, and the harmonic analysis of x_t becomes more problematic. The appearance of nonzero terms in the transform at a given frequency because of a sinusoid of a different frequency is called **leakage**, and is a highly undesirable phenomenon. In the present example, only the values of the transform at frequencies close to ω will be large, and the values decay proportionally to $|\omega - \omega_j|$.

We now show that leakage may be substantially reduced if (before Fourier transformation) we first multiply the data sequence x_t by a sequence of suitably chosen constants w_t (called a **data window**, or **taper**), to form the tapered data sequence $z_t = w_t x_t$. Note that using the raw data corresponds to using the **boxcar** (rectangular) data window, $w_t = 1, t = 0, \dots, n-1$, and $w_t = 0$ for all other t . The motivation for tapering comes from the properties of Fourier transforms. The data window has Fourier transform

$$J_w(\omega) = \frac{1}{n} \sum_{t=0}^{n-1} w_t \exp(-i\omega t) \quad .$$

Suppose that the data sequence is $x_t = \exp(i\lambda t)$. Then the tapered data $z_t = w_t \exp(i\lambda t)$ has Fourier transform

$$J_z(\omega) = \frac{1}{n} \sum_{t=0}^{n-1} w_t \exp(i\lambda t) \exp(-i\omega t) = \frac{1}{n} \sum_{t=0}^{n-1} w_t \exp(-i(\omega - \lambda)t) = J_w(\omega - \lambda) \quad .$$

Thus, the Fourier transform of a tapered sinusoid is just the transform of the data window centered at the frequency of the sinusoid. Hence leakage in the transform of a tapered sinusoid is caused by the sidelobes of the transform of the data window (which is the Dirichlet kernel for the boxcar). It can be shown that the transform of a smooth series decays more rapidly than that of a rough series (see "Smooth Functions"). Thus, it is the smoothness of the data window $\{w_t\}$ which leads to its good leakage characteristics.

A frequently used data window is the **cosine bell**,

$$w_t = \frac{1}{2} \{1 - \cos(2\pi[t + .5]/n)\} \quad .$$

Unlike the boxcar, the cosine bell has no sharp edges. It is a smooth function. Its Fourier transform can be shown to be (Exercise):

$$J_w(\omega) = \exp\{-i\omega(n-1)/2\} \{1/4 D_n(\omega - 2\pi/n) + 1/2 D_n(\omega) + 1/4 D_n(\omega + 2\pi/n)\} \quad .$$

The second bracketed term is called the **Hanned** version of $D_n(\omega)$. It is a frequency domain running mean of the Dirichlet kernel with weights 1/4, 1/2, 1/4. Since for ω not too near zero, we have (Exercise)

$$D_n(\omega - 2\pi/n) \approx -D_n(\omega) \approx D_n(\omega + 2\pi/n) \quad ,$$

we see that the operation of **Hanning** $D_n(\omega)$ produces a quantity which is approximately zero for $\omega \neq 0$. Thus, the sidelobes of $J_w(\omega)$ die off much more rapidly than those of the Dirichlet Kernel. It follows that tapering the data with a cosine bell produces much smaller leakage than we would get by using the raw data (boxcar taper).

The cosine bell modifies all of the data. This may be considered undesirable, since we would like to use as much of the original data as possible. Thus, we might prefer a window that leaves the bulk of the data unmodified, and just tapers the ends. A convenient window is obtained by separating the two halves of the cosine bell and inserting a stretch of ones. This gives the **split cosine bell**,

$$w_t = \begin{cases} (1/2)(1 - \cos(\pi(t + .5)/m)) & t = 0, \dots, m-1 \\ 1 & t = m, \dots, n-m-1 \\ (1/2)(1 - \cos(\pi(n - t - .5)/m)) & t = n-m, \dots, n-1 \end{cases} \quad .$$

The split cosine bell tapers a proportion $2m/n$ of the data. m is usually chosen to make this proportion 10% or 20%. Its leakage properties are intermediate between those of the boxcar and those of the cosine bell.