CONTINUOUS TIME LONG MEMORY MODELS:
FRACTIONAL BROWNIAN MOTION
AND FRACTIONAL GAUSSIAN NOISE

The **fractional Brownian motion (fBm)** described by Mandelbrot and Van Ness is a Gaussian stochastic process $V_H(t)$ in continuous time with the property that for any t_1, t_2,

$$var\,[V_H(t_2) - V_H(t_1)] \propto |t_2 - t_1|^{2H} \quad ,$$

where $0 < H < 1$. When $H = 1/2$, we get the standard Brownian motion: The increment $V_{1/2}(t_2) - V_{1/2}(t_1)$ has a variance proportional to the time difference $t_2 - t_1$. The fractional Brownian motion $V_H(t)$ is a nonstationary process, but its increments form a stationary process. The average of the increments $V_H(1) - V_H(0), \ldots, V_H(n) - V_H(n-1)$ is $\bar{x} = \frac{1}{n}[V_H(n) - V_H(0)]$, so $var\,\bar{x} \propto n^{2H-2}$. We see that the usual Central Limit Theorem does not apply for the sample mean of increments of $V_H(t)$ unless $H = 1/2$, in which case the increments are a (continuous time) white noise process.

The "derivative" of $V_H(t)$ (which can be heuristically defined, even though $V_H(t)$ is not actually differentiable anywhere), is called **fractional Gaussian noise.** The fractional Gaussian noise is stationary, and is essentially a continuous-time version of the Fractional $ARIMA(0,d,0)$ process we studied earlier.

The process $V_H(t)$ is said to be **statistically self-affine,** because if we change the time scale by a factor of r, we find that, apart from a multiplicative constant, the stochastic behavior remains unchanged:

$$V_H(rt) \overset{D}{=} r^H\, V_H(t) \quad ,$$

where "$\overset{D}{=}$" denotes equality in distribution. The curve traced out by $V_H(t)$ may be thought of as a random fractal. It is not purely self-similar, but, as shown above, it is statistically self-affine, and the following rough argument shows how we might assign a fractal dimension to the curve. Consider the curve traced out between $t = 0$ and $t = 1$. Divide this unit interval into N equal subintervals of length $1/N$. Within each subinterval, we cover the corresponding part of the curve $V_H(t)$ with boxes of side

$1/N$. Since

$$var\ [V_H(1/N) - V_H(0)] \propto (1/N)^{2H} \quad ,$$

we can expect that the range of values taken on by $V_H(t)$ in the subinterval will be of order $(1/N)^H$. Thus, we should be able to cover the part of the curve $V_H(t)$ in the subinterval with $(1/N)^H / (1/N) = N/N^H$ boxes, and therefore we can cover the entire curve with $N^2/N^H = N^{2-H}$ boxes. It follows that the dimension of the curve $V_H(t)$ is

$$D = \frac{\log N^{2-H}}{\log N} = 2 - H \quad .$$

For example, the curve traced by ordinary Brownian motion has dimension 1.5. (See Figure 12, p. 44.)

Since fBm with $0 < H < 1$ is not stationary, $V_H(t)$ does not, strictly speaking, have a spectral density. Nevertheless, if we define the "spectral density" as the limiting expectation of the periodogram (as the interval of observation goes to infinity), we obtain a "spectral density"

$$f_H(\omega) \propto |\omega|^{-2(H+1/2)} \quad \text{as} \quad \omega \to 0 \quad .$$

Also, if we define the "covariance function" $C(\tau)$ as the limiting expectation of the sample covariance, we obtain

$$C(\tau) \propto \tau^{2H} \quad \text{as} \quad \tau \to \infty \quad .$$

The fractional Gaussian noise process, which is stationary, is obtained by taking the "derivative" of fBm, and may be thought of as another fBm, with H reduced by 1. It is also possible to "integrate" fBm, which has the effect of increasing H by 1. We may integrate and differentiate as many times as we wish. In this way, we can define an fBm $V_H(t)$ for any real value of H. The fractional Gaussian noises are the fBm processes with $H \le 0$. If we differentiate $V_{1/2}(t)$, which is Brownian motion, we obtain the white noise process, $V_{-1/2}(t)$. So white noise corresponds to $H = -1/2$.

For any $d \in (-1/2, 1/2)$, if we define a fBm with $H = d - 1/2$, then the discretized process $\{V_H(t)\}_{t=-\infty}^{\infty}$ is a long-memory process with memory parameter d. The formulas above for the spectral density and covariance sequence then reduce to $f(\omega) \propto |\omega|^{-2d}$ as $\omega \to 0$ and $c_r \propto r^{2d-1}$ as $r \to \infty$, as expected.

Three important special cases are as follows.

- Brownian Motion: $H = 1/2$, $d = 1$, "Spectral density" $\propto |\omega|^{-2}$.

- White Noise: $H = -1/2$, $d = 0$, "Spectral density" is constant.

- "$1/f$ noise": $H = 0$, $d = 1/2$, "Spectral density" $\propto 1/|\omega|$.

See the attached pages of "The Science of Fractal Images" (pages 40-43) for a connection between music and "$1/f$ noise". Finally, note that the discretized fBm $\{V_H(t)\}_{t=-\infty}^{\infty}$ for $0 < H < 1$ provides a well-defined class of nonstationary processes which may be thought of as long memory, with memory parameter $d \in (1/2\,, 3/2)$.