

Term Structures of Asset Prices and Returns

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(With thanks to Ian Martin and Stan Zin)

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Look for entropy

Paul Samuelson (“Gibbs in economics,” 1989):

I have limited tolerance for the perpetual attempts to fabricate for economics concepts of “entropy.”

Look for logarithms

- ▶ Think about $\log m_{t,t+1}$ rather than $m_{t,t+1}$
- ▶ Sums more user-friendly than products

$$\log(m_{t,t+1}r_{t,t+1}) = \log m_{t,t+1} + \log r_{t,t+1}$$

more user-friendly than

$$m_{t,t+1}r_{t,t+1}$$

$$\log(m_{t,t+1}m_{t+1,t+2}) = \log m_{t,t+1} + \log m_{t+1,t+2}$$

more user-friendly than

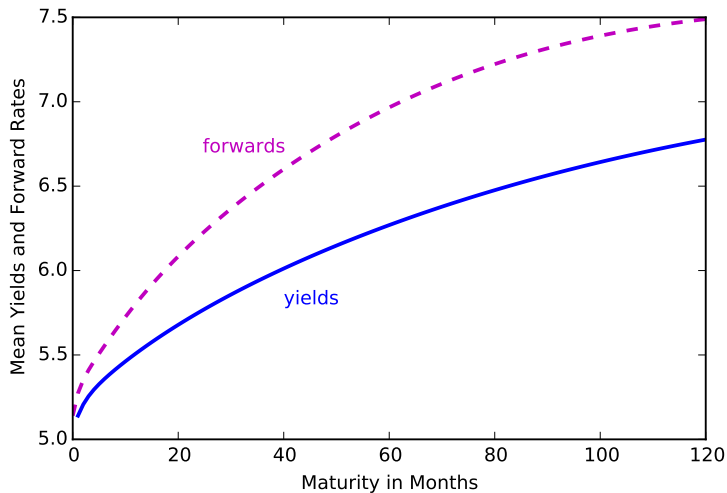
$$m_{t,t+1}m_{t+1,t+2}$$

Excess returns

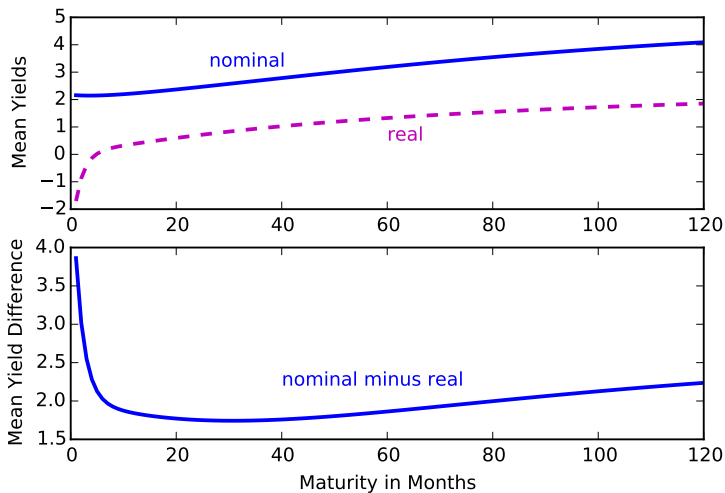
Monthly excess log returns in dollars: $\log r_{t,t+1} - \log r_{t,t+1}^1$

Asset	Mean	Standard Deviation	Skewness	Excess Kurtosis
S&P 500	0.0040	0.0556	-0.40	7.90
Fama-French (small, low)	-0.0030	0.1140	0.28	9.40
Fama-French (small, high)	0.0090	0.0894	1.00	12.80
Pound Sterling	0.0035	0.0316	-0.50	1.50
5-year bond	0.0015	0.0190	0.10	4.87
10-year bond	0.0019			

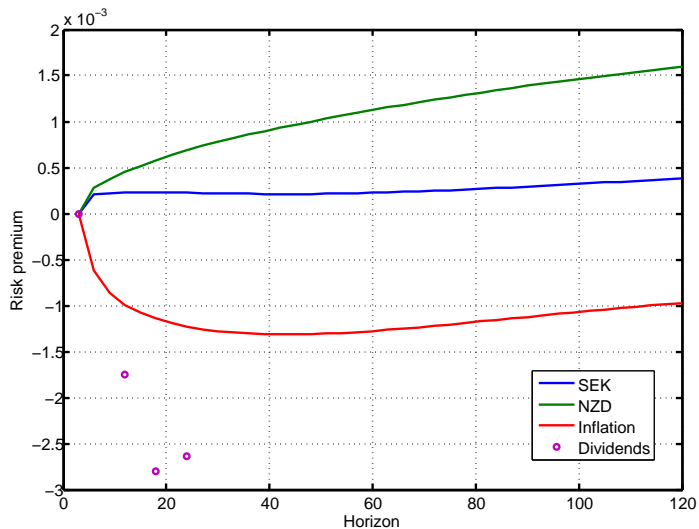
Term structure data: US nominal



Term structure data: US nominal and real



Term structure data: other assets relative to US nominal



Where we're headed

What makes these term structures different?

Plan of attack

- ▶ **Entropy:** dispersion in the pricing kernel
- ▶ **Risk premiums:** entropy bound
- ▶ **Term structure:** pricing kernel dynamics
- ▶ **Coentropy:** risk premiums revisited
- ▶ **Other term structures:** cash flow dynamics, more coentropy

Entropy

Entropy

- ▶ Entropy is a measure of dispersion: for rv $x > 0$

$$L(x) \equiv \log E(x) - E(\log x) \geq 0$$

- ▶ Invariant to scale: $L(\alpha x) = L(x)$ for $\alpha > 0$

- ▶ Lognormal example: if $\log x \sim \mathcal{N}(\kappa_1, \kappa_2)$, then

$$\log E(x) = \kappa_1 + \kappa_2/2$$

$$E(\log x) = \kappa_1$$

$$L(x) = (\kappa_1 + \kappa_2/2) - \kappa_1 = \kappa_2/2$$

Cumulants

- ▶ Cumulant generating function (cgf) of rv y

$$k(s; y) = \underbrace{\log E(e^{sy})}_{\text{mgf}} = \sum_{j=1}^{\infty} \kappa_j s^j / j!$$

- ▶ Cumulants are close relatives of moments

$$\text{mean} = \kappa_1$$

$$\text{variance} = \kappa_2$$

$$\text{skewness} = \kappa_3 / \kappa_2^{3/2}$$

$$\text{excess kurtosis} = \kappa_4 / \kappa_2^2$$

- ▶ If y is normal: $k(s; y) = \kappa_1 s + \kappa_2 s^2 / 2$

Entropy and cumulants

- ▶ Cumulants of $y = \log x$

$$k(s; \log x) = \log E(e^{s \log x}) = \sum_{j=1}^{\infty} \kappa_j (\log x) s^j / j!$$

- ▶ Entropy and cumulants (set $s = 1$)

$$\begin{aligned} L(x) &= k(1; \log x) - E(\log x) = \sum_{j=2}^{\infty} \kappa_j (\log x) / j! \\ &= \underbrace{\kappa_2 (\log x) / 2!}_{(\log)\text{normal term}} + \underbrace{\kappa_3 (\log x) / 3! + \kappa_4 (\log x) / 4! + \dots}_{\text{high-order cumulants}} \end{aligned}$$

Entropy of a stationary stochastic process

- ▶ Conditional entropy defined for conditional distribution

$$L_t(x_{t+1}) = \log E_t(x_{t+1}) - E_t(\log x_{t+1})$$

- ▶ We define entropy as the mean $E[L_t(x_{t+1})]$

- ▶ Connected to entropy for unconditional distribution

$$L(x_{t+1}) = \underbrace{E[L_t(x_{t+1})]}_{\text{entropy}} + L[E_t(x_{t+1})]$$

Risk premiums: the entropy bound

Entropy bound (Alvarez-Jermann)

- ▶ Returns satisfy the pricing relation $E_t(m_{t,t+1}r_{t,t+1}) = 1$
- ▶ Entropy bound: maximize $E_t(\log r_{t,t+1} - \log r_{t,t+1}^1)$
- ▶ Maximization leads to the bound

$$\overbrace{E_t(\log r_{t,t+1} - \log r_{t,t+1}^1)}^{\text{risk premium?}} \leq L_t(m_{t,t+1})$$
$$E(\log r_{t,t+1} - \log r_{t,t+1}^1) \leq \underbrace{E[L_t(m_{t,t+1})]}_{\text{entropy}}$$

- ▶ High return is

$$\log r_{t,t+1} = -\log m_{t,t+1}$$

Hansen-Jagannathan bound

- ▶ HJ bound: maximize Sharpe ratio
- ▶ Maximization leads to the bound

$$\begin{aligned} \text{SR}_t &\equiv E_t(r_{t,t+1} - r_{t,t+1}^1) / \text{Var}_t(r_{t,t+1} - r_{t,t+1}^1)^{1/2} \\ &\leq \text{Var}_t(m_{t,t+1})^{1/2} / E_t(m_{t,t+1}) \end{aligned}$$

- ▶ High return is

$$r_{t,t+1} = \frac{1 + \text{Var}_t(m_{t,t+1})^{1/2}}{E_t(m_{t,t+1})} - \frac{m_{t,t+1} - E_t(m_{t,t+1})}{\text{Var}_t(m_{t,t+1})^{1/2}}$$

Stan's "never a dull moment" machine

- ▶ Entropy of pricing kernel

$$\begin{aligned} L(m) &= \log E(e^{\log m}) - E(\log m) \\ &= k(1; \log m) - E(\log m) = \sum_{j=2}^{\infty} \kappa_j (\log m) / j! \end{aligned}$$

- ▶ Stan's entropy machine (but ask about Lukacs)

$$L(m) = \underbrace{\kappa_2 (\log m) / 2!}_{(\log)\text{normal term}} + \underbrace{\kappa_3 (\log m) / 3! + \kappa_4 (\log m) / 4! + \dots}_{\text{high-order cumulants}}$$

- ▶ Kraus and Litzenberger revisited?

Why is this entropy?

- ▶ Humpty Dumpty (in “Through the Looking Glass”)

“When I use a word,” Humpty Dumpty said, “it means just what I choose it to mean — neither more nor less.”

- ▶ Hans-Otto Georgii (quoted by Hansen and Sargent):

When Shannon had invented his quantity and consulted von Neumann on what to call it, von Neumann replied: “Call it entropy. It is already in use under that name and, besides, it will give you a great edge in debates because nobody knows what entropy is anyway.”

Why is this entropy?

- ▶ Notation: states z have (true) probabilities $\pi(z)$
- ▶ Risk-neutral probabilities π^*

$$\pi^*(z) = \pi(z)m(z)/p^1$$

$$m(z) = p^1\pi^*(z)/\pi(z)$$

$$p^1 = E(m) \quad (1\text{-period bond price})$$

- ▶ Entropy (aka “relative entropy” or “Kullback-Leibler divergence”)

$$L(m) = L(\pi^*/\pi) = E \log(\pi/\pi^*)$$

$$(\pi^* = \pi \Rightarrow L(m) = 0, \text{ risk premiums} = 0)$$

Vasicek model

- ▶ Pricing kernel

$$\log m_{t,t+1} = \log \beta + x_t + \lambda w_{t+1}$$

with $\{w_t\}$ iid, mean zero, variance one, and cgf $k(s)$

- ▶ Conditional entropy

$$\begin{aligned} L_t(m_{t,t+1}) &= k(\lambda) \\ &= \lambda^2 \kappa_2 / 2! + \lambda^3 \kappa_3 / 3! + \lambda^4 \kappa_4 / 4! + \dots \end{aligned}$$

- ▶ Entropy: the same (maximum risk premium is constant)

State-dependent price of risk

- ▶ Pricing kernel

$$\log m_{t,t+1} = \log \beta + x_t + (\lambda_0 + \lambda_1 x_t) w_{t+1}$$

with $\{w_t\} \sim \text{NID}(0, 1)$, $k(s) = s^2/2$

- ▶ Conditional entropy

$$L_t(m_{t,t+1}) = k(\lambda_0 + \lambda_1 x_t) = (\lambda_0 + \lambda_1 x_t)^2/2$$

- ▶ Entropy

$$E[L_t(m_{t,t+1})] = E[(\lambda_0 + \lambda_1 x_t)^2/2]$$

Power utility

- ▶ Consumption growth $g_{t,t+1} = c_{t+1}/c_t$ iid
- ▶ Pricing kernel

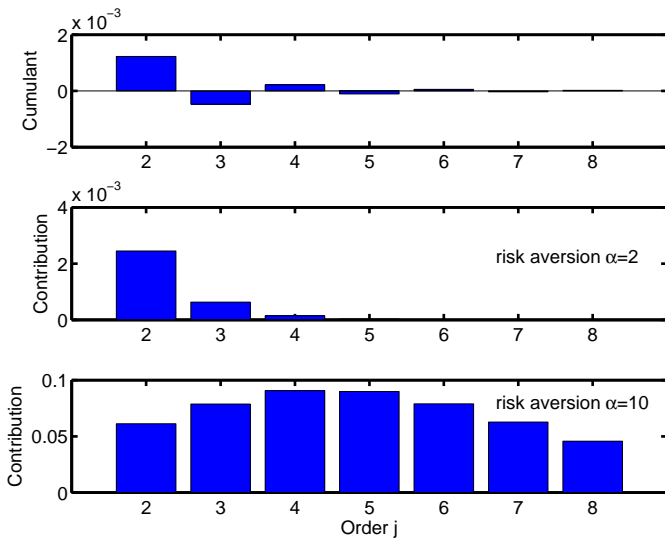
$$\log m_{t,t+1} = \log \beta - \alpha \log g_{t,t+1}$$

(Vasicek with $x_t = 0$, $\lambda = -\alpha$, and $w_{t+1} = \log g_{t,t+1}$)

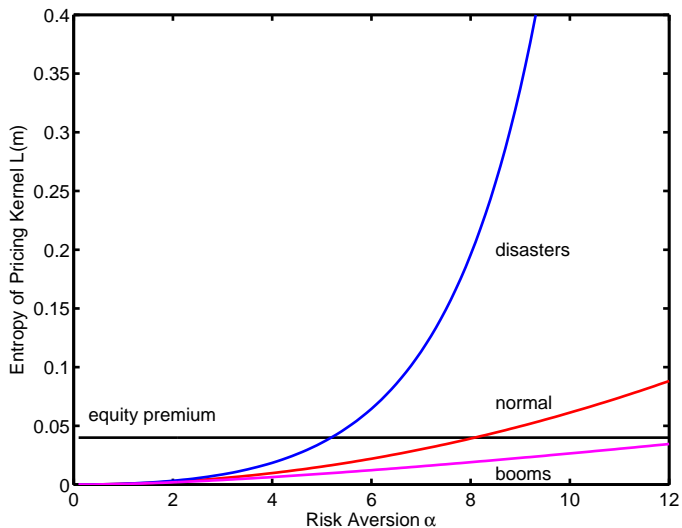
- ▶ Yaron's bazooka

$$\begin{aligned} L_t(m_{t,t+1}) &= k(-\alpha) \\ &= (-\alpha)^2 \kappa_2 / 2! + (-\alpha)^3 \kappa_3 / 3! + (-\alpha)^4 \kappa_4 / 4! + \dots \end{aligned}$$

Power utility: the bazooka



Power utility: the bazooka



Term structure: pricing kernel dynamics

The idea

- ▶ In an iid world, entropy is proportional to time interval
- ▶ Deviations from proportionality reflect pricing kernel dynamics
- ▶ Detectable from mean yields and forward rates

Bond prices, yields, and forward rates

- ▶ Bond price: p_t^n is price at t of a claim to one (dollar?) at $t + n$
- ▶ Bond yield: $y_t^n = -n^{-1} \log p_t^n$
- ▶ Forward rate: $f_t^n = \log(p_t^n / p_t^{n+1}) \Rightarrow y_t^n = \sum_{j=1}^n f_t^{j-1}$
- ▶ One-period return:

$$\log r_{t,t+1}^{n+1} = \log(p_{t+1}^n / p_t^{n+1}) \Rightarrow E(\log r^{n+1}) = E(f^n)$$

- ▶ Cross sections reflect pricing kernel dynamics

Bond pricing fundamentals

- ▶ Markov environment with state variable x
- ▶ Bond pricing is recursive

$$p^n(x_t) = E_t[m(x_t, x_{t+1})p^{n-1}(x_{t+1})]$$

starting with $p^0 = 1$

- ▶ Equivalent to

$$p^n(x_t) = E_t[m(x_t, x_{t+1})m(x_{t+1}, x_{t+2}) \cdots m(x_{t+n-1}, x_{t+n})]$$

- ▶ Definitions give us yields $y^n(x_t)$ and forward rates $f^n(x_t)$

Entropy and the term structure

- ▶ Entropy over n periods

$$\begin{aligned}m_{t,t+n} &= m_{t,t+1}m_{t+1,t+2} \cdots m_{t+n-1,t+n} \\L_t(m_{t,t+n}) &= \underbrace{\log E_t(m_{t,t+n}) - E_t(\log m_{t,t+n})}_{\log p_t^n = -ny_t^n} \\ \mathcal{L}(n) &\equiv E[L_t(m_{t,t+n})] = -nE(y^n) - nE(\log m_{t,t+1})\end{aligned}$$

- ▶ Two measures of horizon dependence

$$\begin{aligned}H(n) &\equiv \underbrace{n^{-1}\mathcal{L}(n)}_{\text{avg over } n \text{ periods}} - \underbrace{\mathcal{L}(1)}_{\text{one period}} = -E(y_t^n - y_t^1) \\ F(n) &\equiv \mathcal{L}(n+1) - \mathcal{L}(n) - \mathcal{L}(1) = -E(f_t^n - f_t^0)\end{aligned}$$

Entropy and the term structure

- ▶ The iid benchmark: $\{m_{t,t+1}\}$ iid \Rightarrow

$$\mathcal{L}(n) = n \mathcal{L}(1)$$

$$H(n) = 0$$

$$F(n) = 0$$

- ▶ Also: yields and forwards constant, same at all maturities
- ▶ Anything different from this reflects dynamics in m

Vasicek model: dynamic structure

- ▶ Pricing kernel

$$\log m_{t,t+1} = \log \beta + x_t + \lambda w_{t+1}$$

$$x_{t+1} = \varphi x_t + \sigma w_{t+1}$$

x is (persistent or long-run) risk, λ is price of risk

- ▶ Moving average representation

$$\begin{aligned} \log m_{t,t+1} &= \log \beta + x_t + \lambda w_{t+1} \\ &= \log \beta + \lambda w_{t+1} + \underbrace{\sigma w_t + \sigma \varphi w_{t-1} + \dots}_{x_t} \end{aligned}$$

Vasicek model: entropy

- ▶ Pricing kernel dynamics inherited from x
- ▶ Term structure of entropy

$$\mathcal{L}(1) = k(\lambda)$$

$$\mathcal{L}(2) = k(\lambda) + k(\lambda + \sigma)$$

$$\mathcal{L}(3) = k(\lambda) + k(\lambda + \sigma) + k(\lambda + \sigma + \sigma\varphi)$$

- ▶ What makes this non-iid?

Vasicek model: parameter values

- ▶ Short rate

$$y_t^1 = f_t^0 = -\log E_t(m_{t,t+1}) = -[\log \beta + k(\lambda)] - x_t$$

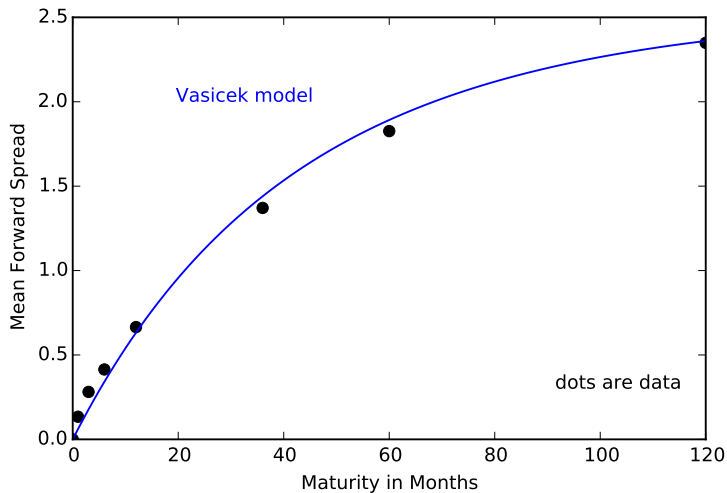
- ▶ Choose

- ▶ $(\varphi, \sigma) = (0.98, -0.006)$ match variance and autocorrelation
- ▶ w normal $\Rightarrow k(s) = s^2/2$
- ▶ $\lambda = 0.088$ matches mean forward spread $E(f^n - f^0)$
(ask how this works)

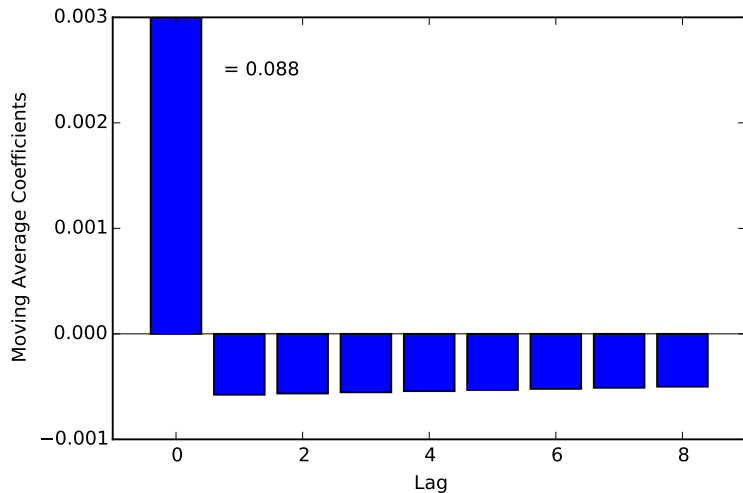
- ▶ Features

- ▶ σ and λ must have opposite signs for curve to slope up
- ▶ λ **much** greater than σ in absolute value

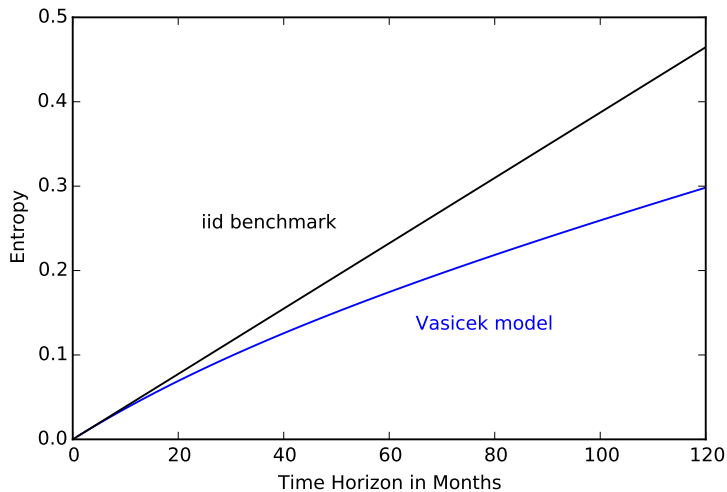
Vasicek model: mean forward spreads



Vasicek model: moving average coefficients



Vasicek model: entropy



Coentropy: risk premiums revisited

The idea

- ▶ Expected excess returns differ across assets
- ▶ Reflects dependence of pricing kernel and cash flows
- ▶ We measure dependence with coentropy
- ▶ Extend shortly to long time horizons

Coentropy

- ▶ Coentropy is a measure of dependence: for $x_1, x_2 > 0$

$$C(x_1, x_2) \equiv L(x_1 x_2) - L(x_1) - L(x_2)$$

- ▶ Features

- ▶ Invariant to scaling
- ▶ Equals zero if x_1 and x_2 are independent

- ▶ Related to (joint) cgf $k(s_1, s_2) = \log E(e^{s_1 \log x_1 + s_2 \log x_2})$

$$C(x_1, x_2) = \underbrace{k(1, 1)}_{x_1 x_2} - \underbrace{k(1, 0)}_{x_1} - \underbrace{k(0, 1)}_{x_2}$$

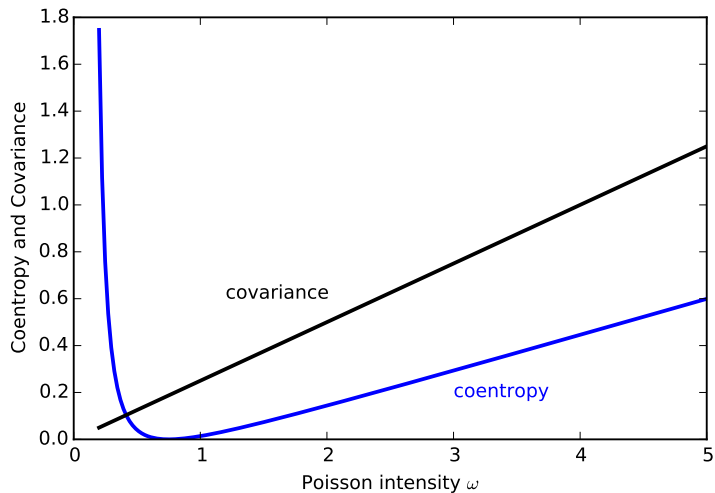
Coentropy (continued)

- ▶ If $\log x = (\log x_1, \log x_2)$ is normal, coentropy = covariance
- ▶ Can also be much different
- ▶ Example: Poisson mixture (“jump process”)
 - ▶ Poisson jumps: probability $e^{-\omega} \omega^j / j!$ of $j = 0, 1, 2, \dots$
 - ▶ Conditional on j , $\log x \sim \mathcal{N}(j\theta, j\Delta)$
- ▶ Properties

$$\text{Cov}(\log x_1, \log x_2) = \omega(\theta_1\theta_2 + \delta_{12})$$

$$C(x_1, x_2) = E2C2E$$

Coentropy and covariance



Coentropy and excess returns

- ▶ Consider a claim to the cash flow $g_{t,t+1}$
- ▶ Return is cash flow over price
- ▶ Invariance to scaling implies

$$\begin{aligned}L_t(r_{t,t+1}) &= L_t(g_{t,t+1}) \\ C_t(m_{t,t+1}, r_{t,t+1}) &= C_t(m_{t,t+1}, g_{t,t+1})\end{aligned}$$

- ▶ Expected excess returns (“risk premiums”)

$$\begin{aligned}E_t(\log r_{t,t+1} - \log r_{t,t+1}^1) &= -L_t(g_{t,t+1}) - C_t(m_{t,t+1}, g_{t,t+1}) \\ E(\log r_{t,t+1} - \log r_{t,t+1}^1) &= \underbrace{-E[L_t(g_{t,t+1})]}_{\text{entropy}} - \underbrace{E[C_t(m_{t,t+1}, g_{t,t+1})]}_{\text{coentropy}}\end{aligned}$$

KLV model (streamlined version)

- ▶ Add another disturbance to Vasicek

$$\log m_{t,t+1} = \log \beta + x_t + \lambda_1 w_{1t+1} + \lambda_2 w_{2t+1}$$

$$x_{t+1} = \varphi x_t + \sigma w_{1t+1}$$

$$(w_{1t}, w_{2t}) \sim \text{NID}(0, I)$$

- ▶ Stir in some cash flow growth

$$\log g_{t,t+1} = \log \gamma + \theta x_t + \eta_1 w_{1t+1} + \eta_2 w_{2t+1}$$

- ▶ Entropy and coentropy

$$E[L_t(m_{t,t+1})] = (\lambda_1^2 + \lambda_2^2)/2$$

$$E[L_t(g_{t,t+1})] = (\eta_1^2 + \eta_2^2)/2$$

$$E[C_t(m_{t,t+1}, g_{t,t+1})] = \lambda_1 \eta_1 + \lambda_2 \eta_2$$

KLV model: numerical example

- ▶ Choose $(\varphi, \sigma, \lambda_1) = (0.98, -0.0006, 0.088)$ as before to fit yields/forwards
- ▶ Choose $(\eta_1, \eta_2) = (-0.005, -0.050)$ to match
 - ▶ Variance of excess return on equity (0.05)
 - ▶ Correlation of excess returns on equity and bonds (0.1)
- ▶ Choose $\lambda_2 = 0.097$ to match equity premium
- ▶ Results (monthly):
 - ▶ Bond premium (10 years): 0.002
 - ▶ Equity premium: $0.004 = 0.005$ (coentropy) $- 0.001$ (entropy)
 - ▶ Entropy of m : 0.009 (upper bound)

Other term structures: cash flow dynamics

The idea

- ▶ Consider claims to currencies, equity indexes, dividends, ...
- ▶ How do their term structures compare?
- ▶ The time horizon of coentropy
- ▶ Explorations with the KLV model

Prices, yields, and forward rates

- ▶ Let \hat{p}_t^n be price at t of claim to cash flow growth $g_{t,t+n}$
- ▶ Term structure

$$\hat{y}_t^n = -n^{-1} \log \hat{p}_t^n$$

$$\hat{f}_t^n = \log(\hat{p}_t^n / \hat{p}_t^{n+1})$$

$$\log \hat{r}_{t,t+1}^{n+1} = \log \hat{p}_{t+1}^n - \log \hat{p}_t^{n+1} + \log g_{t,t+1}$$

$$E(\log \hat{r}^{n+1}) = E(f^n + \log g)$$

- ▶ Forward price $q_t^n = \hat{p}_t^n / p_t^n \Rightarrow$

$$n^{-1} \log q_t^n = y_t^n - \hat{y}_t^n$$

$$\log q_t^{n+1} - \log q_t^n = f_t^n - \hat{f}_t^n$$

Pricing fundamentals

- ▶ Pricing is recursive

$$\begin{aligned}\hat{p}^n(x_t) &= E_t[m(x_t, x_{t+1})g(x_t, x_{t+1})\hat{p}^{n-1}(x_{t+1})] \\ &= E_t[\hat{m}(x_t, x_{t+1})\hat{p}^{n-1}(x_{t+1})]\end{aligned}$$

with $\hat{p}^0 = 1$ and $\hat{m}(x_t, x_{t+1}) = m(x_t, x_{t+1})g(x_t, x_{t+1})$

- ▶ Definitions give us yields $\hat{y}^n(x_t)$ and forward rates $\hat{f}^n(x_t)$
- ▶ Think of this as a change of units: dollars to yen
- ▶ Empirical strategy changes: we observe g

Entropy, coentropy, and term structures

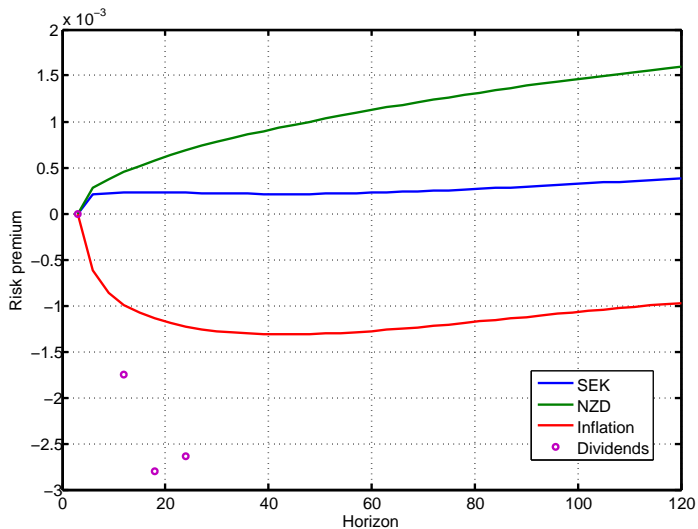
- ▶ Entropy and coentropy

$$\begin{aligned}L_t(\widehat{m}_{t,t+n}) &= L_t(m_{t,t+n}g_{t,t+n}) \\ &= C_t(m_{t,t+n}, g_{t,t+n}) + L_t(m_{t,t+n}) + L_t(g_{t,t+n})\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{\widehat{m}}(n) &= \mathcal{L}_m(n) + E[C_t(m_{t,t+n}, g_{t,t+n})] + E[L_t(g_{t,t+n})] \\ &= \mathcal{L}_m(n) + \underbrace{C_{mg}(n)}_{\text{coentropy}} + \underbrace{\mathcal{L}_g(n)}_{\text{entropy}}\end{aligned}$$

- ▶ Same connection to mean yields and forward rates as before

Term structure data: other assets relative to US nominal



KLV model

► Model

$$\log m_{t,t+1} = \log \beta + x_t + \lambda_1 w_{1t+1} + \lambda_2 w_{2t+1}$$

$$x_{t+1} = \varphi x_t + \sigma w_{1t+1}$$

$$\log g_{t,t+1} = \log \gamma + \theta x_t + \eta_1 w_{1t+1} + \eta_2 w_{2t+1}$$

► Transformed pricing kernel

$$\begin{aligned} \log \hat{m}_{t,t+1} = & (\log \beta + \log \gamma) + \overbrace{(1 + \theta) x_t}^{\text{long-run risk}} \\ & + \underbrace{(\lambda_1 + \eta_1)}_{\text{price of lr risk}} w_{1t+1} + \underbrace{(\lambda_2 + \eta_2)}_{\text{price of iid risk}} w_{2t+1} \end{aligned}$$

► Roles of: $\eta_1, \lambda_2, \eta_2, \theta$

KLV model: currencies

- ▶ Model

$$\log m_{t,t+1} = \log \beta + x_t + \lambda_1 w_{1t+1} + \lambda_2 w_{2t+1}$$

$$x_{t+1} = \varphi x_t + \sigma w_{1t+1}$$

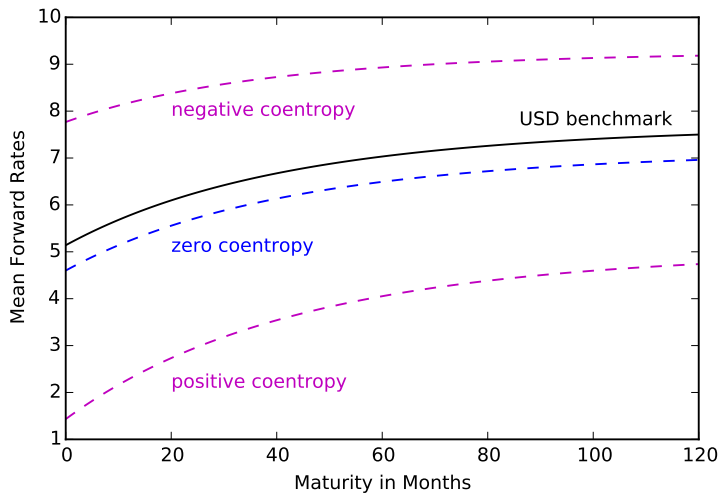
$$\log g_{t,t+1} = \log \gamma + \theta x_t + \eta_1 w_{1t+1} + \eta_2 w_{2t+1}$$

- ▶ Currencies: $\theta \approx 0$, $\eta_1^2 + \eta_2^2 \approx 0.03^3$, $\lambda_2 \approx 0$ (for now)

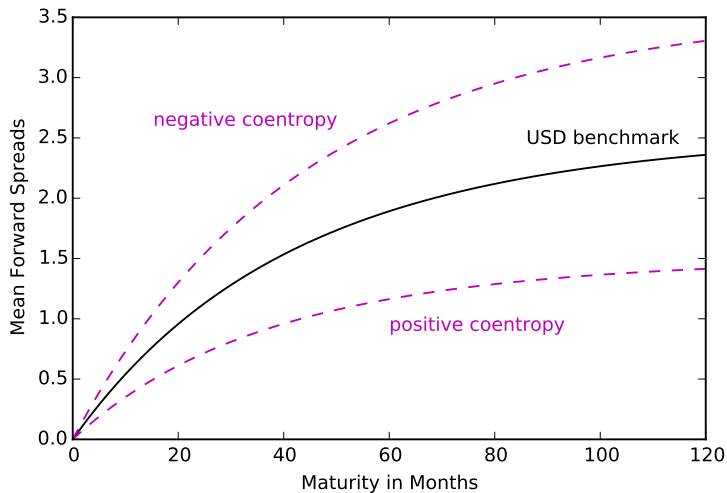
- ▶ Then (η_1, η_2) control coentropy

- ▶ $(\eta_1, \eta_2) = (0, 0.3)$: coentropy is zero
- ▶ $(\eta_1, \eta_2) = (0.3, 0)$: coentropy is positive
- ▶ $(\eta_1, \eta_2) = (-0.3, 0)$: coentropy is negative

KLV model: currencies



KLV model: currencies



KLV model: equity

► Model

$$\log m_{t,t+1} = \log \beta + x_t + \lambda_1 w_{1t+1} + \lambda_2 w_{2t+1}$$

$$x_{t+1} = \varphi x_t + \sigma w_{1t+1}$$

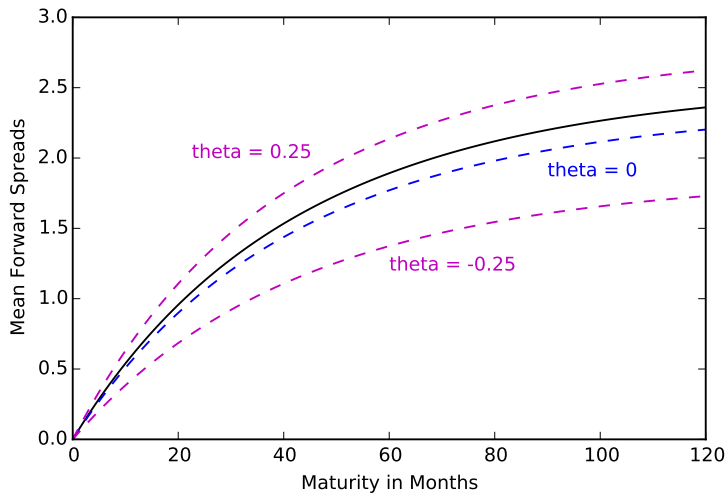
$$\log g_{t,t+1} = \log \gamma + \theta x_t + \eta_1 w_{1t+1} + \eta_2 w_{2t+1}$$

► Reminder: transformed pricing kernel

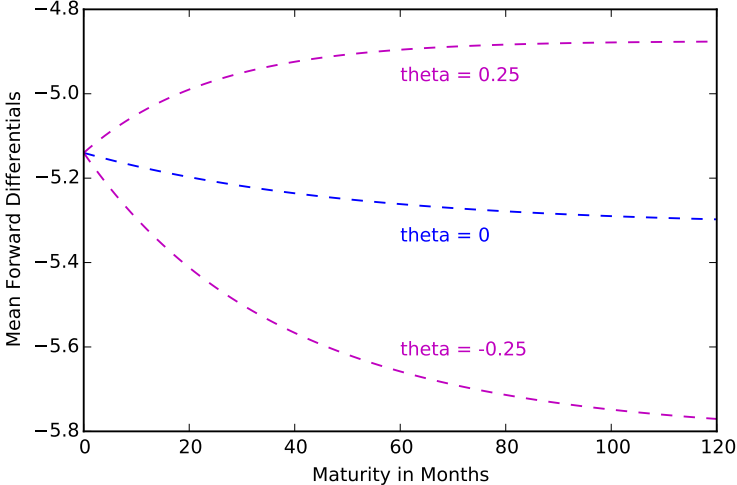
$$\begin{aligned} \log \hat{m}_{t,t+1} = & (\log \beta + \log \gamma) + \overbrace{(1 + \theta) x_t}^{\text{long-run risk}} \\ & + \underbrace{(\lambda_1 + \eta_1)}_{\text{price of lr risk}} w_{1t+1} + \underbrace{(\lambda_2 + \eta_2)}_{\text{price of iid risk}} w_{2t+1} \end{aligned}$$

► Roles of: $\lambda_2 + \eta_2$, η_1 (-0.005), θ (± 0.25)

KLW model: equity



KLV model: equity



Where were we?

Summary and open questions

Summary

- ▶ Significant variation in average term structures across assets
- ▶ Connected to entropy and coentropy
- ▶ Large quantitative effects in simple models (still no bazooka!)

Open questions

- ▶ What would you do with these ingredients?
- ▶ Would predictability interest you?