J Popul Econ (2003) 16:135–160

Journal of
Population Economics

Overlapping generations models with realistic demography

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Received: 27 March 2000/Accepted: 25 April 2001

Abstract. This paper revisits the literature on overlapping generations models in the demographic context of a continuous age distribution and a general age schedule of mortality. We show that most of the static results known for the 3 or N age-group models can be extended to the continuous model. Some results, previously established for economies without capital, are extended to productive economies. We also make some progress on the existence of some steady states as well as on the dynamic properties.

JEL classification: D9, E13, H55, H62, 010, 016

Key words: Overlapping generations, Pay-As-You-Go systems, demography

1. Introduction

Samuelson (1958) used a simple demo-economic model to raise fundamental questions such as whether a market economy could reach an optimal equilibrium. His rich analysis included both static and dynamic aspects. First, he described different steady states consistent with certain economic constraints. Second, he considered which of these steady states might or might not be reached by a market economy.

Since then, many static (that is, comparative steady state) results have been established for the two age group model (Diamond 1965; Gale 1972; Balasko

All correspondence to Antoine Bommier. We would like to think Robert Willis for many helpful suggestions and acknowledge comments by two anonymous referees. Our research was supported by a grant from the National Institute on Aging, R37-AG11761. *Responsible editor:* Alessandro Cigno.

et al. 1980; Balasko and Shell 1980, 1981a, b; Esteban et al. 1993). Some static results have also been established for the discrete N age-group case (Starrett 1972; Gale 1973; Kim and Willis 1982; Willis 1988; Augusztinovics 1992).

By contrast, there have been few studies of the dynamics of overlapping generations models, which is a much more complex topic. Gale (1973) made some progress with the two age-group pure exchange economy, and made conjectures for the N age-group case. A number of results have been obtained for the two age-group productive economy as well, as in Gale (1972), Tirole (1985), Weil (1987), Galor and Ryder (1989, 1991), and Galor (1992).

All these works are based on very simplistic demographic assumptions, with the population divided into a finite number of age groups, typically only two, and with everybody dying at the end of the last age group. Results often depend strongly on such assumptions. In Samuelson's article, for example, the three age-group model can support steady states which are impossible in the two age-group model. Two age group models are not capable of representing the most basic feature of the human economic life cycle: that it begins and ends with periods of dependency, separated by a long intermediate period of consuming less than is produced. Models in which all survive to the end of the last age group cannot be used to investigate the consequences of mortality change.

Blanchard (1985) introduced some variation and uncertainty concerning the timing of death. However, his paper is based on the unrealistic assumption that the probability of dying does not change over the life-cycle[1]. This a very convenient assumption since it implies that all the individuals have the same life expectancy, regardless of their ages, and therefore they all have the same propensity to consume. As acknowledged by Blanchard, this approach is unable to capture the life-cycle aspect of life, which is the essence of overlapping generation models.

Calvo and Obstfeld (1988) considered a dynamic continuous model with realistic demographic assumptions. Their paper analyzed the properties of optimal steady states associated with particular social welfare functions and the existence of a time consistent fiscal policy allowing the economy to reach these optimal steady-states.

Other researchers with particular interest in demographic issues have extensively studied the properties of Golden Rule steady-states (Arthur and McNicoll 1978; Lee 1980, 1994a, b). These works explored important questions such as population aging and the consequences of mortality decline. They also lend themselves more readily to empirical implementation (Lee 1994a, b and in press).

In this article we consider a continuous demographic model, with a general mortality pattern, and study mainly the static but also some dynamic properties of a productive market economy. Most of the static results known for the N age-group models are extended to the continuous model. Some results, previously established for economies without capital, will be extended to productive economies. We also make some progress on the dynamic properties. Little has been known about the dynamics of models with more than two age-groups, even for the simplest N age-group model. In the most general case of our continuous model we are able to obtain some results conjectured by Gale about the stability of steady-states.

Although we do not discuss empirical applications in this paper, this model has been used as a framework for empirical studies in Bommier and Lee

(2000), and we view this paper as helping to bridge the gap between theoretical Overlapping Generations models and the empirical analysis of the economic effects of changing age distributions.

The remainder of the article is organized as follows. In Sect. 2 we discuss the interest of adding some demographic complexity to the theory of overlapping generation models. Section 3 presents our theoretical accounting framework. In Sect. 4 we set up our market economy model, and then study its static and dynamic properties, respectively. Section 5 will examine more general economies with intergenerational transfers or government taxes. The main technical proofs are in an Appendix.

2. Does demography really matter?

As noted in the introduction, overlapping generations models have mainly been developed with two, three or N age-group models. Most of the models do not consider any uncertainty in the age at which death occurs. People are assumed to live for 2, 3 or N periods and to die at the end of the last one. In this article we develop a continuous model, which therefore includes an infinite number of age-groups, with uncertainty in the time of death characterized by a smooth survival function. Such a model would be expected to be more complex to develop than the simple stylized two age-group model (although use of an appropriate accounting framework will allow us to circumvent most of the complexity). One might wonder, then, whether the added complexity is worth the effort.

A minimum of two age groups is necessary to encompass the fact that a population includes people with different ages, and therefore with different planning horizons. Two age-group models have yielded very interesting insights and raised important questions, such as problems of efficiency. However, the two age-group model is very restrictive since it does not allow for periods of dependence in both youth and old-age. This is a major theoretical restriction. Indeed, in a three (or more) age-group model it is possible to have a Pay-As-You-Go transfer system which is actuarially neutral, while in a two age group model this is impossible in a steady state, unless the rate of economic growth equals the rate of interest (Samuelson 1958). Thus the three age-group models support exchanges that are impossible in two age-group models. The existence of actuarially neutral transfers is not only interesting for theoretical research but is also a central element in more applied research. For example, Becker and Murphy (1988) claim that, if transfers for education and transfers towards the elderly are considered as a whole, the development of the public transfer systems during these last decades approximately corresponds to the introduction of such a neutral transfer system.

The extension of overlapping generation models from two age-groups to three age-groups is fundamental as it supports new economic interpretations. As we will see, in this paper the extension from 3 to N or an infinity of age groups is not as innovative. In particular the reader should not expect to find revolutionary results such as new market inefficiency following from the introduction of a continuum of age-groups. We will show that most of the results known for two or three age-group models can simply be extended to this more general framework. Small differences occur – for example, the necessary conditions for the existence of steady state are weaker – but they are minor.

There are nonetheless some advantages in using a continuous framework because with an appropriate accounting framework the continuous model appears to be easier to write and to solve. This esthetic improvement also implies what one might call a better "traceability" of the results, by making more transparent the logic underlying some of the analytic results. To take a concrete example, if one considers a three age-group model with ages 1, 2 and 3, then the mean age in a stationary population is simply "2"[2]. Once lost in a mathematical formula, it is sometimes difficult to realize that this "2" stands for a demographic variable and does not result from a casual mathematical calculation (such as $1 + 1$ or $\frac{2}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-x^2}\,dx$.). When we simplify too much, it is possible to lose the meaning of some analytic results. This argument might seem secondary, but it is not when it comes to interpreting the theoretical results or to evaluating the theoretical concepts from empirical data.

Yet, we believe that the most compelling reason for developing overlapping generation models with realistic demography is that they provide a ready-to-use basis for empirical research. We argue that this is a fundamental point since we believe that the gap between theoretical and empirical research is at the origin of many false or very imprecise conclusions. A quick look at a few well-known articles in the economic literature shows how surprisingly easily people use misleading shortcuts to go from a simple theoretical model to empirical estimates. For example, in the controversy about the role of bequests in capital accumulation, Kotlikoff and Summers (1981) and Modigliani (1988) used wealth estimates computed from flows of bequests assuming no dispersion of the age at death. This method was again used by Gale and Scholz (1994) although their data actually included the age dispersion, so the simplifying assumption was not necessarily. In a more recent article Gale (1998), analyzing the effect of pensions on saving, estimated pension wealth, which was the main explanatory variable, relying on similar assumptions. Even numerical simulations often rely on such simplistic assumptions. For example, in Auerbach and Kotlikoff (1987) or in Hviding and Mérette (1998) there is no uncertainty in the age at death.

These simplifications can have important consequences. For example, Bommier and Lee (2000) calculated the hypothetical effect on the steady state capital stock of eliminating various public transfer programs. When age dispersion was not taken into account, the estimated effect of eliminating Social Security was 14% greater than the result with age dispersion. For Medicare the discrepancy was 12%; for Institutional Medicaid (nursing home care) there was a sign reversal; and for K-12 public education the effect without aged dispersion was 78% greater than it was with age dispersion. Bommier et al. (1995) show that the standard mean age approximation for calculating bequest wealth understates it by 8% if the interest rate exceeds the rate of economic growth by 1%, by 18% if the excess is 2%, and by 33% if the excess is 3%. Some of these distortions are quite substantial.

Most of the time, these rough approximations could be avoided at very low cost, but the common reliance on simple theoretical models makes researchers easily forget that reality is not that simple. Theoreticians are, of course, not responsible for the inappropriate use of their simple illustrative models (unless they themselves use them to derive empirical results!), but we argue that if it is possible to support their thought with more realistic models, this should be done. This is precisely the aim of this article which can be seen as a theoretical exercise which aims to add a demographic framework to the existing literature on overlapping generations models. Bommier and Lee (2000) contains applications showing how this demographically enriched framework can be used for empirical research.

3. Accounting framework

As noted by Blanchard (1985), aggregation is a major difficulty that arises when considering an economy with finitely lived agents. Individuals behave according to life cycle (or longitudinal) constraints while the economy faces cross-sectional constraints[3]. Except for very particular cases, such as the golden rule steady state and the demographically unrealistic case study of Blanchard (1985), these constraints have very different analytic expressions and aggregation is problematic. This question of aggregation cannot be avoided, however, if we aim to make the link between individual behavior and aggregate variables, as we would do, for example, to analyze the effect of a Pay-as-you-go pension system on aggregate capital accumulation.

Cross sectional and longitudinal constraints, although they are of a different nature, are not completely independent and the object of this section is to develop an accounting framework too see how cross sectional and longitudinal accounts are dynamically linked.

Of course, demographic variables play a central role in this problem of aggregation and we believe it is important to develop our accounting framework without making restrictive demographic assumptions. We will thus consider a population closed to migration, but not necessary stable in the demographer's sense, that is not necessarily in steady state. We will suppose that the probability that an individual born x years ago (at time $t-x$) is still alive at time t is a function of age x and time t that we denote $p(x,t)$[4]. We will also assume, as most analyses do implicitly, that there exists a "maximum age" ω such that $p(x,t)=0$ for all $x \geq \omega$. The flow of births at time t is denoted $B(t)$ and the size of the population is $P(t)$. Such assumptions are quite general since they allow for changes in mortality and fertility.

We will now develop a notation and accounting framework for describing at a very general level the age and time specific flows into and out of the average individual's budget. Let $g(x,t)$ be a function of age x and time t, which we will call by the generic name "system of reallocation". $g(x,t)$ might be the consumption by people of age x at time t, their labor income, their savings, or any other flow or stock of resources. To make our discussion easier to follow, we will often illustrate our accounting framework by taking $g(x,t)$ to describe net social security transfers, so that $g(x,t)$ equals $g^+(x,t)-g^-(x,t)$, where $g^+(x,t)$ are the benefits received and $g^-(x,t)$ the taxes paid by people of age x at time t. To see the redistribution over time for a cohort born at t_0, (taking a longitudinal perspective) we can focus on the appropriate values of $g(x,t_0+x)$, for $x=0$ to ω. To see the reallocation across age in a given calendar year t_0, we can focus on $g(x,t_0)$, for all ages x.

The dashed lines in Fig. 1 plot the cross sectional distribution of taxes (g^-) and benefits (g^+) for 1995, and g, which is not shown, is simply the difference of these two. Values of g referring to future time periods $t>1999$ are the average expectations of actors, conditional on their survival to that future year. The continuous lines plots Social Security taxes and benefits from a longitudinal perspective for the cohort that is age 62 in 1997.

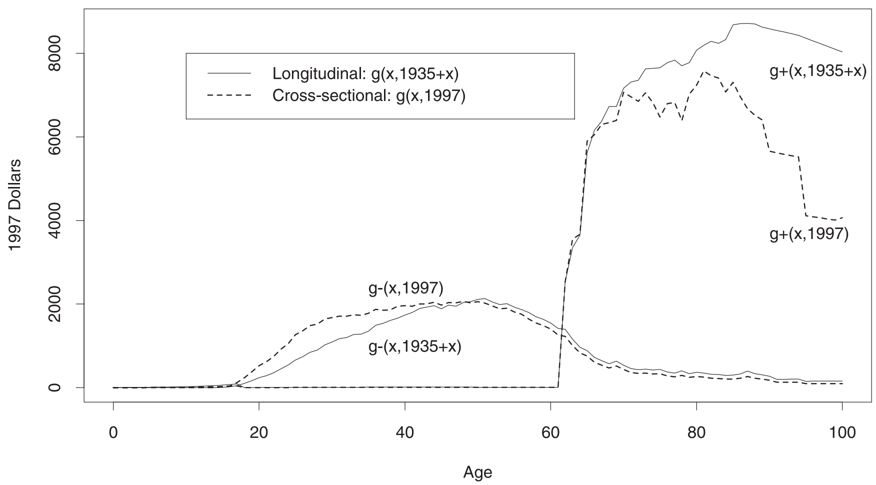

Fig. 1. The US Social Security System (OAI): Cross-sectional taxes and benefits in 1997 and longitudinal taxes and benefits for the cohort born in 1935

3.1. Some relevant characteristics of reallocation functions

We begin by defining some concepts relative to any function g of the sort discussed above. Viewing g from a life cycle or cohort or longitudinal perspective, we define the present value of expectation of net receipt at birth by:

$$PV(g,t) = \int_0^{\omega} e^{-\int_t^{t+x} r(a)\,da} p(x, t+x) g(x, t+x)\,dx$$

where $r(t)$ denotes the rate of interest at time t.

Viewing g from a cross-sectional perspective, at a given time t, we define the population-weighted average flow by:

$$Pop(g,t) = \frac{1}{P(t)} \int_0^{\omega} B(t-x) p(x,t) g(x,t)\,dx$$

(note that the number of people of age x at time t is $B(t-x)p(x,t)$, and $P(t)$, the size of the population, is the integral of this over x.)

Definition 1. *We will say that a system of reallocation, g, is:*
– *life cycle balanced if $PV(g,t) = 0$ for every t.*
– *population balanced if $Pop(g,t) = 0$ for every t.*

Of course, whether the function is in life cycle balance or in population balance will depend on the expected levels of mortality and of r as well as on g itself.

A system g is life cycle balanced if the present value of receipt at birth through g is zero. If a system g is population balanced, then all the flows aggregate to zero at any time. This would be true, for example, of any pure Pay-

As-You-Go public transfer system, for which the specific tax revenues exactly cover the expenditures, or for any familial transfers. In general, some systems which are in life cycle balance are not in population balance (such as private investment), while some systems in population balance are not in life cycle balance (such as a Pay-As-You-Go pension system). Borrowing and lending at market rates of interest, viewed as a system of reallocation, is in both life cycle and population balance.

As an illustration, we use the values shown in Fig. 1 to compute $PV(g, 1935)$ and $Pop(g, 1997)$ when g are social security transfers, assuming a discount rate of 4%. We found $PV(g, 1935) = -\$4700$ which says that the cost of participating in the social security transfer system, for the 1935 cohort, has been equivalent to a lump-sum tax at birth of \$4700. $Pop(g, 1997)$ is found to be $-\$100$. In a pure Pay As You Go system this would be zero, but taxes have been raised to permit the US system to accumulate a fund in anticipation of the retirement of the Baby Boom generations in the early 21st Century.

3.2. Wealth associated with a system of reallocation

The concept of the average wealth associated with g is fundamental to our theoretical accounting framework. For individuals at some given age, their wealth is the expected present value of future flows of g. For the population, the average wealth is the population weighted average across all ages of these expected present values. More formally, individual or cohort wealth is defined by:

$$w_g(x,t) = \int_x^{\omega} \frac{p(u, t+u-x)}{p(x,t)} g(u, t+u-x) e^{-\int_t^{t+u-x} r(a)\,da}\, du \tag{1}$$

Cohort or individual wealth at age 0, $w_g(0, t)$, equals $PV(g)$ as defined above.

Aggregate wealth per capita at time t, $W(g, t)$, equals $Pop(w_g)$ as defined above; that is:

$$W(g,t) = Pop(w_g) = \frac{1}{P(t)} \int_0^{\omega} B(t-x) p(x,t) w_g(x,t)\, dx \tag{2}$$

When g refers to some specific transfer system, then $Pop(w_g)$ corresponds to the usual definition of transfer wealth held through that system. For example, if g refers to the US Social Security system, then $Pop(w_g)$ is the average Social Security wealth for the population, and $P(t) * Pop(w_g)$ is the aggregate Social Security wealth, around 11 trillion dollars in the US in 1995 (see Feldstein 1996). This amount depends sensitively on the interest rate used to discount the future. Lee (1994a) provides estimates under certain assumptions of transfer wealth held in various forms in the US as of 1987, expressed per household. For example, Social Security wealth was \$70,000; wealth through public health care was \$35,000; wealth through bequests was $-\$44,000$; wealth through public education was $-\$17,000$; and wealth through AFDC (aid to poor families) was \$800. Negative wealth occurs in the foregoing when the average person expects to pay more into a system in the future then they expect to receive from it, which happens when a particular kind of transfer typically occurs earlier in life and the payment later, as with bequests or public education.

When g refers to $c - y_l$, consumption minus labor income, then $w_g(x, t)$ is what we will call "life cycle wealth". It is the average wealth necessary to achieve the life cycle consumption path defined by c, given the life cycle earning path defined by y_l. W_g is then life cycle wealth per capita in the population. In the general case, life cycle wealth can be held either as capital or as transfer wealth. Lee (2000) reports that per capita life cycle wealth in pre-industrial economies, including both hunter/gatherers and agriculturalists, is typically negative. In the US, and perhaps in other industrial economies, per capita life cycle wealth is positive. The difference is stems both from their population age distributions and from their age patterns of consuming and earning. In pre-industrial economies, high fertility and high mortality together give the populations a very young age distribution, with many children and few elderly. This age distribution gives a heavier weight to children who are recipients of downward transfers and holders of negative net wealth, than to the elderly, who may have either positive or negative life cycle wealth depending on the setting. Also, in pre-industrial settings, the elderly tend to remain economically active, and so receive relatively smaller transfers or in some settings may even continue to make transfers to their children. In the United States, by contrast, low fertility and mortality generate an older population age distribution. Retirement age is relatively young, having declined considerably during the 20th century, while the growing cost of health care has raised consumption of the elderly relative to earlier ages.

This notion of life cycle wealth, which follows Lee's definition (1994a, b), is an expectation, therefore forward looking. It contrasts with the notions of assets used by Gale (1973), Kotlikoff and Summers (1981) and Willis (1988) which look backward at prior accumulation[5]. If there are transfers present, then forward looking and backward looking definitions of wealth give different results. When considering the behavioral implications of transfers, it is the forward looking definition of wealth that is pertinent, since individuals make plans according to the constraints they will have to face in the future, but not according to their past.

We can now give an important accounting identity for the evolution of wealth over time, linking this to the properties of life cycle balance and population balance of an economy.

Proposition 1. *The wealth held through the system of reallocation g satisfies the equation:*

$$\frac{dW}{dt}(g, t) = (r(t) - n(t))W(g, t) + b(t)PV(g, t) - Pop(g, t)$$

where $n(t) = \dfrac{P'(t)}{P(t)}$ *is the rate of population growth and* $b(t) = \dfrac{B(t)}{P(t)}$ *is the flow of births per capita.*

This equation encompasses the links between demographic variables, and longitudinal and cross sectional accounts. It says that the wealth per capita increases because it earns a return at rate $r(t)$, decreases through dilution due to population growth, and increases (or decreases) because new entrants to the population, that is births, may arrive with a positive (or negative) wealth[6], and decreases (or increases) as a consequence of the aggregate net flow of wealth. The mathematical proof is written in Appendix A.

3.3. *First example: The steady states*

In steady states the dependence on t disappears and the definitions of PV and Pop may be simply written:

$$PV(g) = \int_0^{\omega} e^{-rx} p(x) g(x)\, dx \tag{3}$$

$$Pop(g) = b \int_0^{\omega} e^{-nx} p(x) g(x)\, dx \tag{4}$$

where b is the crude birth rate, that is the flow of births per capita, and n is the population growth rate. By inspection of (4) and comparison to (3) we can see that in any system in population balance ($Pop(g) = 0$), the rate of population growth is always a solution for the implicit rate of return earned through participation in the system g, and in particular n is the implicit rate of return for any intergenerational transfer system. This is well-known for Social Security, but it is equally true for rearing children, making bequests, or any other perpetually recurring pattern of transfers[7].

Proposition 1 implies that in a steady-state the wealth held through a system of allocation g must satisfy the equation:

$$(r - n)W(g) + bPV(g) - Pop(g) = 0$$

This equation shows the simple relationships among the concepts of wealth, life cycle balance, and population balance in this particular case.

Splitting g into $g = g^+ - g^-$, with g^+ and g^- being non negative, and using the limit $(r - n) \to 0$ of this latter equation it is easy to obtain results known for the Golden Rule case (Lee 1994a, b):

$$W(g) = Pop(g^+)A_{g^+} - Pop(g^-)A_{g^-}$$

where A_{g^+} and A_{g^-} are the average ages of in-flows and out-flows of g. In particular, life cycle wealth in golden rule is given by $y(A_c - A_{y_l})$ where y is per capita income.

3.4. *Second example: Auerbach, Gokhale and Kotlikoff generational accounting*

Call g the system of government taxes, transfers and services[8]. Instead of the per-capita accounting equation of Proposition (1) we use the aggregate version of it:

$$\frac{d}{dt}(P(t)W(g,t)) = r(t)P(t)W(g,t) + B(t)PV(g,t) - P(t)Pop(g,t)$$

Following Auerbach et al. (1991), assume that r will be constant in the future ($t \geq 0$).

Integrating this equation leads to:

$$\int_0^{+\infty} \frac{d}{dt}(P(t)W(g,t)e^{-rt})\,dt$$
$$= \int_0^{+\infty} B(t)PV(g,t)e^{-rt} - \int_0^{+\infty} e^{-rt}P(t)Pop(g,t)\,dt \tag{5}$$

Auerbach, Gokhale and Kotlikoff assume that the government debt must remain bounded, and therefore that $\lim_{t\to+\infty} P(t)W(t)e^{-rt} = 0$. Then Eq. (5) leads to the following budget constraint:

$$-W(g,0)P(0) = \int_0^{+\infty} B(t)PV(g,t)e^{-rt}\,dt - \int_0^{+\infty} P(t)e^{-rt}Pop(g,t)\,dt \tag{6}$$

The left hand term is the negative of the wealth held by the population currently alive. The term $\int_0^{+\infty} B(t)PV(g,t)e^{-rt}\,dt$ gives what is to be paid by the future generations. $P(t)Pop(g,t)$ is the government net expenditure on transfers at time t, so $\int_0^{+\infty} e^{-rt}P(t)Pop(g,t)\,dt$ must equal the present value of the future government consumption, minus its wealth at present.

Auerbach, Gokhale and Kotlikoff want to compare the present value of taxes to be paid by the generation just born, $PV(g,0^-)$, to what should be paid by the next generation to be born, $PV(g,0^+)$, if in the future the average lifetime net tax payment were to rise at the economy's rate of productivity growth, λ. In other words they assume that for $t > 0$, $PV(g,t) = PV(g,0^+)e^{\lambda t}$. The budget constraint Eq. (6) then leads to:

$$PV(g,0^+) = \frac{1}{\int_0^{+\infty} B(t)e^{-(r-\lambda)t}\,dt}(\text{Future Gvt. consumption}$$
$$- \text{Gvt. wealth} - P(0)W(g,0))$$

Not surprisingly, we have come to the same accounting equation as in Auerbach, Gokhale and Kotlikoff generational accounting. Note that our framework allows us to extend this equation to the case where r or λ are no longer constant. In this case the factor $\frac{1}{\int_0^{+\infty} B(t)e^{-(r-\lambda)t}\,dt}$ should be replaced by $\frac{1}{\int_0^{\infty} B(t)e^{-\int_0^t (r(a)-\lambda(a))\,da}\,dt}$ in the formula above, and of course the estimate of life cycle wealth, and government consumption recomputed with the new values of $r(t)$ and $\lambda(t)$. Empirically speaking, there is probably not much to gain from this generalization, since the evolution of $r(t)$ and $\lambda(t)$ are quite difficult to predict.

4. A closed market economy with capital

From now on we will assume that the population, but not necessarily the economy, is in a steady state, that is that $p(x,t)$, $n(t)$ and $b(t)$ do not depend on t. We also assume that we are in a productive world (not pure exchange) where there is no technical progress and where production is an homogeneous function of Capital and Labor. Because the population is stable Labor is proportional to population size, and the production function can be written as:

$$F(t) = P(t)f(k(t))$$

where $k(t) = \dfrac{K(t)}{P(t)}$ is capital per capita at time t. f is assumed to satisfy the usual conditions, $f \geq 0, f' > 0$ and $f'' < 0$ plus the Inada conditions:

$$\lim_{k\to 0} f'(k) = +\infty \quad \text{and} \quad \lim_{k\to +\infty} f'(k) = 0$$

We will assume that agents are selfish and that their satisfaction exclusively derives from their consumption. There are therefore no altruistic transfers in our model. More precisely we assume that for any length of life T and any consumption pattern $c_T(x)$, $x = 0$ to T, agents have a utility $U(c_T)$. At this point we only assume that U is continuous and is an increasing function in the sense that $U(\lambda c_T) > U(c_T)$ for any $\lambda > 1$. Some additional assumptions will be needed to derive the existence of a steady state equilibrium, in which r and k are endogenously determined, but these assumptions will be given later on, when necessary.

We will say that an economy is a closed market economy if the four following conditions are fulfilled:

- All that is produced is consumed or invested.
- Labor is paid its marginal product.
- The rate of interest equals the marginal product of capital.
- Agents behave rationally, maximizing their expected utility (the length of life being uncertain) under the constraint $PV(c) \leq PV(y_l)$.

This latter condition coincides with Case C of Yaari (1965, p. 141). It implicitly assumes that there exist free and actuarially fair life insurance and annuities. Mortality risk can then be perfectly shared and agents can choose their consumption profile under the constraint that the expected present value of consumption at birth does not exceed their expected income.

With our notation the first condition may be written as:

$$f(k(t)) = \frac{dk}{dt}(t) + nk(t) + Pop(c,t) \tag{7}$$

which is the basic dynamic equation of Solow's growth model.

Since we do not want to make additional assumptions about how Labor earnings vary with age, we will use the second condition only at the aggregate level:

$$Pop(y_l,t) = f(k(t)) - r(t)k(t) \tag{8}$$

with $r(t) = f'(k(t))$.

In the absence of nonmarket transfers, rational behavior implies:

$$PV(y_l, t) = PV(c, t) \tag{9}$$

the following we note:

$$\xi(x, t) = c(x, t) - y_l(x, t)$$

We know from Eq. (9) that $\xi(x, t)$ is life cycle balanced in the sense we defined in Sect. 3. $W(\xi, t)$ corresponds to the usual notion of life cycle wealth as noted earlier.

Public and private transfers, which are ruled out in this section, will be introduced in Sect. 5. Even without them, however, these assumptions are more general than those in numerous articles such as in Galor and Ryder (1989) and Galor (1992). Following Samuelson (1958), Diamond (1965), Gale (1972, 1973), Balasko et al. (1980), Tirole (1985), Weil (1987) and Lee (1994a, b), among others, we allow the aggregate wealth held by individuals to differ from the amount of capital, through the presence of "asset bubbles" (see Tirole, 1985)[9]. By "balance" we will mean the difference between the aggregate wealth per capita and the capital per capita, which is $W(\xi, t) - k(t)$. This concept of balance, which corresponds to Tirole's notion of asset bubble, will play a crucial role throughout our analysis.

The following result will be useful in subsequent analysis.

Proposition 2. *In a closed market economy we have:*

$$\frac{d}{dt}(W(\xi, t) - k(t)) = (r(t) - n)(W(\xi, t) - k(t)) \tag{10}$$

Proof. Combining Eqs. (7) and (8) we find:

$$Pop(\xi, t) = Pop(c, t) - Pop(y_l, t) = -\frac{dk}{dt}(t) + (r(t) - n)k(t) \tag{11}$$

Also as a consequence of Proposition 1 we have:

$$Pop(\xi, t) = -\frac{dW}{dt}(\xi, t) + (r(t) - n(t))W(t) + b(t)PV(\xi, t)$$

Since $PV(\xi, t) = 0$ by hypothesis (Eq. (9)), the subtraction of these two equations give the desired result.

4.1. The steady-states

In a steady state Eq. (10) becomes simply:

$$(r - n)(W(\xi) - k) = 0$$

Therefore, using Gale's classification:

Theorem 1. *A steady state of a closed market economy is always either "balanced" (no bubble) ($W(\xi) = k$) or "Golden-rule" ($r = n$).*

This result is well-known for the case of finite discrete age groups[10].

Although we have described some properties that must be satisfied by any steady-state, we have not yet shown that any such steady states exist. Here we will establish existence under an additional assumption:

Assumption A1. The utility and the production functions are such that in the hypothetical limit $k \to 0$ (and $r \to +\infty$), the aggregate wealth implied by rational behavior of the individuals would exceed the value of capital.

Assumption $A1$ links together the properties of the individual's preferences and the properties of the production function. As k goes to zero the rate of interest goes to $+\infty$ and therefore we expect people to postpone their consumption, so that the wealth would be greater than the capital. However, at the same time that k tends to zero labor income decreases. It may happen, in some particular cases, that preferences for present consumption increase as income decreases in such a way as to offset the first effect. Assumption $A1$ is made in most articles on productive two age-group models (as in Diamond 1965; Tirole 1985; Weil 1987; etc.). Konishi and Perera-Tallo (1997) established a sufficient condition for $A1$ to hold. Their condition merely says that marginal rate of substitution between present and future consumption must remain bounded as consumption goes to zero and that as k goes to zero the labor share of production does not go to zero.

In the continuous case, this sufficient condition may be replaced by the following one:

Proposition 3. *If preferences are additive and homothetic and if for any $\varepsilon > 0$ there exists k_0 such that:*

$$\frac{f(k) - kf'(k)}{k} > e^{-\varepsilon f'(k)}$$

for $k < k_0$, then assumption A1 is always satisfied.

Note that this condition is less restrictive for the production function than the one assumed by Konishi and Perera-Tallo. Indeed, instead of assuming that the labor share of production does not vanish when k goes to zero, we only need to assume here that this share does not vanish too quickly. The proof is in Appendix B.

The following theorem extends Gale's result (1973):

Theorem 2. *If assumption A1 is fulfilled, there always exist both a balanced steady-state and a Golden Rule steady state.*

Moreover, if in the Golden-Rule $W(\xi) > k$ (resp: $W(\xi) < k$), then there exists a balanced equilibrium with $r < n$ (resp: $r > n$)[11].

The proof is in Appendix C. Gale (1973) used the term "Samuelson" for the case where $W(\xi) > k$ in the Golden Rule steady-state, and "Classical" for the case where $W(\xi) < k$ in the Golden Rule steady-state. Golden Rule steady states of the former kind may be supported by the existence of money (with positive value) or by transfers from younger to older members of the population (which are ruled out by assumption in this section, however). A Golden-Rule steady-state of the latter kind would require some other institutional support allowing the society to keep a surplus of capital, since money of negative value contradicts its free disposal.

There are several results concerning the welfare of agents. The following theorem, proved in Appendix D, extends the first result of Starrett (1972):

Theorem 3. *A Golden Rule steady state is optimal.*

This result is comparative static. It says that the best steady-state is the Golden-Rule steady-state. Such a result was also obtained by Calvo and Obstfeld (1988) for a model similar to ours.

It may happen coincidentally that the Golden-Rule steady state is also a balanced steady state. This case corresponds to the "Goldenest Golden Rule" of Samuelson (1975). Indeed, more generally, we claim that:

Theorem 4. *If we denote by $U(n)$ the lifetime utility of individuals in the Golden-Rule steady state with rate of population growth n, and if $U(n)$ is continuously differentiable, then the first derivative of U always has the sign of the balance $W(\xi) - k$ of this Golden-Rule steady state.*

In other words we have:

$$(W(\xi_{gr}) - k_{gr})\frac{dU}{dn} \geq 0$$

where the subscripts gr indicate reference to the Golden-Rule steady state. Following Lee (1994a), this may also be written as:

$$(Pop(c)(A_c - A_{y_l}) - k)\frac{dU}{dn} \geq 0$$

where A_c and A_{y_l} are the average ages of consumption and labor income.

In particular if the Golden Rule is balanced then $\frac{dU}{dn} = 0$, which is the first order derivative condition that should be satisfied for an optimal population growth[12]. More precisely we may say that if the society is in a Classical Golden Rule or, equivalently, if the average age of labor income is greater than the average age of consumption, the welfare of individuals could be improved, in the long term, by a slower population growth. The reverse result holds if the economy is in a Samuelson Golden Rule steady state or if average age of consumption exceeds the average age of labor income. This result has been proved by Arthur and McNicoll (1978) for the case of an additive and atemporal utility function and is proved in a more general context in Appendix E[13].

Coming back to the situation where the rate of population growth is exogenously fixed at a value n, we have the following properties which generalize Starrett's second result (1972):

Theorem 5. *A balanced steady state is efficient if $r \geq n$ and inefficient if $r < n$.*

We know from Theorem 3 that the optimal steady-state is Golden-Rule. Thus a balanced steady-state with $r \neq n$ leads to a lower welfare level than does the Golden-Rule steady state. However, the transition from such a steady-state towards the Golden-Rule steady state may be costly for some generations whose welfare may fall below what they would have experienced in the balanced equilibrium. This theorem, proved in Appendix F, shows precisely that if the economy is initially in a balanced steady-state with $r > n$, then a transition from the balanced steady state to the optimal one would necessarily be costly for some individuals.

These static properties do not imply that a market economy will converge to the optimal equilibrium, or even to an efficient one. Samuelson's numerical example (1958), in a pure exchange economy with three age-groups, gives a situation where from almost every initial condition the economy will converge to an inefficient balanced equilibrium. Such a result shows the interest of studying the dynamic properties of overlapping generations market economies in general, as we begin to do in the following section.

4.2. Dynamics

Our framework allows us to gain some insight into the dynamics of closed market economies of this sort, retaining the assumption of steady state population.

Let us begin with the result of Proposition 2 which says that in market economies we have:

$$\frac{d}{dt}(W(\xi,t) - k(t)) = (r(t) - n)(W(\xi,t) - k(t))$$

This can be solved to yield:

$$W(\xi,t) - k(t) = (W(\xi,0) - k(0)) \exp\left(\int_0^t (r(a) - n)\, da\right) \tag{12}$$

Therefore we see that the evolution of the balance $W(\xi,t) - k(t)$ depends exclusively on the nature of the generalized integral:

$$\int_0^{+\infty} (r(t) - n)\, dt$$

This integral depends on the evolution of the rate of interest and thus indirectly on the agents' behavior. Nonetheless, we can obtain some fairly general results.

Theorem 6.
1) *If a program is balanced at some time then it remains balanced for ever.*
2) *The sign of the balance $W(\xi,t) - k(t)$ of a program is constant.*
3) *A balanced equilibrium with $r > n$ is not stable.*

All these results are well known for the two age-group model. The first point of this theorem is a generalization of the impossibility theorem of

Samuelson (1958). It is also in Gale (1973) for a N age group model in a pure exchange economy. It states that a market economy cannot support the transition from a balanced state to an unbalanced one.

The second point generalizes the first. Together with the third point they give a partial answer to Gale's conjecture[14]. Indeed, if we are in what Gale calls the classical case (that is with $W(\xi) < k$ in the Golden-Rule steady-state) we know that there exists at least one balanced steady state with $r > n$. The theorem says that in this case the economy will tend to move away (at least locally) from this balanced steady state. Moreover, if the initial conditions are such that $W(\xi, 0) \geq k(0)$ then the balance will remain non positive (from point 2 of the proposition) and the economy will not converge towards the Golden-Rule steady states which would be characterized by a negative balance.

In the third point, by "not stable" we mean that for at least some initial conditions infinitely close to the steady-state the economic path will move away from the steady-state.

Proof of Theorem 6. Points 1 and 2 come directly from Eq. (12). To prove point 3 we have to show that for some initial conditions infinitely close to the steady-state the economic path will move away from the steady-state. Now suppose that there exists a balanced steady state with a rate of interest $r^* > n$. Let us choose some initial conditions where the balance is close, but not equal, to zero, and where the rate of interest is close to r^* (so that $r - n > 0$). From Eq. (12) we know that an evolution where r remains close to r^* and the balance tends to zero is impossible. Thus we see that in this case, the economy which starts from initial conditions arbitrarily close to the balanced steady-state will move away from this steady-state.

These dynamic results are quite incomplete. In particular results on convergence to particular steady states known for the two age-group model could not be extended. We have nonetheless made progress since the results that we derived on instability of some steady-states were only available for the two age group model.

5. General economies

Although the study of market economies has been the main preoccupation for economists, all real world economies in fact include many kinds of transfers, such as child rearing, Pay-As-You-Go pension systems, familial intergenerational transfers and government taxes and transfers. Indeed, such non-market transfers comprise by far the most important source of, or institutional support for, unbalanced economies (see Lee 1994a for a quantification).

The aim of this section is to show how our previous analysis can be easily extended to these general economies. In particular, we will describe the possible steady states and show how their characteristics are linked to the properties of the transfer systems.

Let us call τ the sum of all the non-market systems of reallocation, familial and governmental. Purely for expositional convenience, we may think of τ as operationalized by some abstract (or real) unproductive institution, which gives $\tau^+(x, t)$ to any individual of age x at time t and collects $\tau^-(x, t)$, the net transfers being $\tau(x, t) = \tau^+(x, t) - \tau^-(x, t)$. In a closed economy, it is the nature of transfers that the amount given by some individuals must be received by others, so that $\tau(x, t)$ must satisfy population balance, that is $Pop(\tau, t) = 0$.

However, we may also think of different situations, such as an open economy where $\tau(x,t)$ may include some government taxes used to pay some foreign economies. For example, this would be the case for a country which has to pay interest on its external debt as in Diamond (1965). In this case τ does not need to satisfy population balance, and to avoid any loss of generality we will not make any assumption here on the value of $Pop(\tau,t)$.

Let us define here:

$$\theta(x,t) = \xi(x,t) - \tau(x,t) = c(x,t) - y_l(x,t) - \tau(x,t)$$

In the absence of capital depreciation we have:

$$\frac{dk}{dt}(k) + nk(t) = rk(t) - Pop(\theta,t)$$

The new life cycle budget constraint is $PV(c) \leq PV(y_l) + PV(\tau)$, so rational behavior implies that $PV(\theta,t) = 0$. Thus using proposition 1 we get:

$$\frac{d}{dt}(W(\theta,t) - k(t)) = (r(t) - n)(W(\theta,t) - k(t)) \tag{13}$$

In a steady-state all the dependence in t disappears and this equation simply becomes $(r-n)(W(\theta) - k) = 0$. Therefore:

Theorem 7. *A steady-state must be either:*

– Golden-Rule

– Non-Golden Rule, with balance equaling the transfer wealth associated with τ, $W(\tau)$:

$$W(\xi) - k = W(\tau) = \frac{1}{n-r}(bPV(\tau) - Pop(\tau)) \tag{14}$$

Theorem 7 is obviously the generalization of Theorem 1. Most of the previous results can be extended in the same way. Indeed we see at a glance that Eqs. (10) and (13) are identical. In the case of a general economy with intergenerational transfers, the important variable is no longer the balance $W(\xi,t) - k$, but $W(\theta) - k$, which is also $W(\xi) + W(\tau) - k$, the difference between the aggregate wealth and the sum of the capital per capita and the (institutional) transfer wealth. We will call this difference the "residual balance" which includes money, bonds or other asset bubbles. Theorems 3, 5 and Theorem 6 become then:

Theorem 8. *In a general economy with fixed (and unchangeable) institutional transfers and taxes:*

(i) *A Golden-Rule steady-state is optimal.*

(ii) *A steady state with no residual balance is efficient if $r \geq n$ and inefficient if $r < n$.*

(iii) *If a program has no residual balance at some time then it has none forever.*

(iv) *The sign of the residual balance is constant.*

(v) *A steady-state with no residual balance and with $r > n$ is not stable.*

Proof. The proof of (i) is exactly the same as the proof of Theorem 3. For the proof of (ii) we can literally follow the proof of Theorem 5 replacing $c(x,t)$

and $c_0(x)$ by $c(x,t) - \tau(x)$ and $c_0(x) - \tau(x)$, respectively. Substituting ξ for θ in the proof of Theorem 6 proves (iii), (iv) and (v).

6. Conclusion

The extensive literature on overlapping generation models is rich and productive, yet it suffers from its reliance on simplistic demographic assumptions which are largely unnecessary. The past literature has mainly assumed only two age groups and perfect survival until the end of the second of these. Theoretical results for two age groups sometimes do not generalize, and without mortality one cannot study the implications of its change. Such a crude model cannot even simultaneously accommodate dependent childhood, productive mid years, and retirement. Any kind of empirical implementation of these models is virtually impossible.

This paper aimed to revisit the literature on overlapping generations models in the demographic context of a continuous age distribution and a general age schedule of mortality. The core economic model was standard.

The proofs of our results make use of an accounting framework that handles the aggregation problem relatively simply. Once this accounting framework is developed, our proofs are mostly no longer than those for the two age-group model. The accounting framework is of interest in itself, and we showed it could be used to derive the fundamental equation of generational accounting. We also provided some illustrative empirical estimates of various quantities in the accounting framework for the US.

We showed that most static results known for the two or N age group model could be extended to this more realistic demographic setting. Only a slight difference appeared, when considering the existence of a balanced steady state equilibrium. The sufficient condition obtain by Konishi and Perera-Tallo (1997) was replaced by a weaker condition. Dynamic results could be extended to a lesser extent, with results on instability being much easier to prove than convergence results. In a final section we showed that our results could be easily extended to a general economy with non-market transfers. This extension does not introduce any technical innovation, nor complication, and should be of interest for future empirical applications.

Economists should not be put off by the apparent complexity of realistic demographic models, models which in principle should permit a much greater degree of generality and relevance to real world phenomena and policy problems. We have shown that such models remain tractable, and that comparative static, dynamic and welfare theoretic results can be obtained.

Endnotes

1 Demographic studies show that this assumption is not realistic, and that, after age 30, the probability of dying increases more or less exponentially with age, doubling every 9 years or so.

2 This typically corresponds to the framework of most three age-group models where all the periods and age groups are assumed to have the same length.

3 The most common longitudinal constraint is the life cycle budget constraint which says that in a market economy an individual cannot consume more than he earns. A typical cross sectional constraint is that "all that is produced is consumed or invested" (in a closed economy) or for transfers, that "all that is given is received".

[4] We therefore assume that the population is homogeneous in the sense that all individuals of the same cohort have the same expectations, although most results hold for heterogeneous populations as well. Indeed all our results, except Proposition 3 and Theorems 2, 3, 4 and 5, hold for heterogeneous populations if we assume that demographic and economic heterogeneity are independent. All the results, without exception, hold if we also assume that the utility function is homothetic.

[5] In particular the notion of life cycle wealth of Kotlikoff and Summers (1981), defined as accumulated earnings minus accumulated consumption, differs from ours. These notions will only coincide when all individuals spend as much as they earn during their life. This is the case only if there are no intergenerational transfers or if transfers are life-cycle balanced, in the sense defined above.

[6] Non zero wealth arises, for example, when people have to participate in a Pay as You Go pension system. The rate of return for such systems is n (if productivity growth is zero) and therefore social security wealth at birth, discounted at rate r, $PV(g)$, is negative whenever the rate of interest is larger than the rate of economic growth.

[7] More generally, if transfers are rising at a constant rate λ, then the implicit rate of return of any transfer system in steady state is $n + \lambda$.

[8] g could be limited to taxes and transfers, as was done in the original versions of generational accounting, or a more expansive definition of government services could be adopted.

[9] Tirole defines a "bubble" as the difference between the market price of an asset and its market fundamental (i.e. the expected present discounted value of its dividends) (Tirole 1985:1071). Assets with bubbles include money, bonds, rights for intergenerational transfers, etc. For Tirole, bubbles cannot be negative, since he assumes that agents can freely dispose of them. However it's clear that bubbles such as intergenerational transfers can also have a negative value.

[10] See Gale (1972) for the two age-group model and Kim and Willis (1982) for the N age-group model.

[11] If assumption $A1$ does not hold then there may not exist a (non trivial) balanced steady-state when $W(\xi) < k$ in the Golden-Rule (see Galor and Ryder 1989).

[12] Samuelson (1975) intuitively interpreted this necessary condition, $\frac{dU}{dn} = 0$, as a sufficient condition for the existence of an optimal population growth rate. But Deardoff (1976), with Samuelson's acknowledgment (1976), showed that this rate of growth may also correspond to a welfare minimum. In general it is not obvious that there exists a finite rate of population growth which is optimal. The existence of such a rate has been proved, under certain conditions, by Kim and Willis (1982) in the three age-group case.

[13] In fact Arthur and McNicoll expressed their result in terms of the difference in mean ages of consumption and labor income and Willis (1988) was the first to connect this difference to the balance measure, $W - k$, and therefore to the notion of Classical and Samuleson golden rule steady-states.

[14] Gale (1973) conjectured, for an economy with no durable good, that in the Classical case the balanced steady-state is unstable, the economy converging toward the Golden-Rule only if the initial balance is negative. He conjectured also that in the Samuelson case the economy always converges toward the balanced steady-state.

[15] In fact we could have shown that (17) is true for any ε_1 smaller than the difference between the maximum age at death and the minimum age of non-zero productivity. We would obtain a weaker sufficient condition supposing only that (18) has to be true for some ε smaller than this age gap.

Appendix

A. Proof of Proposition 1

From Eq. (1) we compute the variation of individual wealth along the life-cycle:

$$\left(\frac{d}{dx} + \frac{d}{dt}\right) w_g(x,t) = \left(-g(x,t) + r(t) w_g(x,t) - \frac{p'_x + p'_t}{p}(x,t) w_g(x,t)\right) \quad (15)$$

At the aggregate level, or in other words, multiplying both sides of this equations by $B(t-x)p(x,t)$ and integrating between 0 and ω we get:

$$\int_0^{\omega} B(t-x)\left(\frac{d}{dt}+\frac{d}{dx}\right)(p(x,t)w_g(x,t))\,dx$$
$$= -P(t)Pop(g,t) + r(t)P(t)W(g,t) \tag{16}$$

On the other hand let us note $f(t)$ the aggregate wealth of the population:

$$f(t) = P(t)W(g,t) = \int_0^{\omega} B(t-x)p(x,t)w_g(x,t)\,dx$$

A simple derivation shows that:

$$f'(t) = \int_0^{\omega}\left(B'(t-x)p(x,t)w_g(x,t)\,dx + B(t-x)\frac{d}{dt}(p(x,t)w_g(x,t))\right)dx$$

Integrating by parts the first term of the integral yields:

$$f'(t) = B(t)p(0,t)w_g(0,t) + \int_0^{\omega} B(t-x)\left(\frac{d}{dt}+\frac{d}{dx}\right)(p(x,t)w_g(x,t))\,dx$$

The first term on the right hand side of this equation is precisely $B(t)PV(g,t)$. Using Eq. (16) we obtain:

$$f'(t) = \frac{d}{dt}(P(t)W(g,t)) = r(t)P(t)W(g,t) - P(t)Pop(g,t) + B(t)PV(g,t)$$

which is the aggregate version of the accounting equation of Theorem 2.2. Indeed, writing $\frac{d}{dt}(P(t)W(g,t)) = \frac{dP(t)}{dt} + \frac{dW(g,t)}{dt}$ and dividing the equation by $P(t)$ gives the per capita accounting equation announced in Theorem 2.2.

B. Proof of Proposition 3

As remarked by Kessler and Masson (1988) any additive homothetic intertemporal utility function leading to time consistent individual choices may be written as:

$$U(c) = \int_0^{\omega} p(x)e^{-\lambda x}u(c(x))\,dx$$

where $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$. A simple calculation shows that the consumption pattern that follows from utility maximization satisfies $c(x) = c(0)e^{(r-\lambda)x/\gamma}$. The value of $c(0)$ is determined by the constraint $PV(c) = PV(y_l)$ and we have:

$$\frac{1}{b}Pop(c) = \frac{\int_0^\omega e^{-nx} p(x) e^{(r-\lambda)x/\gamma}\, dx}{\int_0^\omega e^{-rx} p(x) e^{(r-\lambda)x/\gamma}\, dx} PV(y_l)$$

It follows that for any age a such that $p(a) \neq 0$ there exists r_0 such that $\frac{1}{b}Pop(c) \geq e^{(r-n)a} PV(y_l)$ for $r \geq r_0$. In particular, since the income profile (given by the labor productivity) is supposed to be fixed, we know that there exists $\varepsilon > 0$ such that for r large enough $Pop(c) \geq e^{\varepsilon r} Pop(y_l)$.

Assumption $h2$ says that $W(\xi)$ should be greater than k in the hypothetical limit $k \to 0$. We know from Theorem 1 that $(r-n)W(\xi) = Pop(\xi)$ so assumption $h2$ is also equivalent to: "In the hypothetical limit $r \to +\infty$ we should have $Pop(c) - Pop(y_l) > (r-n)k$."

We have seen that for some positive ε we have in the limit $r \to +\infty$:

$$Pop(c) - Pop(y_l) > (e^{\varepsilon r} - 1)Pop(y_l) \geq e^{\varepsilon_1 r} Pop(y_l) = e^{\varepsilon_1 f'(k)}(f(k) - kf'(k)) \tag{17}$$

Thus, assuming that preferences are additive and homothetic, the condition which says that for any $\varepsilon > 0$ we must have:

$$\frac{f(k) - kf'(k)}{k} > e^{-\varepsilon f'(k)} \tag{18}$$

for k small enough is enough to ensure assumption $h2$.[15]

C. *Proof of Theorem 2*

Assuming that agents behave rationally, for any hypothetical steady state, characterized by a value of capital per capita k and a rate of interest $r = f'(k)$, there corresponds an income profile $y_l(x)$ determined by the marginal productivity of Labor and a consumption profile $c(x)$ which maximize $U(c(x))$ under the constraint $PV(c) = PV(y_l)$. We will note $\xi(x) = c(x) - y_l(x)$ where $-\xi(x)$ is in some sense the "rational" investment of individuals of age x from their labor income. An hypothetical steady state will actually be a feasible steady state if the desired investment, $-Pop(\xi)$, is equal to the investment needed to maintain the capital per capita at his level. This means that a steady-state is feasible if and only if:

$$-Pop(\xi) = (n-r)k$$

Let us call $z(r) = -Pop(\xi) - (n-r)k$ which is the difference between the rational investment and the necessary investment to support a steady state. We will show that $z(r)$ has always at least two roots when r varies in $[0, +\infty]$, and therefore that there are always two possible steady-states.

The root $r = n$ corresponding to the Golden-Rule steady state is obvious. Indeed when $r = n$ the application PV and Pop are proportional and:

$$Pop(\xi) = Pop(c) - Pop(y_l) = bPV(c) - bPV(y_l) = 0$$

since $PV(c) = PV(y_l)$ is the constraint of the individual maximization program.

It remains to see that there is at least one other root. The consumption being non negative we have $-Pop(\xi) \leq Pop(y_l) = f(k) - rk$. Thus we see that $z(r) \leq f(k) - nk$ and since $f'(k) \to 0$ when $k \to +\infty$ we have:

$$\lim_{r \to 0} z(r) = -\infty \tag{19}$$

From Theorem 1 we know that in a steady state we have $Pop(\xi) = (r - n)W(\xi)$ (since $PV(\xi) = 0$ from the individual budget constraint). Therefore:

$$z(r) = (n - r)(W(\xi) - k) \tag{20}$$

and hypothesis $h2$ implies that $z(r) \leq 0$ when $r \to +\infty$.

The function $z(r)$ is continuous, non positive when $r \to +\infty$ and when $r \to 0$ and is equal to zero when $r = n$. Moreover we know from Eq. (20) that the derivative of z in $r = n$ is given by:

$$\left.\frac{dz}{dr}\right|_{r=n} = -(W(\xi) - k)|_{r=n}$$

which is the opposite of the balance in the Golden Rule equilibrium. Thus in the classical case where $W(\xi) < k$ in the Golden rule we know that there exists at least one root of $z(r)$ greater than n and in the "Samuelson's case", when $W(\xi) > k$ in the Golden rule we know that there exists one root smaller than n.

D. Proof of Theorem 3

By claiming that a Golden-Rule steady-state is optimal we mean that agents in such a steady-state have a higher lifetime utility that in any other steady state. Indeed let us call k_g, $c_g(x)$, y_{l_g} the capital per capita, labor income and consumption profiles of the Golden-Rule steady state and k and $c(x)$, $y_l(x)$ the values in another steady-state. To show that $U(c(x)) \leq U(c_g(x))$ we only need to prove that $Pop(c) \leq Pop(y_{l_g})$ since $c_g(x)$ is by assumption a solution of the program:

$$\underset{Pop(c_g) \leq Pop(y_{l_g})}{\text{Max}} E(U(c_g(x)))$$

(in the Golden Rule the function Pop and PV are proportional). But it is well known that the Golden Rule steady state is the steady state that maximizes the aggregate consumption. So we have $Pop(c) \leq Pop(c_g) \leq Pop(y_{l_g})$, which completes the proof.

E. Proof of Theorem 4

Let us note $c_n(x)$ and y_{l_n} the consumption an labor income of individuals of age x in the Golden-Rule of rate of population growth (and rate of interest) n. We know that c_n maximizes the utility function under the constraint $PV(c_n) =$

$PV(y_{l_n})$. Thus, any consumption pattern such that $PV(c) < PV(c_n)$ would make individuals worse off and any better consumption pattern will have to satisfy $PV(c) > PV(c_n)$. Consequently, as we assumed $U(n)$ to be continuously differentiable, the sign of $\frac{dU}{dn}(n)$ will be the same as the sign of $PV\left(\frac{\partial c}{\partial n}\right)$.

We know that for every Golden Rule steady state we have $PV(c_n) = PV(y_{l_n})$. Differentiating both sides of this equality and using the fact that for a positive system of reallocation $\frac{d}{dn}(PV(g)) = -A_g PV(g) + PV\left(\frac{\partial g}{\partial n}\right)$ we obtain:

$$-A_c PV(c) + PV\left(\frac{\partial c}{\partial n}\right) = -A_{y_l} PV(i) + PV\left(\frac{\partial y_l}{\partial n}\right) \tag{21}$$

$bPV\left(\frac{\partial y_l}{\partial n}\right)$ or also $Pop\left(\frac{\partial y_l}{\partial n}\right)$ is the variation of aggregate income at constant structure by age and is given by:

$$bPV\left(\frac{\partial y_l}{\partial n}\right) = Pop\left(\frac{\partial y_l}{\partial n}\right) = \frac{d}{dn}(f(k) - nk) = (f'(k) - n)\frac{\partial k}{\partial n} - k = -k \tag{22}$$

which corresponds to the classic effect of capital dilution of the growth model of Solow. Using the result of Lee (1994a), we know that:

$$W(\xi) = (A_c - A_{y_l})Pop(c) \tag{23}$$

From (21) (22) (23) we get $PV\left(\frac{\partial c}{\partial n}\right) = \frac{1}{b}(W(\xi) - k)$ which completes the proof.

F. Proof of Theorem 5

Let us prove first that a balanced equilibrium program with $r > n$ is efficient, or in other words that there does not exist any transition that makes nobody worse off and at least one individual better off. For this we show that the existence of such a transition would lead to some inconsistency.

Let us call k_0, r_0, c_0 the values of the capital per capita, rate of interest and consumption of the steady state, before the transition, and call PV_0 the function PV obtained for $r = r_0$. Applying the result of theorem (1) we know that:

$$\frac{dW_0}{dt}(c,t) = (r_0 - n)W_0(c,t) + bPV_0(c,t) - Pop(c,t)$$

where we call W_0 the notion of wealth defined as in Eq. (2) for $r = r_0$. With Eq. (7) we get:

$$\frac{d(W_0(c,t) - k)}{dt} = (W_0(c,t) - k)(r_0 - n) + r_0 k - f(k) + bPV_0(c,t)$$

Calling:

$$\alpha(t) = (W_0(c,t) - k) - (W_0(c_0) - k_0)$$

we have:

$$\frac{d}{dt}(\alpha e^{(n-r_0)t}) = e^{(n-r_0)t}(r_0 k - f(k) + bPV_0(c,t) - (n - r_0)(W_0(c_0) - k_0)) \tag{24}$$

or also given the fact that $\alpha(0) = 0$, per definition, and that $(n - r_0) \cdot (W_0(c_0) - k_0) = r_0 k_0 - f(k_0) + bPV_0(c_0)$ (since the zero indices correspond to a steady state program) we obtain:

$$\alpha(t) = \int_0^t e^{(r_0-n)(t-u)}(r_0 k(u) - f(k(u)) - (r_0 - f(k_0)) + bPV_0(c,u) - bPV(c_0))\,du \tag{25}$$

Now note that a simple study of the variation of the function $r_0 k - f(k)$ shows that for all k we have $r_0 k - f(k) \geq r_0 k_0 - f(k_0)$ since $r_0 = f'(k_0)$ and $f''(k) \leq 0$. Remark also that a transition that makes one person better off and nobody worse off must pass by a point where $PV_0(c,t) > PV_0(c_0)$ and be such that $PV_0(c,t) \geq PV_0(c_0)$ for every t. Thus, since $r_0 > n$, we see from Eq. (25) that such a transition would lead to:

$$\lim_{t \to +\infty} (W_0(c,t) - k)(t) = +\infty$$

which is physically impossible (it would mean that the expected consumption of agents, discounted at the rate r_0, would tend to infinity).

To prove that a balanced steady-state with $r < n$ is not efficient we must construct a transition that makes nobody worse off and some persons better off. The idea is very simple. Imagine that starting from this balanced steady-state we add after some time $t = t_0$ an infinitesimal population balanced intergenerational transfer going from the younger to the elderly. The capital per capita is not affected by such a transition since we make only a reallocation of resources between generations. It is easy to check that after the transition is achieved people will have a higher utility than those who were alive before the transition since from an individual viewpoint this intergenerational transfer is like an infinitesimal investment at a rate $n > r$. For people alive during the transition the situation is even better since they receive all the benefit of this kind of "investment" without having paid all the contributions.

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